for all $v \in S \cup \{x_0\}$. Thus $(x, v) \in \tilde{E}$ for all $x \in B$ and $v \in S \cup \{x_0\}$, which gives that $\tilde{G}(S \cup B)$ is a complete subgraph. Therefore, for all $v \in S \cup B$, $d_{\tilde{G}}(v) \geqslant |S \cup B| - 1 = n - r + |B| - 1 \geqslant n - r + t$. Consequently $\tilde{G}$ is a complete graph and so $\tilde{G}$ is hamiltonian.

It can be checked that the class of sequences in Theorem 2 does not satisfy the condition given by Chvátal.

Put $t = 1$. The value of l can be any integer from 1 to $r - 2$. In particular, let $l = r - 2$; then Theorem 1 of [1] is obtained.

If $l = \left[\frac{r}{t+1}\right]$, then we have the following

Corollary 2.1. Let

$$d_i = \begin{cases} r - t, & 1 \leqslant i \leqslant \left[\frac{r}{t+1}\right], \\ r, & \left[\frac{r}{t+1}\right] < i \leqslant r, \\ n - r - 1, & r < i \leqslant n \end{cases}$$

where n, r and t are integers such that $1 \leqslant t \leqslant r - 2 < \left[\frac{n}{2}\right] - 2$. Then any graph with $(d_1, \cdots, d_n)$ as its degree sequence is hamiltonian.

REFERENCES

[1] Zhu Yongjin and Liu Zhenhong, Two Kinds of Hamiltonian Degree Sequences, *J. Sys. Sci. & Math. Scis.*, 2: 2(1982), 131—135.

[2] Chvátal, V., On Hamilton's Ideals, *J. Comb. Theory, Ser.* B 12 (1972), 163—168.

[3] Bondy, J. A and Chvátal, V., A Method in Graph Theory, *Dis. Math.*, 15(1976), 111—135.

[4] Bollobás, B., Extremal Graph Theory, Academic Press, London, 1978, 78—79.

具有 Hamilton 性的次序列

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摘 要

朱永津、刘振宏在[1]中给出了两类次序列，它们不满足 Chvátal 条件[2]，其中一类甚至不满足 Bondy 和 Chvátal[3] 的 n-闭包是完全图的条件，但它们却都保证了图的 Hamilton 圈的存在。本文推广了[1]的结果，得到了更为广泛的两大类具有前述性质的次序列。

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ON THE DEGREE SEQUENCES ENSURING THE GRAPHS TO BE HAMILTONIAN

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Two kinds of graphic degree sequences which ensure the graphs to be hamiltonian are presented in [1]. They do not satisfy the condition given by Chvátal[2], and one of them does not satisfy the condition given by Bondy and Chvátal[3]. This paper extends the results of [1].

All graphs considered in the paper will be simple. A graph will be denoted by $G=(V,E)$, where V represents the set of vertices and E the set of edges. If u and v are vertices in G, then (u, v) denotes an edge with u and v as its end points. The set of vertices adjacent to a vertex $x\in V$ in G is denoted by $\Gamma_G(x)$ and $d_G(x)=|\Gamma_G(x)|$ is said to be the degree of x in G. The subgraph of G generated by subset S of V is denoted by $G(S)$. The closure of G is the graph obtained from G by recursively joining pairs of nonadjacent vertices, whose degree sum is at least $|V|$, until no such pair remains.

Lemma 1[3]. *A simple graph is hamiltonian if and only if its closure is hamiltonian.*

A sequence $(d_1, d_2, \cdots, d_n)$ of non-negative integers is said to be graphic if it is a degree sequence of a graph.

