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# A BIVARIATE WEIBULL AND ITS COMPETING RISKS MODELS

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**Summary**
Competing risks analysis with two causes of failure was considered. Since independence of causes are not realistic in many situations, bivariate dependent model was considered for the underlying causes of failure. The absolutely continuous bivariate Weibull model proposed by Ryu (1993) was modified in order to derive a Weibull competing risks model with crude and net hazards equals. This condition allow identifiability of the marginals corresponding to each cause of failure. Identifiability and estimation of its parameters by maximum likelihood method are investigated. Also, tests for some hypotheses of interest were studied. Simulation studies for comparison of the proposed, Ryu's and independent Weibulls bivariate models and of proposed with independent competing risks models were performed. Applications to real data also are presented.

**Key Words:** bivariate distribution; competing risks; dependence; Weibull.

## 1 Introduction

In a variety of fields the time of failure can be associated with a cause of failure. For instance, in the clinical trial for breast cancer from Lagakos (1977), the failures times were associated with either local or metastases causes of failure. This is called competing risks situation. Also, censored observations can occur when the observed time of the experiment is limited (type I censoring) or either by the limitation of the number of failure (type II censoring) both by design or can be random (random censoring). In the competing risks setting, the dichotomous variable $\delta$ indicates censoring when $\delta = 0$ and failure when $\delta = 1$. It can be also be observed the vector of fixed covariates $z' = (z_1, z_2, ..., z_p)$, at the initial time $t_0$ for each individual or to be time-dependent covariates. Thus, the data for each individual is $(t_j, i_j, \delta_j, z_j), j = 1, 2, .., n$.

We may consider the following: the random variables failures times $T_1, T_2, ..., T_k$ are associated with causes $C_1, C_2, .., C_k$, respectively, so that, the observed failure is $T = \min(T_1, T_2, ..., T_k)$ corresponding to the cause of failure $C_i, i = 1, 2, .., k$. Prentice et al.(1978) has approached the competing risks problem considering the observed random variable $T$ for the time of failure (when $\delta = 1$) or for the time of censoring (when $\delta = 0$), in the independent random mechanism. An alternative approach consider parametric model for the underlying times corresponding to the causes of failure. Usually the $k$ failures times were considered independents, in order to avoid the identifiability problems. Since, in some applications, the times of failures corresponding to the causes of failures are generally not independent, in particular for two causes of failures, bivariate dependent parametric model for the underlying times corresponding to the causes of failures could be considered. Thus, given the joint survival distribution of $T_1$ and $T_2$, denoted by $S(t_1, t_2)$, the competing risks model proceed deriving the distribution of $T = \min(T_1, T_2)$ and $I = i$ , $i = 1, 2$. Although this approach appears to be more adequate at the statistical point of view, many problems arises. The first one, is the identifiability problems in most of bivariate distributions; since $T$ and $\delta$ are observed; further, neither the parameters of $S(t_1, t_2)$ nor its marginals may not be identifiable. Also, in competing risks situation there would be more adequate to choose absolutely continuous bivariate distribution since its particular assumption of failure are due to a single cause. Also, there exist a natural practical appealing of choosing a bivariate distribution according to its marginals distributions.

Thus, among the bivariate distributions proposed in the literature, bivariate exponential distributions have meet special attention: Marshall and Olkin (1967), Block-Basu (1974), Raftery (1984), Sarkar (1987) and Ryu (1993), among others. Among them, only Marshall-Olkin bivariate exponential distribution is not absolutely continuous and both, Raftery and Sarkar bivariate exponential distributions presented marginals exponentials. The bivariate exponential distribution of Ryu (1993) (ACBE1), is an extension of a non-absolutely continuous bivariate distribution of Marshal and Olkin (1967) and is absolutely continuous. The Marshall and Olkin's bivariate Weibull distribution (BVW1) is the generalization of Marshall and Olkin's bivariate exponential distribution, with marginals Weibulls and is not absolutely continuous. The bivariate Weibull model suggested by Ryu (ACBW1) is not a generalization of BVW1, but a extension of ACBE1; its marginals are not Weibull. Therefore, it is not bivariate Weibull distribution in strictly sense.

The parametric competing risks model based on bivariate distributions has been studied by Moeschberger (1974), with bivariate Weibull model of Marshall and Olkin, by Wada and Sen (1995), with Sarkar exponential model and Bhattacharyya (1997) with bivariate exponential model of Raftery, among others.

The objective of this study was to investigate a competing risks model with two causes of failure considering a modified bivariate Weibull distribution from Ryu (ACBW1), in order to obtain a competing risks model with identifiable marginals. Identifiability and estimation of the parameters by maximum likelihood method are investigated. Tests of hypotheses of independence and exponential distribution are performed.

In Section 2, the modified bivariate Weibull model (ACBW2) and the derivation of competing risks model (CRW1) based on this modified Weibull model are presented. Section 3 consider estimation and tests of hypotheses of the parameters of this competing risks model. Simulation studies using the bivariate modified model and competing risks model are presented in Section 4. Finally, in Section 5, application of these models are conducted in two real data sets. Conclusions are presented in Section 6.

## 2 Weibull bivariate and competing risks models

In this section, the derivation of a modified Weibull bivariate model based on the bivariate Weibull model formulated by Ryu (1993) and a Weibull competing risks model based on this modified bivariate model are presented.

The joint survival function of the bivariate Weibull model ACBW1 formulated by Ryu (1993) is given by:

$$S(t_1,t_2) = \begin{cases} \exp\{-\lambda_1 t_1^{\alpha_1} - \lambda_{12} t_2 - \lambda_2 t_2^{\alpha_2} + \frac{\lambda_{12}}{\gamma_2}(1-e^{-A_2}) + \frac{\lambda_{12}}{\gamma_0}\left(e^{-A_2}-e^{-A_3}\right)\} \\ \text{if } t_1 \le t_2 \\ \\ \exp\{-\lambda_1 t_1^{\alpha_1} - \lambda_{12} t_1 - \lambda_2 t_2^{\alpha_2} + \frac{\lambda_{12}}{\gamma_1}(1-e^{-A_1}) + \frac{\lambda_{12}}{\gamma_0}\left(e^{-A_1}-e^{-A_3}\right)\} \\ \text{if } t_1 > t_2, \end{cases} \tag{2.1}$$

with $A_1 = \gamma_1(t_1 - t_2)$, $A_2 = \gamma_2(t_2 - t_1)$, $A_3 = \gamma_1 t_1 + \gamma_2 t_2$ and $\gamma_0 = \gamma_1 + \gamma_2$.

We can note that for $\alpha_1 \to 1$ and $\alpha_2 \to 1$, the survival function is the bivariate exponential survival function ACBE1 of Ryu. Thus, it is extension of the exponential Ryu's model. Further, if $\alpha_1 \to 1$, $\alpha_2 \to 1$, $\gamma_1 \to \infty$ and $\gamma_2 \to \infty$, this become the bivariate exponential survival function of Marshall and Olkin (1967). It is an absolutely continuous distribution, unless $\gamma_1 = \gamma_2 = \infty$. When $\lambda_{12} = 0$, $T_1$ and $T_2$, are independent. The association between $T_1$ and $T_2$, can be measured by the correlation coefficient, which was positive for this model and by the measure of dependence introduced by Slud and Rubstein (1983), $\rho(t) > 1$, also indicating positive association. This ACBW1 is not Weibull in the sense that its marginals are not Weibull.

