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TWO KINDS OF HAMILTONIAN DEGREE SEQUENCES

ZHU YONGJIN & LIU ZHENHONG

I. INTRODUCTION

It is well known that to identify whether a graph is hamiltonian is a puzzlling problem. Though there are several sufficient conditions for a graph to be hamiltonian, they all imply that the graph contains many edges. In an undirected graph, there are two important sufficient conditions for a graph to be hamiltonian, one given by Chvátal[2] and the other by Bondy and Chvátal[1]. Most of the sufficient conditions for a graph to be hamiltonian are so far contained in them.

Our paper presents two kinds of graphical degree sequences which ensure the graph to be hamiltonian, but they do not satisfy the condition given by Chvátal, and one of them does not satisfy the condition given by Bondy and Chvátal.

In order to verify our main results, we introduce some concepts first.

All graphs considered in this paper are simple. A graph is said to be hamiltonian if it contains a hamiltonian cycle, otherwise it is called nonhamiltonian. A graph is said to be maximal nonhamiltonian if it is nonhamiltonian and $G+(u,v)$ is hamiltonian for any two nonadjacent vertices u and v in G, where (u,v) denotes an edge with end points u and v.

Lemma 1.[4] *If $G=(V,E)$ is nonhamiltonian, then there exists a maximal nonhamiltonian graph $\tilde{G}=(V,\tilde{E})$, such that $E\subseteq\tilde{E}$.*

Lemma 2.[4] *Let $P(u,v)$ be a hamiltonian chain of graph G, with u and v as its end points, If $d(u)+d(v)\geqslant n$, then G is hamiltonian, where $d(x)$ denotes the degree of vertex x and n the number of vertices of G.*

Lemma 3.[4] *G is maximal nonhamiltonian. If $d(u)+d(v)\geqslant n$ for any vertices u and v in G, then $(u,v)\in E$.*

A nonnegative integer sequence $\pi=(d_1,d_2,\cdots,d_n)$ is called a degree sequence if there exists a graph with π as its degree sequence.

A degree sequence $\pi=(d_1,d_2,\cdots,d_n)$ is said to be hamiltonian if all graphs with π as their degree sequence are hamiltonian.

II. TWO MAIN RESUTS

Theorem 1. Let

$$d_i=\begin{cases} r-1, & 1\leqslant i\leqslant r-2,\\ r, & r-1\leqslant i\leqslant r,\\ n-r-1, & r+1\leqslant i\leqslant n,\end{cases}$$

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where $3 \leqslant r < [n/2]$. *Then* $\pi = (d_1, d_2, \cdots, d_n)$ *is hamiltonian*

Proof. Suppose this is not true and there exists a nonhamiltonian graph $G = (V, E)$ with π as its degree sequence. Let

$$S = \{v \mid v \in V, d(v) = n - r - 1\}$$

and $G(S)$ denote the subgraph of G generated by S. Let $\tilde{G} = (V, \tilde{E})$ be the maximal nonhamiltonian graph of G. By lemma 3 and $r < [n/2]$, $\tilde{G}(S)$ is a complete subgraph of G. Set

$$T = \{v \mid v \in S \text{ and } u \in V - S \text{ such that } (u, v) \in \tilde{E}\}.$$

Then

1) $2 \leqslant |T| \leqslant r$.

In fact, the subgraph $G(V-S)$ of G generated by $V - S$ contains at most $r(r-1)/2$ edges, but by the assumption, the sum of degree of vertices in $V - S$ in G is $r(r-1)+2$, and thus there is at least two edges connecting vertices in S with vertices in $V - S$, which mplies that $|T| \geqslant 2$.

Furthermore, $d_{\tilde{G}}(v) \geqslant n - r$ for all v in T as $\tilde{G}(S)$ is a complete graph and $|T| \geqslant 2$, where $d_{\tilde{G}}(v)$ denotes the degree of vertex v in $\tilde{G}$. From lemma 3 we know that (u, v), $(u', v) \in \tilde{E}$ for all $v \in T$, where u and u' are vertices with degree r in G. This implies that $d_{\tilde{G}}(v) \geqslant n - r + 1$ for all $v \in T$. Again by lemma 3, $(x, v) \in \tilde{E}$ for all $x \in V - S$, and $v \in T$.