Lemma 2[4]. *A sequence $d_1\geqslant d_2\geqslant\cdots\geqslant d_n$ is graphic if and only if $\sum_{i=1}^{n} d_i$ is even and*

$$\sum_{i=1}^{k} d_i\leqslant k(k-1)+\sum_{i=k+1}^{n}\min(k, d_i)\quad \text{for } 1\leqslant k\leqslant n-1.$$

Theorem 1. *Let*

$$d_i=\begin{cases} r-t, & 1\leqslant i\leqslant r,\\ n-r+t-1, & r<i\leqslant n\end{cases}$$

where n, r and t are integers such that

$$2\leqslant 2t<r<\frac{n+2t}{3}.$$

Then, (i) *the class of sequences $(d_1, d_2, \cdots, d_n)$ are graphic if $r^2+t\cdot n$ is even and $n\leqslant\frac{r^2}{t}$.*

(ii) *Any graph with $(d_1, d_2, \cdots, d_n)$ as its degree sequence is hamiltonian.*

Proof. (i) It can be verified by Lemma 2.

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(ii) Let $G = (V, E)$ be a graph with $(d_1, d_2, \cdots, d_n)$ as its degree sequence and let $\tilde{G} = (V, \tilde{E})$ be the closure of G. Set

$$S = \{v \mid d_G(v) = n - r + t - 1,\ v \in V\}.$$

Noting that $r \geq 2$ and $n > 3r - 2t$ we see that $d_G(u') + d_G(u'') = 2(n - r + t - 1) \geq n$ for all u' and u'' in S. This implies that $\tilde{G}(S)$ is a complete subgraph of $\tilde{G}$.

We consider two cases.

Case 1. $G(S)$ is not a complete subgraph of G.

Put

$$T = \{v \mid d_{G(S)}(V) < n - r - 1,\ v \in S\}.$$

Then $T \neq \varnothing$ and $d_{G(S)}(v) = n - r - 1$ for all $v \in S - T$. Thus

(a) $(u, v) \in E$ for all $u \in T$ and $v \in S - T$.

On the other hand, since $\tilde{G}(S)$ is a complete subgraph, $d_{\tilde{G}}(u) \geq d_G(u) + 1 = n - r + t$ for all $u \in T$. This implies

(b) $(u, x) \in \tilde{E}$ for all $u \in T$ and $x \in V - S$.

Consider the subset of $V - S$

$$B = \{x \mid d_{\tilde{G}}(x) \geq r - t + 1,\ x \in V - S\}.$$

We now prove that $|B| \geq t + 1$.

Suppose $|B| \leq t$. Then $|T| \leq r - t$ since otherwise (b) gives $B = V - S$ and $|B| = |V - S| = r \geq t + 1$. Thus

(c) $|S - T| = n - r - |T| \geq n - 2r + t$.

Pick a vertex $u_0 \in T$. From (a), we have

$$|\Gamma_G(u_0) \cap (V - S)| \leq d_G(u_0) - |S - T| \leq r - 1.$$

Hence there exists a vertex $x_0 \in V - S$ such that $x_0 \bar{\in} \Gamma_G(u_0)$. By (b), $d_{\tilde{G}}(x_0) \geq r - t + 1$, which means that $(x_0, v) \in \tilde{E}$ for all $v \in S - T$. On the other hand, there exists a vertex $v_0 \in S - T$ such that $v_0 \bar{\in} \Gamma_G(x_0)$, since the given condition gives $n - 2r + t > r - t = d_G(x_0)$ and (c) implies $|S - T| > d_G(x_0)$. Consequently, $d_{\tilde{G}}(v_0) \geq n - r + t$, which means $(x, v_0) \in \tilde{E}$ for all $x \in V - S$. Therefore if $y \in (V - S) - \Gamma_G(v_0) \cap (V - S)$, then $d_{\tilde{G}}(y) \geq d_{\tilde{G}}(y) + 1 = r - t + 1$, i.e. $y \in B$. It follows that $B \supseteq (V - S) - \Gamma_G(v_0) \cap (V - S)$. But $|\Gamma_G(v_0) \cap (V - S)| = d_G(v_0) - d_{G(S)}(v_0) = t$ and so $|B| \geq |V - S| - t \geq t + 1$, which is contrary to the assumption. The contradiction shows that $|B| \geq t + 1$.