## 2.1 The modified Weibull bivariate model

**Proposition 2.1** *A modified Weibull bivariate distribution (ACBW2) with survival function* $S^*(t_1, t_2)$ *given by:*

$$S^*(t_1,t_2) = \begin{cases} \exp\{-A_0 + \frac{\lambda_{12}}{\gamma_2}\left(1-e^{-A_2}\right) + \frac{4\lambda_{12}}{\gamma_0}(e^{-\frac{A_2}{2}} - e^{-\frac{A_3}{2}})\} \\ \textit{if } t_1 \le t_2 \\ \\ \exp\{-A_0 + \frac{\lambda_{12}}{\gamma_1}\left(1-e^{-A_1}\right) + \frac{4\lambda_{12}}{\gamma_0}(e^{-\frac{A_1}{2}} - e^{-\frac{A_3}{2}})\} \\ \textit{if } t_1 > t_2, \end{cases} \tag{2.2}$$

*with* $A_0 = \lambda_1 t_1^{\alpha_1} + \lambda_2 t_2^{\alpha_2} + \lambda_{12}(t_1+t_2)$*, is obtained from ACBW1 (2.1).*

**Proof:** For the derivation of the bivariate Weibull model ACBW1 given in (2.1), Ryu considered two plants, sharing a commom risk of breakdown, subject to three independent Poisson shock processes $\{N_i(t), t \ge 0\}$ $i = 1,2$ for the plant-specific component and $\{N_{12}(t),\, t \ge 0\}$ for the commom component of the plant with intensity rate $\lambda_i$, $i = 1, 2$ and $\lambda_{12}$, respectively. For these processes, the random variables $X_i$, $i = 1,2$ ($X_i \sim$ Weibull$(\lambda_i, \alpha_i)$) are the times up to the first jump of the process $N_i$ and $Z_i$, $i = 1,2$ are the duration variables having a conditional hazard rates $\gamma_i N_{12}(t)$ at the time $t$ giving a realization $N_{12}$. The hazard rate of $Z_i$ at time $t$ is $\lambda_{12}(1-e^{-\gamma_i t})$ for $i = 1,2$ (Ryu, 1993, p.1459). He stated the joint survival function of $T_1$ and $T_2$ ($T_i = \min(X_i, Z_i), i = 1,2$), the times of failure of $i$th plant, $i = 1,2$ as:

$$S(t_1,t_2) =$$
$$P\left(X_1 > t_1\right) P(X_2 > t_2)\, E\left\{\exp\left[-\gamma_1 \int_0^{t_1} N_{12}(u)du - \gamma_2 \int_0^{t_2} N_{12}(u)du\right]\right\}. \tag{2.3}$$

The expectation term in (2.3) for $t_1 \le t_2$ (the case of $t_1 > t_2$ is similar)was derived as:

$$\begin{aligned} &E\left\{\exp\left[-\gamma_1 \int_0^{t_1} N_{12}(u)\,du - \gamma_2 \int_0^{t_2} N_{12}(u)\,du\right]\right\} \\ &= E\left\{\exp\left[-\gamma_0 \int_0^{t_1} N_{12}(u)du - A_2 N_{12}(t_1)\right]\right\} \\ &\times E\left\{\exp\left[-\gamma_2 \int_{t_1}^{t_2} \left(N_{12}(u) - N_{12}(t_1)\right) du\right]\right\}. \end{aligned} \tag{2.4}$$

At this point of Ryu's derivation we introduced the following argument for the calculation of the first expectation of the expression (2.4): $EW^2 \geq (EW)^2$. Thus,

$$\begin{aligned}
&E\left\{\exp\left[-\gamma_0\int_0^{t_1} N_{12}(u)\,du - A_2 N_{12}(t_1)\right]\right\}\\
&= E\left\{\left(\exp\left[-\frac{\gamma_0}{2}\int_0^{t_1} N_{12}(u)\,du - \frac{A_2}{2}N_{12}(t_1)\right]\right)^2\right\}\\
&\geq \left\{E\left\{\exp\left[-\frac{\gamma_0}{2}\int_0^{t_1} N_{12}(u)\,du - \frac{A_2}{2}N_{12}(t_1)\right]\right\}\right\}^2\\
&= \exp\left[-2\lambda_{12}t_1 + \frac{4\lambda_{12}}{\gamma_0}\left(e^{-\frac{A_2}{2}} - e^{-\frac{A_3}{2}}\right)\right].
\end{aligned}$$

This last expression was solved using the same argument as in Ryu's derivation (Ryu, 1993, p.1464). The second expectation follows the derivation of Ryu:

$$\begin{aligned}
&E\left\{\exp\left[-\gamma_2\int_{t_1}^{t_2} \left(N_{12}(u) - N_{12}(t_1)\right)du\right]\right\}\\
&= \exp\left[-\lambda_{12}(t_2 - t_1) + \frac{\lambda_{12}}{\gamma_2}\left(1 - e^{-A_2}\right)\right].
\end{aligned}$$

Then the product in (2.4) can be written as:

$$\begin{aligned}
&E\left\{\exp\left[-\gamma_1\int_0^{t_1} N_{12}(u)du - \gamma_2\int_0^{t_2} N_{12}(u)du\right]\right\}\\
&\geq \exp\left[-\lambda_{12}(t_1+t_2) + \frac{\lambda_{12}}{\gamma_2}\left(1 - e^{-A_2}\right) + \frac{4\lambda_{12}}{\gamma_0}\left(e^{-\frac{A_2}{2}} - e^{-\frac{A_3}{2}}\right)\right].
\end{aligned}$$

Therefore $S(t_1, t_2) \geq S^*(t_1, t_2)$ given in (2.2).

***Properties of this distribution:***

a) This distribution is absolutely continuous, that is, $P(T_1 = T_2) = 0$.

b) The bivariate hazard functions as defined by Johnson and Kotz (1975) are:

$$\begin{aligned}
\lambda_i^*(t_1, t_2) &= -\frac{\partial \log S^*(t_1, t_2)}{\partial t_i}\\
&= \begin{cases} \lambda_i\alpha_i t_i^{\alpha_i - 1} + \lambda_{12}(1 + e^{-A_2}) + A_{i1} & if\ t_1 \leq t_2 \\ \lambda_i\alpha_i t_i^{\alpha_i - 1} + \lambda_{12}(1 - e^{-A_1}) + A_{i2} & if\ t_1 > t_2 \end{cases} \qquad (2.5)
\end{aligned}$$

where $i = 1, 2$, $A_{11} = -\frac{2\lambda_{12}}{\gamma_0}(\gamma_2 e^{-\frac{A_2}{2}} + \gamma_1 e^{-\frac{A_3}{2}})$, $A_{12} = \frac{2\lambda_{12}\gamma_1}{\gamma_0}(e^{-\frac{A_1}{2}} + e^{-\frac{A_3}{2}})$, $A_{21} = \frac{2\lambda_{12}\gamma_2}{\gamma_0}(e^{-\frac{A_2}{2}} + e^{-\frac{A_3}{2}})$ and $A_{22} = -\frac{2\lambda_{12}}{\gamma_0}(\gamma_1 e^{-\frac{A_1}{2}} + \gamma_2 e^{-\frac{A_3}{2}})$.

c) The marginals $S^*_{T_i}(t_i) = \exp\left[-\lambda_i t_i^{\alpha_i} - \lambda_{12} t_i + \frac{\lambda_{12}}{\gamma_i}\left(1 - e^{-\gamma_i t_i}\right)\right]$ have constant, increasing or decreasing net hazard rates expressed by:

$$h_i^*(t) = -\frac{\partial \log S^*_{T_i}(t_i)}{\partial t_i}\bigg|_{t_i=t} = \lambda_i \alpha_i t^{\alpha_i - 1} + \lambda_{12}(1 - e^{-\gamma_i t}), \text{ for } i = 1, 2, \quad (2.6)$$

depending on the values of $\gamma_i$ and $\alpha_i$. When $\gamma_i \to \infty$, the hazard is constant for $\alpha_i = 1$, is decreasing for $\alpha_i < 1$ and increasing for $\alpha_i > 1$. Note that the marginals are similar as Ryu's model.