If $|T| \geqslant r + 1$, then $d_{\tilde{G}}(x) \geqslant r + 1$ for all x in $V - S$, which implies that $\tilde{G}$ is a complete graph by lemma 3. This contradicts that $\tilde{G}$ is nonhamiltonian.

2) There exists a vertex x in $V - S$ such that $d_{\tilde{G}}(x) \geqslant r + 1$.

Since $d_{\tilde{G}}(v) \geqslant n - r + 1$ for all v in T, $(x, v) \in \tilde{E}$ for all $x \in V - S$ and $v \in T$. Therefore there are exactly $|T| \cdot r$ edges connecting vertices in $V - S$ and the vertices in T in $\tilde{G}$. Among those edges, there are at most $|T| \cdot (|T| - 1)$ edges in G. Thus

$$\sum_{x \in V-S} d_{\tilde{G}}(x) - \sum_{x \in V-S} d_G(x) \geqslant r \cdot |T| - |T|(|T| - 1)$$

which means that the difference of the degree of each vertex of $V - S$ in $\tilde{G}$ and in G is at least $|T| - |T|(|T| - 1)/r$ on average. Let

$$f(|T|) = r \cdot |T| - |T| \cdot (|T| - 1).$$

Obviously, $f(|T|)$ is a concave function, and its minimum value must reach the boundary. It is enough to consider the cases $|T| = 2$ and $|T| = r$.

a) $|T| = 2$.

In this case, $f(|T|) = 2r - 2$, there must be at least one vertex u in $V - S$, such that $d_{\tilde{G}}(u) - d_G(u) \geqslant 2$ as $2/r < 1$, which means that $d_{\tilde{G}}(u) \geqslant r + 1$.

b) $|T| = r$

In this case, $f(|T|) = r$. It is easy to see that there is a vertex u in $V - S$ such that $d_{\tilde{G}}(u) \geqslant r + 1$.

From the cases a) and b), we may conclude that there is a vertex u in $V-S$ such that $d_{\tilde{G}}(u) \geqslant r+1$. By lemma 3, $(u,v) \in \tilde{E}$ for all v in S. Therefore, $T=S$ by the definition of T and $|T|=|S|=n-r>r$, which is contrary to $|T| \leqslant r$. The contradiction proves that G is hamiltonian.

It can be verified that the condition of theorem 1 does not satisfy Chvátal's one.

Theorem 2. Let

$$d_i=\begin{cases} r-1, & 1 \leqslant i \leqslant r, \\ n-r, & r+1 \leqslant i \leqslant n. \end{cases}$$

If $n \neq 3r-2$ *and* $3 \leqslant r<n/2$, *then* $\pi=(d_1, d_2, \cdots, d_n)$ *is hamiltonian.*

Proof. Let $G=(V,E)$ be a graph with π as its degree sequence and

$$S=\{v \mid v \in V, d_G(v)=n-r\}.$$

Case 1. $G(S)$ is a complete graph.

We can partition $V-S$ into three parts, C_0, C_1 and C, where

$C_0=\{u \mid u \in V-S$ and $(u, v) \notin E$ for any $v \in S\}$,

$C_1=\{u \mid u \in V-S$ and u is adjacent to exactly one vertex in $S\}$,

$C=V-S-\{C_0 \cup C_1\}$.

If $|C_0| \neq 0$ or $|C_1| \neq 2$ or $|C_1|=2$ and both vertices in C_1 are adjacent, there is a hamiltonian chain L in $G(C_0 \cup C_1)$ such that the end points of L, u_1 and u_2, are adjacent to at least two vertices v_1 and v_2 in $C \cup S$, and v_1 adjacent to u_1 and v_2 adjacent to u_2. Thus we can easily construct a hamiltonian cycle since for each vertex v in S, there is exactly one edge connecting v with a vertex in $V-S$.

If $|C_0|=0$ and $|C_1|=2$ and both vertices of C_1 are not adjacent, then each vertex in C_1 is adjacent to a vertex in S and a vertex in C. Therefore it is not difficult to construct a hamiltonian cycle of G since each vertex in S is adjacent to only one vertex in $V-S$.

Case 2. $G(S)$ is not a complete graph.