By the definition of B, it is not difficult to see that $\tilde{G}(S \cup B)$ is a complete subgraph. Thus for all $v \in S \cup B$,

$$d_{\tilde{G}}(v) \geq |S \cup B| - 1 = n - r + |B| - 1 \geq n - r + t$$

which gives immediately that $(x, v) \in \tilde{E}$ for all $v \in S \cup B$ and $x \in V - (S \cup B)$. This implies that $\tilde{G}$ is a complete graph. By Lemma 1, G is hamiltonian.

Case 2. $G(S)$ is a complete subgraph of G.

In this case, pick a vertex $x_1 \in V - S$. Evidently there exists a vertex $x_n \in S$ such that $(x_1, x_n)\bar{\in} E$. We construct a new graph $G^* = (V, E^*)$, where $E^* = E \cup \{(x_1, x_n)\}$. Note that $d_{G^*}(x_1) = r - t + 1$. By a proof similar to that in Case 1 (x_1 plays the same role as x_0 there), we can show that G^* is hamiltonian and then the construction of G^* implies that G has a hamiltonian chain P joining x_1 to x_n. Without loss of generality we may assume that

$$P = x_1 \cdot x_2 \cdot \cdots \cdot x_{n-1} \cdot x_n,$$

where $x_i \in V$ for $1 \leqslant i \leqslant n = |V|$.

We will show that G is hamiltonian.

Suppose G is nonhamiltonian and let

$$\Gamma'(x_1) = \{x_i | (x_{i+1}, x_1) \in E, \ x_i \in V\}.$$

Then $(x_i, x_n)\bar{\in} E$ for all $x_i \in \Gamma(x_1)$; otherwise G is hamiltonian. On the other hand, since $|\Gamma'(x_1)| = r - t$ and $d_G(x_n) = n - r + t - 1$, we have that if $x_i \bar{\in} \Gamma'(x_1)$, then it must be adjacent to x_n. So we can obtain that

(a) $x_i \in \Gamma'(x_1)$ if and only if $(x_i, x_n)\bar{\in} E$.

As $G(S)$ is a complete subgraph and $x_n \in S$, x_n must be adjacent to every vertex of S in G. By (a), $\Gamma'(x_1) \subseteq V - S$. For the sake of convenience, put $m = r - t$ and

$$\Gamma'(x_1) = \{x_{i_1}, x_{i_2}, \cdots, x_{i_m}\}$$

where $1 = i_1 < i_2 < \cdots < i_m < n - 1$.

Pick a vertex $x_{i_j} \in \Gamma'(x_1)$ and assume $(x_k, x_{i_j}) \in E$. When $1 \leqslant k < i_j$, if $(x_{k+1}, x_n) \in E$ we have a hamiltonian cycle: $x_{i_j} \cdot x_{i_j-1} \cdots x_{k+1} \cdot x_n \cdot x_{n-1} \cdot \cdots x_{i_j+1} \cdot x_1 \cdots x_k \cdot x_{i_j}$. Thus $(x_{k+1}, x_n)\bar{\in} E$ and $x_{k+1} \in \Gamma'(x_1)$ by (a). Furthermore, $k < i_j$ and $i_1 = 1 \neq k + 1$ gives that $x_{k+1} \in \{x_{i_2}, x_{i_3}, \cdots, x_{i_j}\}$, which implies $x_k \in \{x_{i_2-1}, \cdots, x_{i_j-1}\}$. When $i_j < k \leqslant n$, if $(x_{k-1}, x_n) \in E$, similarly we have a hamiltonian cycle: $x_{i_j} \cdot x_{i_j-1} \cdots x_1 \cdot x_{i_j+1} \cdot \cdots x_{k-1} \cdot x_n \cdot \cdots x_k \cdot x_{i_j}$. Thus $(x_{k-1}, x_n)\bar{\in} E$ and $x_{k-1} \in \Gamma'(x_1)$ by (a). But $i_j < k$ and so $x_{k-1} \in \{x_{i_j}, x_{i_{j+1}}, \cdots, x_{i_m}\}$, which implies $x_k \in \{x_{i_j+1}, x_{i_{j+1}+1}, \cdots, x_{i_m+1}\}$. It follows from the discussions above that