d) When $\gamma_i \to \infty$, the ACBW2 model become independent model with marginals as compound independent survival of Weibull and exponential survivals distributions. When $\lambda_{12} = 0$, the ACBW2 model become an independent Weibull model.

e) The variables $(T_1, T_2)$ following this joint survival function presents the coefficient of correlation negative when $\gamma_1 = \gamma_2$ or when $\gamma_i \to \infty$ and $\lambda_{12} \to \infty$; the coefficient of correlation is positive when $\gamma_i \to \infty$ and $\lambda_{12} \to 0$. The measure of dependence of Slud and Rubstein $\rho(t)$ gave a more general result which was dependent of $\gamma_1$ and $\gamma_2$ : for $\gamma_1 < \gamma_2$, $\rho(t) > 1$ indicating a positive association and for $\gamma_1 > \gamma_2$, $\rho(t) < 1$, indicating a negative association. When setting $\gamma_1 = \gamma_2 = \gamma$, $\rho(t) = 1$ which indicate no association.

f) Comparison between the ACBW1 and ACBW2 models show that the reduction of the model ACBW2 over ACBW1 is a function of $\lambda_{12}$, $\gamma_1$ and $\gamma_2$ as shown bellow. When $\gamma_1 \to \infty$ and $\gamma_2 \to \infty$, then the difference is $\exp(\lambda_{12} t_1)$ in case of $t_1 \le t_2$ and $\exp(\lambda_{12} t_2)$ in case of $t_1 > t_2$. In fact, for $t_1 \le t_2$,

$$\frac{S(t_1, t_2)}{S^*(t_1, t_2)} = \exp\left[\lambda_{12} t_1 + \frac{\lambda_{12}}{\gamma_0}\left(e^{-A_2} - e^{-A_3}\right) - \frac{4\lambda_{12}}{\gamma_0}\left(e^{-\frac{A_2}{2}} - e^{-\frac{A_3}{2}}\right)\right] \ge 1.$$

***Estimation of the parameters:***

In order to construct the likelihood function, including censored observations, we denote $C_1$ the set of observations where $t_1$ and $t_2$ are failure times, $C_2$ the set of observations where $t_1$ is time of failure and $t_2$ is censored times, $C_3$ is the set of observations where $t_1$ is censored times and $t_2$ is failure times and finally, $C_4$ is the set of observations where both are censored times. Considering the observations $(t_{1j}, t_{2j})$, $i = 1, 2, ..., n$, the likelihood function can be written as (Lawless, 1982, p.479):

$$L(\theta) = \prod_{j \in C_1} f^*(t_{1j}, t_{2j}) \prod_{j \in C_2} -\frac{\partial S^*(t_{1j}, t_{2j})}{\partial t_{1j}} \prod_{j \in C_3} -\frac{\partial S^*(t_{1j}, t_{2j})}{\partial t_{2j}} \prod_{j \in C_4} S^*(t_{1j}, t_{2j}), \quad (2.7)$$

where: $S^*(t_{1j}, t_{2j})$ is the survival function given in (2.2); $-\frac{\partial S^*(t_1,t_2)}{\partial t_1} = \int_{t_{2j}}^{\infty} f^*(t_{1j}, v)dv$ is the joint probability of the failure of the first component $t_{1j}$ and censor of the second component; $-\frac{\partial S^*(t_{1j},t_{2j})}{\partial t_{2j}}$ is the joint probability of censor of the first and failure of the second, and finally, $f^*(t_{1j}, t_{2j})$ is the joint density function between $t_{1j}$ and $t_{2j}$. The expressions for $-\frac{\partial S^*(t_{1j},t_{2j})}{\partial t_{1j}}$, for $-\frac{\partial S^*(t_{1j},t_{2j})}{\partial t_{2j}}$ and for $f^*(t_{1j}, t_{2j})$ are:

$$-\frac{\partial S^*(t_{1j}, t_{2j})}{\partial t_{ij}} = S^*(t_{1j}, t_{2j})\lambda_i^*(t_{1j}, t_{2j}), \text{ for } i = 1, 2,$$

where $\lambda_i^*(t_{1j}, t_{2j}),\ i = 1, 2$ are given in (2.5) and

$$f^*(t_{1j}, t_{2j}) = S^*(t_{1j}, t_{2j}) \left[-\frac{\partial S^*(t_{1j}, t_{2j})}{\partial t_{1j}}\right]\left[-\frac{\partial S^*(t_{1j}, t_{2j})}{\partial t_{2j}}\right]\left[\frac{1}{S^*(t_{1j}, t_{2j})}\right] \times E_i, \tag{2.8}$$

where $E_1 = \lambda_{12}\gamma_2 e^{-A_2} - \frac{\lambda_{12}\gamma_2}{\gamma_0}(\gamma_2 e^{-\frac{A_2}{2}} + \gamma_1 e^{-\frac{A_3}{2}}$ if $t_{1j} \leq t_{2j}$ and $E_2 = \lambda_{12}\gamma_1 e^{-A_1} - \frac{\lambda_{12}\gamma_1}{\gamma_0}(\gamma_1 e^{-\frac{A_1}{2}} + \gamma_1 e^{-\frac{A_3}{2}}$ if $t_{1j} \leq t_{2j}$.

The MLE of the parameters of ACBW2 are found, by iterative methods, using the likelihood function given in (2.7).

## 2.2 A Weibull competing risks model

In a competing risks situation, the observed random variables are $(T, I)$ where the observed failure time is the minimum of the hypothetical times of failures $(T_i, i \geq 1)$ from different causes and $I$ is the index relating to the specific cause of failure. Therefore, the data from this situation is $(t_j, \delta_{ij})$, where : $\delta_{ij}$= 1 if the cause of failure is $i$ and 0 otherwise, $i \geq 1$ and $j = 1, 2, ..., n$.

For the current study, we consider the latent parametric approach for the competing risks analysis with two causes of failures. In this setting we need to consider a joint parametric bivariate distribution for the underlying times of failures $(T_1, T_2)$ and derive the distribution of $(T, I)$ and related functions.

We consider $S^*(t_1, t_2)$ given in (2.2) when $\gamma_1 = \gamma_2 = \gamma$ as the survival function of the underlying times $T_1$ and $T_2$. The survival function of observed time $T = \min(T_1, T_2)$ is given by:

$$S_T(t) = \exp\left[-\lambda_1 t^{\alpha_1} - \lambda_2 t^{\alpha_2} - 2\lambda_{12} t + \frac{2\lambda_{12}}{\gamma}(1 - e^{-\gamma t})\right],\ t > 0 \tag{2.9}$$

and the corresponding total hazard function by:

$$\lambda_T(t) = \lambda_1\alpha_1 t^{\alpha_1 - 1} + \lambda_2\alpha_2 t^{\alpha_2 - 1} + 2\lambda_{12}(1 - e^{-\gamma t}). \tag{2.10}$$

It can noted that this hazard function is written as the sum of the hazard functions of Weibull distributions (corresponding to the variables $X_1$ and $X_2$) and the hazard functions of the distribution of $Z_1$ and $Z_2$ (duration variables).