If G were nonhamiltonian, let $\tilde{G}=(V, \tilde{E})$ be a maximal nonhamiltonian graph of G. By lemma 3 there are at least two vertices v and v' in S such that

$$d_{\tilde{G}}(v) \geqslant n-r+1 \quad \text{and} \quad d_{\tilde{G}}(v') \geqslant n-r+1.$$

Let

$$T=\{v \mid v \in S, d_{\tilde{G}}(v) \geqslant n-r+1\}.$$

Evidently $|T| \geqslant 2$ and by lemma 3 $(u, v) \in \tilde{E}$ for all $u \in V-S$ and $v \in T$. If $|T| \geqslant r$, $\tilde{G}$ is a complete graph, which is contrary to that G is nonhamiltonian. Thus $2 \leqslant |T| \leqslant r-1$.

Since $(u, v) \in \tilde{E}$ for all $u \in V-S$ and $v \in T$, there are exactly $r \cdot |T|$ edges connecting the vertices in $V-S$ with those in T. Thus

$$\sum_{x \in V-S} d_{\tilde{G}}(x)-\sum_{x \in V-S} d_G(x) \geqslant r \cdot |T|-|T|-|T|(|T|-1)=|T|(r-|T|),$$

which means that the difference of the degree of each vertex of $V-S$ in $\tilde{G}$ and in G is at least $|T|-|T|^2/r$ on average. It shows that there is at least one vertex u in $V-S$ such that $d_{\tilde{G}}(u) \geqslant r$. Let

$$g(|T|) = |T|(r - |T|), \quad 2 \leqslant |T| \leqslant r - 1.$$

Obviously $g(|T|)$ is a concave function, whose minimum value must reach the boundary. Thus, it is enough to consider the cases $|T| = 2$ and $|T| = r - 1$.

1) $|T| = 2$

(a) If there are two vertices u' and u'' in $V-S$ such that $d_{\bar{G}}(u') \geqslant r$ and $d_{\bar{G}}(u'') \geqslant r$, then $d_{\bar{G}}(v) \geqslant n - r + 1$ for all $v \in S$. Therefore $T = S$ by the definition of T and $|T| = |S| = n - r > r$, which is contrary to $|T| \leqslant r - 1$.

(b) If there is only one vertex u in $V - S$ such that $d_{\bar{G}}(u) \geqslant r$, then $1 < g(2) = 2(r-2) < r$, which means $r = 3$, and all vertices in $V - S - \{u\}$ can not be adjacent to any vertex in $S - T$. Since $d_G(u) = r - 1 \geqslant |S| - |T| = n - r - 2$ and $r < n/2$, then $n = 7$, which is contrary to $n \neq 3r - 2$.

2) $|T| = r - 1$.

(a) There are two vertices u' and u'' in $V - S$ such that

$$d_{\bar{G}}(u') \geqslant r \quad \text{and} \quad d_{\bar{G}}(u'') \geqslant r.$$

The proof is similar to the case 1) (a).

(b) There is only one vertex u in $V - S$ such that $d_{\bar{G}}(u) \geqslant r$.

Its proof is similar to the case 1) (b). We can obtain $n = 3r - 2$, which contradicts $n \neq 3r - 2$.

3) $2 < |T| < r - 1$.

Obviously, there are at least two vertices u' and u'' in $V - S$ such that

$$d_{\bar{G}}(u') \geqslant r \text{ and } d_{\bar{G}}(u'') \geqslant r.$$

Thus the proof is similar to the case 1) (a).

From 1), 2) and 3) above, we can conclude that G is hamiltonian.

It can be verified that the condition of theorem 2 neither satisfies Chvatal's nor the one given by Bondy and Chvátal.

References

[1] Bondy, J. A., and Chvátal, V., A Method in Graph Theory, *Dis. Math.*, **15** (1976), 111—135.

[2] Chvátal, V., On Hamilton's Ideals, *J. Comb. Theory* (B), **12** (1972), 163—168.

[3] Bollobas, B., Extremal Graph Theory, Academic Press INC, London, 1978.

[4] Nash-Williams Hamiltonian Cycles, Studies in Graph Theory II (1978).

两类 Hamilton 次序列

朱永津 刘振宏

摘要

无向图中 Hamilton 圈存在的充分条件，虽有一些，但几乎都包含在 Chvátal 的次序列条件和 Bondy 与 Chvátal 的 n-稳定闭包是完全图的条件里．本文提出两类次序列，虽然它们都不满足 Chvátal 条件，并且也有一个不满足闭包是完全图的条件，但是却都保证了图的 Hamilton 圈的存在性．从而为进一步从次序列方面研究图的 Hamilton 性，提供了新的依据．