$$x_k \in \{x_{i_2-1}, \cdots, x_{i_j-1}, x_{i_j+1}, \cdots, x_{i_m+1}\}$$

for all $x_k \in \Gamma(x_{i_j})$. This means $\Gamma(x_{i_j}) \subseteq \{x_{i_2-1}, \cdots, x_{i_j-1}, x_{i_j+1}, \cdots, x_{i_m+1}\}$. But $|\Gamma(x_{i_j})| = r - t = |\{x_{i_2-1}, \cdots, x_{i_j-1}, x_{i_j+1}, \cdots, x_{i_m+1}\}|$ and so

$$\Gamma(x_{i_j}) = \{x_{i_2-1}, \cdots, x_{i_j-1}, x_{i_j+1}, \cdots, x_{i_m+1}\}$$

which implies $(x_{i_j}, x_{i_m+1}) \in E$ for all $x_{i_j} \in \Gamma'(x_1)$. On the other hand, x_{i_m+1} is also adjacent to x_{i_m+2} on the chain P and so $d_G(x_{i_m+1}) \geqslant |\Gamma'(x_1)| + 1 = r - t + 1$, implying $x_{i_m+1} \in S$. But $\Gamma'(x_1) \subseteq V - S$ and $|\Gamma'(x_1)| = r - t \geqslant t + 1$ gives $d_{G(S)}(x_{i_m+1}) \leqslant d_G(x_{i_m+1}) - |\Gamma'(x_1)| \leqslant n - r - 2$, contradicting that $G(S)$ is a complete subgraph. The contradiction shows that G is hamiltonian.

Put $r - t = k$. We obtain the following reformulation of Theorem 1.

Theorem 1'. *Let*

$$d_i = \begin{cases} k, & 1 \leqslant i \leqslant k + t, \\ n - k - 1, & k + t < i \leqslant n. \end{cases}$$

If n, k and t are integers and $1 \leqslant t < k < \frac{n - t}{3}$, then any graph with $(d_1, d_2, \cdots, d_n)$ as its degree sequence is hamiltonian.

Put $t = k - 1$ in Theorem 1′. We have the following immediate consequence.

Corollary 1.1. *Let*

$$d_i = \begin{cases} k, & 1 \leqslant i \leqslant 2k - 1, \\ n - k - 1, & 2k \leqslant i \leqslant n. \end{cases}$$

If n and k are integers and $2 \leqslant k < \frac{n + 1}{4}$, then any graph with $(d_1, \cdots, d_n)$ as its degree sequence is hamiltonian.

From case 2 of the proof in Theorem 1, we see that the closure of G is not a complete graph, i.e. *G does not satisfy the condition given by Bondy and Chvátal*[3].

Theorem 2. *Let*

$$d_i = \begin{cases} r - t, & 1 \leqslant i \leqslant l, \\ r, & l < i \leqslant r, \\ n - r - 1, & r < i \leqslant n \end{cases}$$

where n, r, l and t are integers such that

$$1 \leqslant l \leqslant r - 1 < \left[\frac{n}{2}\right] - 1 \text{ and } 0 \leqslant t < \min\left\{\frac{r}{t}, r - l\right\}.$$

Then, (i) *the class of sequences $(d_1, \cdots, d_n)$ are graphic when $r^2 - t \cdot l$ is even.*

(ii) *Any graph with $(d_1, \cdots, d_n)$ as its degree sequence is hamiltonian.*

Proof. (i) It is the consequence of Lemma 2.

(ii) Let $G = (V, E)$ be a graph with $(d_1, \cdots, d_n)$ as its degree sequence and let $\tilde{G} = (V, \tilde{E})$ be the closure of G. To prove the theorem, it is sufficient to show that $\tilde{G}$ is hamiltonian by Lemma 1.