In the bivariate hazard functions (property b, section 2.1), if we set $t_1 = t_2 = t$, we have the crude hazard functions, the instantaneous rate of failing from cause $C_i$ at time $t$ where both causes are acting simultaneously (Elandt-Johnson et. al., 1980, p.273):

$$\lambda_i(t) = -\frac{\partial S(t_1,t_2)}{\partial t_i}\bigg|_{t_1=t_2=t}\frac{1}{S_T(t)} = \lambda_i\alpha_i t^{\alpha_i-1} + \lambda_{12}(1-e^{-\gamma t}),\ i=1,2 \tag{2.11}$$

with $\lambda_T(t) = \sum_{i=1}^{2}\lambda_i(t)$, that is, the failure may be due to a unique cause. In this model, the net and crude hazard, given respectively in (2.6) and (2.11), are equal. This is the property we were seeking in a Weibull competing risks model in order to have identifiability of the marginals distributions (Fleming and Harrington, 1991, p.26).

Using (2.11), the total survival function $S_T(t)$ in (2.9) can be expressed by:

$$S_T(t) = \prod_{i=1}^{2}\exp\left\{-\int_0^t \lambda_i(u)du\right\} = \prod_{i=1}^{2} G_i(t) \tag{2.12}$$

where $G_i(t)$ is a pseudo survival function, called by Elandt-Johnson et al. (1980, p.278) as the distribution due to the cause $C_i$ alone and for this model,

$$\begin{aligned} G_i(t) &= \exp\left\{-\int_0^t \lambda_i(u)\,du\right\} = \exp\left\{-\lambda_i t^{\alpha_i} - \lambda_{12}t + \frac{\lambda_{12}}{\gamma}(1-e^{-\gamma t})\right\} \\ &= S^*_{T_i}(t),\ i=1,2. \end{aligned}$$

The distributions of the probability for the time of failure for cause $C_i$ is:

$$\begin{aligned} Q_i(t) &= P(T<t, I=i) = \int_0^t \lambda_i(u)S_T(u)du = \int_0^t f_i(u)du \\ &= \int_0^t \left\{\lambda_i\alpha_i u^{\alpha_i-1} + \lambda_{12}(1-e^{-\gamma u})\right\}\exp\Big\{-\lambda_1 u^{\alpha_1} - 2\lambda_{12}u \\ &\quad -\lambda_2 u^{\alpha_2} + \frac{2\lambda_{12}}{\gamma}(1-e^{-\gamma u})\Big\}du,\ i=1,2 \end{aligned}$$

where $f_i(t)$ is the probability density associated to the cause $C_i$.

The likelihood function for the total time of failure $T$, independent censoring and considering causes of failures, is proportional to

$$L = \prod_{i=1}^{2} \left\{ \prod_{j=1}^{D_i} \lambda_i(t_j) \prod_{j=1}^{n} \left\{ \exp[-\int_0^{t_j} \lambda_i(u)du] \right\} \right\} \tag{2.13}$$

where $D_i$ is the set of individuals that fail for cause $C_i$. For the ACBW2, the likelihood function given in (2.13), can be written as:

$$\begin{aligned} L(\theta) \quad = \quad & \prod_{j=1}^{n}\prod_{i=1}^{2} \Big(\lambda_i \alpha_i t_j^{\alpha_i - 1} + \lambda_{12}(1 - e^{-\gamma t_j})\Big)^{\delta_{ij}} \exp\Big\{-\lambda_i t_j^{\alpha_i} - \lambda_{12} t_j \\ & + \frac{\lambda_{12}}{\gamma}(1 - e^{-\gamma t_j})\Big\} \end{aligned} \tag{2.14}$$

where $\delta_{ij} = 1$ if the cause of failure is $i$ and 0 otherwise, $i = 1, 2$. The parameter space for this model is: $\Theta = \{0 < \lambda_1 < \infty, 0 < \alpha_1 < \infty,\ 0 < \lambda_2 < \infty,\ 0 < \alpha_2 < \infty,\ 0 \leq \lambda_{12} < \infty,\ 0 < \gamma < \infty\}$, and $\boldsymbol{\theta} = (\lambda_1, \alpha_1, \lambda_2, \alpha_2, \lambda_{12}, \gamma)'$ is the vector of the parameters. They are all identifiable.

The maximum likelihood estimates (MLE) of $\boldsymbol{\theta}$, $\hat{\boldsymbol{\theta}} = (\hat{\lambda_1}, \hat{\lambda_2}, \hat{\lambda_{12}}, \hat{\alpha_1}, \hat{\alpha_2}, \hat{\gamma})$ are found maximizing the likelihood (2.14) by iterative numerical method.

# 3 Estimation and tests of hypotheses of the parameters of the Weibull competing risks model

In this section we studied the regularity conditions for the large sample properties of the MLE and some hypotheses tests of interest.

As in Lehmann and Casella (1998, p.463), the model must satisfies the conditions given in Lemma 3.1 and 3.2 bellow. Consider the vector of the scores statistics $U(\boldsymbol{\theta}) = (U(\theta_1), U((\theta_2), ..., U(\theta_s))'$, where $U(\theta_i) = \frac{\partial \log f(t,\delta|\boldsymbol{\theta})}{\partial \theta_i}$, $i = 1, 2, ..., s$ and the $s \times s$ matrix $I(\boldsymbol{\theta})$ of the information matrix whose elements are $I_{jk} = cov\left[\frac{\partial \log f(t,\delta|\boldsymbol{\theta})}{\partial \theta_j}, \frac{\partial \log f(t,\delta|\boldsymbol{\theta})}{\partial \theta_k}\right]$, $j, k = 1, ..., s$.

**Lemma 3.1** *The elements $I_{jk}$, $j, k = 1, 2, ...., s$ of $I(\boldsymbol{\theta})$ are finite and $I(\boldsymbol{\theta})$ is a positive definite for all $\boldsymbol{\theta}$ in $\omega \subset \Omega$, a open set of which the true parameter value $\boldsymbol{\theta}^0$, is an interior point.*

**Lemma 3.2** *There exist functions $M_{jkl}$ such that $\left|\frac{\partial^3}{\partial\theta_j \partial\theta_k \partial\theta_l} \log\left(f(t, \delta|\theta)\right)\right| \leq M_{jkl}(t)$ for all $\boldsymbol{\theta} \in \omega$ where $m_{jkl} = E_{\boldsymbol{\theta}}[M_{jkl}(t)] < \infty$ for all $j, k, l$.*

The proofs are outlined as follows: Denoting

$$\begin{aligned} B &= B_1 \times B_2 = \left[ \prod_{i=1}^{2} \left( \lambda_i \alpha_i t^{\alpha_i - 1} + \lambda_{12}(1 - e^{-\gamma t}) \right)^{\delta_i} \right], \\ C &= \exp\left\{ - \left[ \prod_{i=1}^{2} \lambda_i t^{\alpha_i} \right] - 2\lambda_{12} t + \frac{2\lambda_{12}}{\gamma}(1 - e^{-\gamma t}) \right\}, \quad \text{and} \\ D &= \exp\left\{ -\lambda_1 t^{\alpha_1} + \frac{2\lambda_{12}}{\gamma} \right\} \end{aligned}$$

with the following restrictions: $a)0 < (1 - e^{-\gamma t}) \leq 1, \gamma > 0$ and $t > 0$, b) $|\log(t)| \leq t^{-1}$ if $t \leq 1$ and $|\log(t)| \leq t$ if $t \geq 1$, it can be show that in Lemma 3.1: $I_{jk}(\boldsymbol{\theta}) = E\left( \left| -\frac{\partial^2 \log f(t,\delta|\boldsymbol{\theta})}{\partial\theta_j \partial\theta_k} \right| \right) < \infty$ and in Lemma 3.2, $E_{\boldsymbol{\theta}}[M_{jkl}(t)] < \infty$. Also, $I(\boldsymbol{\theta})$ is a positive definite since it is a covariance matrix. The score statistics $U$ are linearly independents, so that $E(U'U)$ is not singular and $I(\boldsymbol{\theta})$ is a full rank matrix. Therefore, $I(\boldsymbol{\theta})$ is positive definite.

The regularity conditions for the large sample derivations of MLE was satisfied by Lemma 3.1 and 3.2 above. Therefore, the following theorem was satisfied:

**Theorem 3.1** *Let $X_1, X_2, ..., X_n$ be iid, each with density $f(x; \boldsymbol{\theta}), \boldsymbol{\theta} \in \Theta^p \subset \mathbb{R}^p$ which satisfies the regularity conditions shown in Lemma 1 and 2. Then, the MLE of $\boldsymbol{\theta}$, which maximize the log $L(\boldsymbol{\theta})$, present the following asymptotic properties:*

$$a)\hat{\boldsymbol{\theta}_n} \xrightarrow{P} \boldsymbol{\theta}_0; \qquad b)\sqrt{n}(\hat{\boldsymbol{\theta}_n} - \boldsymbol{\theta}_0) \xrightarrow{D} N_p\left(0, I(\boldsymbol{\theta})^{-1}\right)$$

*Moreover, $I_0(\hat{\boldsymbol{\theta}}) = \left( -\frac{\partial^2}{\partial\boldsymbol{\theta}\partial\boldsymbol{\theta}'} lnL(\boldsymbol{\theta}) \right) \Big|_{\theta = \hat{\boldsymbol{\theta}}} \xrightarrow{P} I(\boldsymbol{\theta})$, where $I_0(\hat{\boldsymbol{\theta}})$ is called observed information matrix.*

The hypotheses of interest in the Weibull competing risks model are related to reduction of the model, that is, whether or not a bivariate exponential fit the data ($H_{01} : \alpha_1 = \alpha_2 = 1$) and whether or not the independent Weibull model fit the data ($H_{02} : \lambda_{12} = 0$). Another hypothesis of interest may be the hypothesis of equality of marginals or equality of hazard functions ($H_{03} : \alpha_1 = \alpha_2$ and $\lambda_1 = \lambda_2$).

Asymptotic tests were considered for the hypotheses above, stated as $H_0 : \boldsymbol{\theta} = \boldsymbol{\theta}_1$ considering $\boldsymbol{\theta}_2$ as nuisance parameters (Lawless, 1982, p.523). They have $\chi^2_k$ distribution under the null hypotheses ($k$ is the dimension of $\boldsymbol{\theta}_1$). For the hypothesis $H_{01} : \boldsymbol{\theta}_1 = (\alpha_1 \quad \alpha_2)' = (1 \quad 1)'$ the usual alternative is $H_{a1}$ : at least one of $\alpha_i \neq 1$, the vector $\boldsymbol{\theta}_2 = \{\lambda_1, \lambda_2, \lambda_{12}, \gamma\}$ is the vector of nuisance parameters. For the hypothesis $H_{02} : \theta_1 = \lambda_{12} = 0$ against the alternative $H_{a2} : \lambda_{12} \neq 0$, the vector $\boldsymbol{\theta}_2 = \{\lambda_1, \lambda_2, \alpha_1, \alpha_2, \gamma\}$ is the vector of nuisance parameters. The parameter $\gamma$ is not identifiable, under $H_{02}$ and hence, there is no the score statistic test for this hypothesis. For the hypothesis $H_{03} : \alpha_1 = \alpha_2$ and $\lambda_1 = \lambda_2$ or $H_{03} : \alpha_1 - \alpha_2 = \alpha = 0$ and $\lambda_1 - \lambda_2 = \lambda = 0$, the vector of nuisance parameters is $\boldsymbol{\theta}_2 = \{\alpha_2, \lambda_2, \lambda_{12}, \gamma\}$.

# 4 Simulation studies

In this section, we present a simulation study in order to assess the performance of the standard likelihood based tests. First, the studies were based on proposed ACBW2 and compared with Ryu's bivariate Weibull (ACBW1) and independent Weibull (ACBW3) models. Second, the studies were based on the proposed Weibull (CRW1) and independent Weibull competing risks (CRW2) models.

For these studies, 500 samples of size 30, 50, 100, 200, 300 and 500 were generated by Gibbs and Metropolis-Hasting methods from bivariated Weibull model ACBW2. The true parameters values are $\lambda_1 = \lambda_2 = 0.1, \lambda_{12} = 0.2, \alpha_1 = \alpha_2 = \gamma_1 = \gamma_2 = 0.5$ following the Ryu's paper (1993).

This method consist in generating random samples of the following conditional densities written as:

$$f^*(t_1|t_2) = e^{-\lambda_1 t_1^{\alpha_1}} \psi_{1i}(t_1),$$

where $\psi_{1i}(t_1) = \frac{f^*(t_1,t_2)}{f^*(t_2)} \times F_i$, $F_1 = \exp\left[-\lambda_2 t_2^{\alpha_2} - \lambda_{12}(t_1 + t_2) + A_2 + B_2\right]$ if $t_1 \leq t_2$ and $F_2 = \exp\left[-\lambda_2 t_2^{\alpha_2} - \lambda_{12}(t_1 + t_2) + A_1 + B_1\right]$ if $t_1 > t_2$ and

$$f^*(t_2|t_1) = e^{-\lambda_2 t^{\alpha_2}} \psi_{2j}(t_2),$$

where $\psi_{2i}(t_2) = \frac{f^*(t_1,t_2)}{f^*(t_2)} \times G_i$, $G_1 = \exp\left[-\lambda_1 t_1^{\alpha_1} - \lambda_{12}(t_1 + t_2) + A_2 + B_2\right]$ if $t_1 \leq t_2$ and $G_2 = \exp\left[-\lambda_1 t_1^{\alpha_1} - \lambda_{12}(t_1 + t_2) + A_1 + B_1\right]$ if $t_1 > t_2$, with

$$A_2 = \frac{\lambda_{12}}{\gamma_2}\left(1 - e^{-\gamma_2(t_2 - t_1)}\right), \quad A_1 = \frac{\lambda_{12}}{\gamma_1}\left(1 - e^{-\gamma_1(t_1 - t_2)}\right),$$
$$B_2 = \frac{4\lambda_{12}}{(\gamma_1 + \gamma_2)}\left(e^{-\frac{\gamma_2}{2}(t_2 - t_1)} - e^{-\frac{\gamma_1}{2}t_1 - \frac{\gamma_2}{2}t_2}\right),$$
$$B_1 = \frac{4\lambda_{12}}{(\gamma_1 + \gamma_2)}\left(e^{-\frac{\gamma_1}{2}(t_1 - t_2)} - e^{-\frac{\gamma_1}{2}t_1 - \frac{\gamma_2}{2}t_2}\right),$$

$f^*(t_1, t_2)$ is given in (2.9) and $f^*(t_i) = h_i^*(t_i) S_{T_i}^*(t_i)$, where $h_i^*(t_i)$ and $S_{T_i}^*(t_i)$ are given in property c) (section 2.1).

The method proceed generating samples units $(t_1, t_2)$ from Weibull distribution $(\lambda_i, \alpha_i)$ $(i = 1, 2)$ which has to be accepted with probability

$\min\left\{\frac{\psi_i(t_i^{j+1})}{\psi_i(t_i^j)}, 1\right\}$, $i = 1, 2$ and $j = 1, 2, \ldots, k$. For each simulation, we generated five Metropolis-Hastings chains each of which ran for 6000 iterations and burn-in the first 1000 iterations. We monitored the convergence of the Metropolis-Hastings samplers using the Gelman and Rubin (1992) method that uses the analysis of variance technique to determine if further iterations are needed.