We first partition V into the following three parts

$$S = \{v \mid d_G(v) = n - r - 1, \ v \in V\},$$
$$B = \{v \mid d_G(v) = r, \quad v \in V\},$$
$$W = \{v \mid d_G(v) = r - t, \quad v \in V\}.$$

When $t = 0$, we take arbitrarily l vertices from those of degree r as W and the remains as B. So we always have

$$B \cap W = \varnothing, \ |B| = r - l > t \text{ and } |W| = l \geqslant 1.$$

By the definition of closure and $r < \left[\frac{n}{2}\right]$, $\tilde{G}(S)$ is a complete subgraph. But $G(S)$

is not, since $B \neq \varnothing$. Set

$$T = \{v \mid d_{G(S)}(v) < n - r - 1, \ v \in S\}.$$

Then $T \neq \varnothing$ and $d_{G(S)}(v) = n - r - 1$ for all $v \in S - T$. Thus

(a) $(v, u) \in E$ for all $u \in T$ and $v \in S - T$.

Noting that $d_{\tilde{G}}(u) \geqslant d_G(u) + 1 = n - r$ for all $u \in T$ we also have

(b) $(u, x) \in \tilde{E}$ for all $u \in T$ and $x \in B$.

Combining this with the fact that $\tilde{G}(S)$ is a complete subgraph, we see that $d_{\tilde{G}}(u) \geqslant |S| - 1 + |B| \geqslant n - r + t$ for all $u \in T$, which gives immediately

(c) $(u, w) \in \tilde{E}$ for all $u \in T$ and $w \in W$.

We now show that there exists a vertex $x_0 \in V - S$ such that $d_{\tilde{G}}(x_0) \geqslant r + 1$.

Suppose this is not true, i.e., $d_{\tilde{G}}(x) \leqslant r$ for all $x \in V - S$. Then

(1) $(x, u) \in E$ for all $x \in B$ and $u \in T$.

This is because if there exist $x \in B$ and $u \in T$ such that $(x, u) \bar{\in} E$, then, by (b), $d_{\tilde{G}}(x) \geqslant r + 1$, contradicting the assumption.

(2) $t + 2 \leqslant |T| \leqslant r$.

In fact, let $u \in T$. Then, by (a) and (1), $n - r - 1 = d_G(u) \geqslant |S - T| + |B| = n - r - |T| + r - l$, and so $|T| \geqslant r - l + 1 \geqslant t + 2$.

If $|T| \geqslant r + 1$, then, by (b), $d_{\tilde{G}}(x) \geqslant r + 1$ for all $x \in B$, which is contrary to the assumption.

Consider now the vertices in W. Let $w \in W$. If there exist $t + 1$ vertices in T nonadjacent to w, then, by (c), $d_{\tilde{G}}(w) \geqslant d_G(w) + t + 1 = r + 1$, a contradiction. Thus w is nonadjacent to at most t vertices of T. This means that w is adjacent to at least $|T| - t$ vertices of T. Therefore there are at least $|W| \cdot (|T| - t)$ edges of G connecting T with W. On the other hand, there are exactly $|T| \cdot |S - T|$ edges of G connecting T with $S - T$ by (a) and $|T| \cdot |B|$ edges of G connecting T with B by (1). It follows that

$$\sum_{u \in T} d_G(u) \geqslant |W| \cdot (|T| - t) + |T| \cdot |S - T| + |T| \cdot |B|$$
$$= |T| \cdot (n - r - 1) + |T| - |T|^2 + r \cdot |T| - t \cdot l.$$

With $\sum_{u \in T} d_G(u) = |T| \cdot (n - r - 1)$ we find that

$$0 \geqslant |T| - |T|^2 + r \cdot |T| - t \cdot l = (|T| - 1)(r - |T|) + r - t \cdot l.$$

But by (2), $(|T| - 1)(r - |T|) \geqslant 0$. Consequently $t \geqslant \frac{r}{l}$, contradicting that $t < \min\left\{\frac{r}{l}, r - l\right\}$. The contradiction shows that there exists a vertex $x_0 \in V - S$ such that $d_{\tilde{G}}(x_0) \geqslant r + 1$.

In this case, $G(S \cup \{x_0\})$ is clearly a complete subgraph of $\tilde{G}$ and $d_{\tilde{G}}(v) \geqslant n - r$