**Table 1**

*Mean, standard error, bias and MSE of MLE ($\times 10$) for the proposed ACBW2, Ryu ACBW1 and independent ACBW3 bivariate models.*

| n | models | $\hat{\lambda}_1(S.D.)$ bias(mse) | $\hat{\lambda}_2(S.D.)$ bias(mse) | $\hat{\lambda}_{12}(S.D.)$ bias(mse) | $\hat{\alpha}_1(S.D.)$ bias(mse) | $\hat{\alpha}_2(S.D.)$ bias(mse) | $\hat{\gamma}_1(S.D.)$ bias(mse) | $\hat{\gamma}_2(S.D.)$ bias(mse) |
|---|---|---|---|---|---|---|---|---|
| 30 | ACBW2 | 0.92(0.61) | 0.93(0.65) | 2.07(0.79) | 5.37(3.47) | 5.53(3.88) | 6.86(5.44) | 6.75(4.93) |
| | | 0.08(0.03) | 0.07(0.04) | 0.07(0.06) | 0.37(1.21) | 0.53(1.53) | 1.86(3.30) | 1.75(2.73) |
| | ACBW1 | 1.37(0.63) | 1.40(0.69) | 2.15(1.68) | 6.90(3.19) | 6.66(2.99) | 6.90(11.0) | 7.39(12.6) |
| | | 0.37(0.05) | 0.40(0.06) | 0.15(0.28) | 1.90(1.37) | 1.66(1.16) | 1.90(12.4) | 2.39(16.4) |
| | ACBW3 | 1.74(0.66) | 1.82(0.71) | | 10.6(1.73) | 10.4(1.74) | | |
| | | 0.74(0.09) | 0.82(0.11) | | 5.65(3.43) | 5.48(3.21) | | |
| 50 | ACBW2 | 0.89(0.52) | 0.91(0.49) | 2.08(0.83) | 5.48(3.33) | 5.20(2.78) | 6.16(3.55) | 6.28(4.20) |
| | | 0.11(0.02) | 0.09(0.02) | 0.08(0.06) | 0.48(1.13) | 0.20(0.77) | 1.16(1.39) | 1.28(1.92) |
| | ACBW1 | 1.36(0.56) | 1.34(0.50) | 2.12(1.81) | 6.98(2.84) | 6.94(2.79) | 5.91(9.57) | 6.19(9.94) |
| | | 0.36(0.04) | 0.34(0.03) | 0.12(0.32) | 1.98(1.19) | 1.94(1.15) | 0.91(9.24) | 1.19(10.0) |
| | ACBW3 | 1.68(0.52) | 1.68(0.50) | | 10.6(1.39) | 10.6(1.32) | | |
| | | 0.68(0.07) | 0.68(0.07) | | 5.69(3.32) | 5.68(3.31) | | |
| 100 | ACBW2 | 0.93(0.40) | 0.92(0.38) | 1.99(0.30) | 5.22(2.25) | 5.16(2.14) | 5.73(2.42) | 5.62(2.20) |
| | | 0.07(0.01) | 0.08(0.01) | 0.01(0.01) | 0.22(0.51) | 0.16(0.46) | 0.73(0.63) | 0.62(0.52) |
| | ACBW1 | 1.35(0.44) | 1.35(0.43) | 1.85(1.15) | 7.40(2.53) | 7.23(2.54) | 5.59(9.32) | 5.13(7.79) |
| | | 0.35(0.03) | 0.35(0.03) | 0.15(0.13) | 2.40(1.21) | 2.23(1.14) | 0.59(8.72) | 0.13(6.07) |
| | ACBW3 | 1.63(0.40) | 1.64(0.38) | | 10.6(1.03) | 10.6(1.06) | | |
| | | 0.63(0.05) | 0.64(0.05) | | 5.68(3.24) | 5.68(3.24) | | |
| 200 | ACBW1 | 0.98(0.26) | 0.97(0.24) | 2.00(0.20) | 5.17(1.19) | 5.14(1.20) | 5.21(1.26) | 5.21(1.29) |
| | | 0.02(0.01) | 0.03(0.01) | 0.00(0.01) | 0.17(0.14) | 0.14(0.14) | 0.21(0.16) | 0.21(0.17) |
| | ACBW2 | 1.40(0.32) | 1.34(0.33) | 1.79(0.84) | 7.62(2.03) | 7.21(2.10) | 3.97(6.06) | 4.33(6.22) |
| | | 0.40(0.02) | 0.34(0.02) | 0.21(0.07) | 2.62(1.09) | 2.21(0.92) | 1.03(3.77) | 0.67(3.91) |
| | ACBW3 | 1.61(0.27) | 1.60(0.26) | | 10.7(0.73) | 10.7(0.72) | | |
| | | 0.61(0.04) | 0.60(0.04) | | 5.70(3.30) | 5.70(3.30) | | |
| 300 | ACBW2 | 0.99(0.20) | 0.98(0.19) | 1.98(0.16) | 5.08(1.00) | 5.13(1.01) | 5.17(0.96) | 5.16(0.98) |
| | | 0.01(0.00) | 0.02(0.00) | 0.02(0.00) | 0.08(0.10) | 0.13(0.10) | 0.17(0.09) | 0.16(0.09) |
| | ACBW1 | 1.40(0.28) | 1.38(0.28) | 1.70(0.50) | 7.38(1.82) | 7.46(1.91) | 3.68(3.94) | 3.72(4.37) |
| | | 0.40(0.02) | 0.38(0.02) | 0.30(0.03) | 2.38(0.89) | 2.46(0.96) | 1.32(1.72) | 1.28(2.07) |
| | ACBW3 | 1.61(0.23) | 1.59(0.22) | | 10.6(0.61) | 10.6(0.61) | | |
| | | 0.61(0.04) | 0.59(0.03) | | 5.65(3.17) | 5.68(3.17) | | |
| 500 | ACBW2 | 0.99(0.15) | 0.99(0.13) | 1.99(0.11) | 5.03(0.72) | 5.03(0.66) | 5.07(0.68) | 5.06(0.67) |
| | | 0.01(0.00) | 0.01(0.00) | 0.01(0.00) | 0.03(0.05) | 0.03(0.04) | 0.07(0.04) | 0.06(0.04) |
| | ACBW1 | 1.41(0.22) | 1.43(0.23) | 1.70(0.41) | 7.31(1.58) | 7.43(1.53) | 3.24(3.00) | 3.19(3.98) |
| | | 0.41(0.02) | 0.43(0.02) | 0.30(0.02) | 2.31(0.78) | 2.43(0.82) | 1.76(1.20) | 1.81(1.91) |
| | ACBW3 | 1.61(0.17) | 1.60(0.17) | | 10.6(0.46) | 10.6(0.45) | | |
| | | 0.61(0.04) | 0.60(0.03) | | 5.64(3.15) | 5.64(3.15) | | |

The real values of the parameters are: $\lambda_1 = \lambda_2 = 1, \lambda_{12} = 2, \alpha_1 = \alpha_2 = \gamma_1 = \gamma_2 = 5$.

Maximum likelihood estimators were calculated by Quasi-Newton method; their standard deviation, bias and mean square error are calculated also. It was used for the calculations, the observed information matrix instead of the Fisher information matrix. Table 1 shows these results under the three models studied.

For the samples generated by the bivariate model ACBW2, two facts could be noticed when sample sizes increase: the MLE of the parameters converge to the true values and both bias and standard deviations decrease, specially when $n \geq 200$. Even for the sample sizes 30 up to 100, the bias and MSE of the parameters: $\lambda_1$, $\lambda_2$, $\lambda_{12}$, $\alpha_1$ and $\alpha_2$ are small. The fitness of the Weibull bivariate model ACBW1 for these 500 samples seems to produce MLE which overestimates the true values when sample sizes are 30 and 50 and underestimate, when samples size are greater than 100. When independent Weibull model ACBW3 was fitted to these samples, the convergence of MLE of all the parameters was bad, showing how we could mislead the true values, when we treat dependent data as independent.

Some plots of the 500 MLE based on samples generated by the bivariate model ACBW2 with size 500, were performed in order to study the asymptotic normality; histogram and normal probability plots gives a partial view of the approximation. The plots below suggest a reasonable fit to the normal distribution.

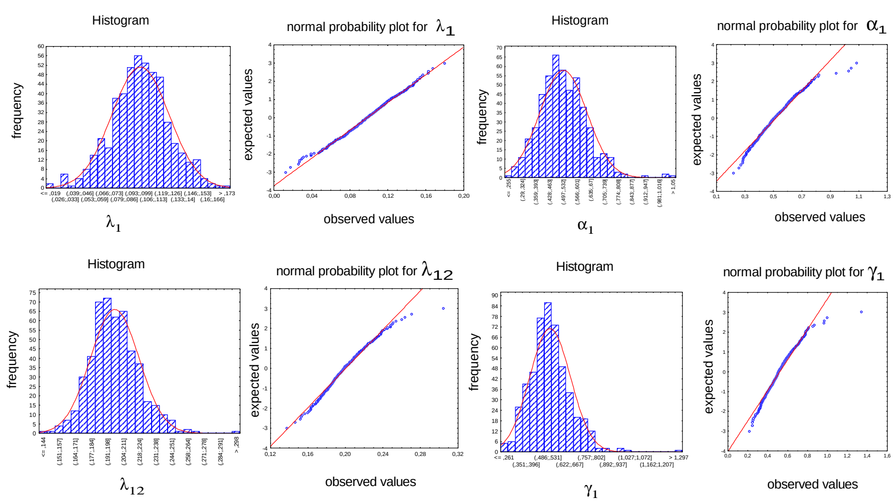


Table 2 shows the means, standard deviations, bias and MSE of 500 samples for the Weibull competing risks model CRW1 and independent model CRW2. These 500 samples of competing risks setting were derived from the generated samples of bivariate Weibull ACBW2, that is, the data $(t_j, \delta_{ij})$ where $t_j = \min(t_{1j}, t_{2j})$, $\delta_{ij} = 1$ if the cause of failure is $i$ and $\delta_{ij} = 0$ otherwise, $i = 1, 2$ and $j = 1, 2, ..., n$.

The MLE of the parameters of the competing risks model converges also for the true values when sample size increase; the bias are small for samples size up to 50 for all estimates, except for estimate of $\gamma$, which bias become small only for sample sizes up to 200. The estimates of $\lambda_{12}$ and $\gamma$ presented large standard deviations even decreasing with the increasing of sample size. It seems that the approximation to normality seems reasonable for the estimates of the parameters for sample size up to 200, with exception to the parameter $\gamma$. Weibull independent competing risks model with these 500 samples produced overestimates the true values for all samples sizes.

**Table 2**

*Mean, standard error, bias and MSE of MLE* ($\times 10$) *for the proposed CRW1 and independent CRW2 competing risks models.*

| n | mo-dels | $\hat{\lambda}_1(S.D.)$ bias(mse) | $\hat{\lambda}_2(S.D.)$ bias(mse) | $\hat{\lambda}_{12}(S.D.)$ bias(mse) | $\hat{\alpha}_1(S.D.)$ bias(mse) | $\hat{\alpha}_2(S.D.)$ bias(mse) | $\hat{\gamma}(S.D.)$ bias(mse) |
|---|---|---|---|---|---|---|---|
| 30 | CRW1 | 1.13(0.66) | 1.10(0.63) | 2.26(1.64) | 5.78(3.49) | 5.66(3.41) | 9.13(1.14) |
| | | 0.13(0.04) | 0.10(0.04) | 0.26(0.27) | 0.78(1.27) | 0.66(1.20) | 4.13(1.83) |
| | CRW2 | 1.92(0.77) | 1.87(0.72) | | 9.60(2.43) | 9.70(2.46) | |
| | | 0.92(0.14) | 0.87(0.12) | | 4.60(2.70) | 4.70(2.81) | |
| 50 | CRW1 | 1.04(0.48) | 1.05(0.50) | 2.26(1.35) | 5.58(3.20) | 5.48(2.65) | 7.75(9.42) |
| | | 0.04(0.02) | 0.05(0.02) | 0.26(0.18) | 0.58(1.05) | 0.48(0.72) | 2.75(9.62) |
| | CRW2 | 1.78(0.55) | 1.44(0.57) | | 9.88(1.87) | 9.90(1.95) | |
| | | 0.78(0.09) | 0.44(0.05) | | 4.88(2.73) | 4.90(2.78) | |
| 100 | CRW1 | 1.01(0.38) | 1.01(0.42) | 2.29(1.11) | 5.38(2.64) | 5.45(2.24) | 6.68(8.61) |
| | | 0.01(0.01) | 0.01(0.02) | 0.29(0.13) | 0.38(0.71) | 0.45(0.52) | 1.68(7.69) |
| | CRW2 | 1.72(0.43) | 1.72(0.46) | | 9.99(1.45) | 10.0(1.50) | |
| | | 0.72(0.07) | 0.72(0.07) | | 4.99(2.70) | 5.00(2.72) | |
| 200 | CRW1 | 1.02(0.28) | 1.03(0.32) | 2.35(0.95) | 5.28(1.33) | 5.33(1.35) | 5.01(3.67) |
| | | 0.02(0.01) | 0.03(0.01) | 0.35(0.10) | 0.28(0.18) | 0.33(0.19) | 0.01(1.34) |
| | CRW2 | 1.69(0.30) | 1.69(0.31) | | 10.0(1.02) | 10.1(1.06) | |
| | | 0.69(0.05) | 0.69(0.05) | | 5.00(2.60) | 5.10(2.71) | |
| 300 | CRW1 | 1.01(0.24) | 1.03(0.25) | 2.14(0.57) | 5.24(1.17) | 5.20(1.13) | 5.02(2.44) |
| | | 0.01(0.01) | 0.03(0.01) | 0.14(0.03) | 0.24(0.14) | 0.20(0.13) | 0.02(0.59) |
| | CRW2 | 1.68(0.26) | 1.70(0.27) | | 10.0(0.87) | 10.0(0.88) | |
| | | 0.68(0.05) | 0.70(0.05) | | 5.00(2.57) | 5.00(2.57) | |
| 500 | CRW1 | 1.02(0.19) | 1.02(0.19) | 2.09(0.41) | 5.15(0.83) | 5.12(0.83) | 4.91(1.71) |
| | | 0.02(0.00) | 0.02(0.00) | 0.09(0.01) | 0.15(0.07) | 0.12(0.07) | 0.09(0.29) |
| | CRW2 | 1.70(0.19) | 1.69(0.20) | | 9.98(0.67) | 10.0(0.70) | |
| | | 0.70(0.05) | 0.69(0.05) | | 4.98(2.52) | 5.00(2.54) | |

The real values of the parameters are: $\lambda_1 = \lambda_2 = 1, \lambda_{12} = 2, \alpha_1 = \alpha_2 = \gamma = 5$.

In order to study the asymptotic statistic tests for the hypotheses $H_{01}$, $H_{02}$ and $H_{03}$, 250 samples of size 200 were generated in competing risks setting. The true values were $\lambda_1 = \lambda_2 = 0.1, \lambda_{12} = 0.2$ and $\gamma = 0.5$ for $H_{01}$; $\lambda_1 = \lambda_2 = 0.1$ and $\alpha_1 = \alpha_2 = \gamma = 0.5$, for the hypothesis $H_{02}$.

Table 3 shows the true size and the power of the asymptotic tests when the nominal level of significance are 5% and 10%. The Wald statistic test (W) presented true size larger than the nominal size in all the hypotheses and the power is higher than the other tests. The Rao's score statistic test (LM) presented true size close to the nominal size for $H_{01}$ but high for $H_{03}$ and the power is smaller than the other tests. For the likelihood ratio statistic test (LR) the true size is close of the nominal size in all cases, except in $H_{01}$. The hypothesis $H_{02} : \lambda_{12} = 0$ is testing the parameter on the boundary of the parameter space, which the statistic test could not result in $\chi^2$ with 1 d.f. under $H_{02}$.

**Table 3**
*Estimates of true size and power (in parentheses) of the asymptotic tests for competing risks model CRW1.*

| estimates | null hypotheses | 5% | | | 10% | | | correlation | | |
|---|---|---|---|---|---|---|---|---|---|---|
| | | W | LM | LR | W | LM | LR | W-LM | W-LR | LM-LR |
| true size (power) | $H_{01} : \alpha_1 = \alpha_2 = 1$ | 11.2 | 6.0 | 3.2 | 16.4 | 8.0 | 7.6 | 0.68 | 0.97 | 0.79 |
| | | (93.6) | (77.6) | (88.0) | (96.4) | (84.0) | (94.0) | (0.68) | (0.84) | (0.89) |
| | $H_{02} : \lambda_{12} = 0$ | 7.2 | – | 4.0 | 14.8 | – | 12.0 | – | 0.98 | – |
| | | (99.6) | – | (95.6) | (99.6) | – | (98.0) | – | (0.82) | – |
| | $H_{03} : \lambda = \alpha = 0$ | 8.0 | 9.6 | 5.6 | 13.6 | 16.8 | 11.2 | 0.95 | 0.99 | 0.97 |
| | | (92.8) | (94.4) | (97.5) | (97.8) | (97.7) | (98.4) | (0.65) | (0.69) | (0.70) |

# 5 Applications to real data

In order to illustrate the methodology above, we studied two competing risks data:

**Example 5.1** The first data analysed was extracted from Lagakos (1977). This data is from the lung cancer clinical trial being conducted by Eastern Cooperative Oncology group at that time. Among the 194 patients, 83 died from local spread of disease (cause 1) and 44 died with metastatic spread of disease (cause 2) and 67 were censored. Covariates were considered in that study.

The Weibull competing risks model CRW1 (section 2.2) was considered for the analysis of this data. The hypotheses of interest are in Table 4.

From these results we may conclude that exponential bivariate dependent model may fit the data since $H_{01}$ was not significant at 5%. The crude hazard functions (2.11) of deaths from local (higher) an metastic (lower) spread of lung cancer may be increasing functions as show in the Figure 1.

**Table 4**
*Tests of hypotheses of interest.*

| hypotheses | asymptotic tests | | | | | |
|---|---|---|---|---|---|---|
| | W | | LR | | LM | |
| | value | p-value | value | p-value | value | p-value |
| $H_{01}: \alpha_1 = \alpha_2 = 1$ | 2.18 | 0.34 | 2.37 | 0.31 | 2.50 | 0.29 |
| $H_{02}: \lambda_{12} = 0$ | 6.09 | 0.01 | 7.31 | 0.006 | | |
| $H_{03}: \lambda = \alpha = 0$ | 5.84 | 0.05 | 9.75 | 0.007 | 6.67 | 0.04 |

**Example 5.2** The second data is from a follow-up study considered in patients with dilated cardiomyopathy with heart congestive failure. In this study, 95 patients with heart congestive failure were observed from august/92 to august/94 and the time of failures (in weeks) are from the hospital entrance until death. Thus patients alive at the end of the study were considered as censored. Two causes of death were observed: deaths from shock and unexpected deaths. It was verified that among the 95 patients of the study, 23 had survival to the final of the study; 63 died due to shock and only 9 died unexpectedly. The median survival times from shock is 15 weeks and from unexpected causes is 20 weeks for failures times and is 64 weeks for censored times.

**Table 5**
*Tests of hypotheses of interest*

| hypotheses | asymptotic tests | | | | | |
|---|---|---|---|---|---|---|
| | W | | LR | | LM | |
| | value | p-value | value | p-value | value | p-value |
| $H_{01}: \alpha_1 = \alpha_2 = 1$ | 3.72 | 0.16 | 3.94 | 0.14 | 3.74 | 0.15 |
| $H_{01}^{*}: \alpha_1 = 1$ | 4.28 | 0.04 | 4.44 | 0.03 | 3.73 | 0.05 |
| $H_{01}^{**}: \alpha_2 = 1$ | 4.1e-5 | 0.99 | 2.33e-6 | 0.99 | 2.4e-6 | 0.99 |
| $H_{02}: \lambda_{12} = 0$ | 0.92 | 0.34 | 0.62 | 0.43 | | |
| $H_{03}: \lambda = \alpha = 0$ | 14.41 | $< 0.001$ | 46.44 | $< 0.001$ | 38.79 | $< 0.001$ |

The test of the reduction to exponential bivariate model ($H_{01}$) resulted non-significant at 5% because the high standard deviation (0.2686). This may be due the small sample size verified for unexpected deaths. Tests for reduction of each marginal to exponential resulted significant only for cause due to shock deaths ($H_{01}^{*}$), that is, for this cause of death the hazard functions is not constant. We may conclude that Weibull bivariate independent model would fit the data ($H_{02}$). The crude hazard functions can be shown in the Figure 2.

We used the software SAS (proc IML) to perform all the simulations and calculations.

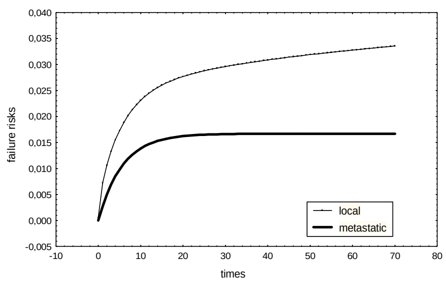


Fig.1: Crude Hazard functions (example 1)

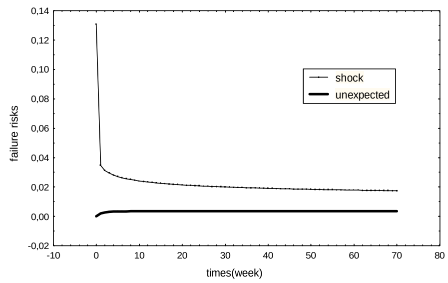


Fig.2: Crude hazard functions (example 2)

# 6 Conclusions

The modified bivariate Weibull model has some importants requirements for a derivation of competing risks model: 1) the modified bivariate Weibull model is absolutely continuos and 2) it is a dependent model. The competing risks model derived from this distribution presents (in the only possible case of $\gamma_1 = \gamma_2 = \gamma$), the net and crude hazards equals which makes the marginal distributions of competing Weibull model identifiable. This property is very interesting since usually the marginals are not identifiable in competing risks models.

In the modified bivariate and competing risks model, the MLE seems to have asymptotic desirable properties needed for the tests of hypotheses for all the parameters, with exception to the parameter $\gamma$ in competing risks model. Simulations studies were performed for these models in order to study MLE distributions and tests of hypotheses in the bivariate and competing risks models. There was need of heavy use of computer programs for the study of the proposed bivariate and competing risks models, since many expressions could not be written in closed form as some integrals and the information matrix and even so the observed information matrix.

Tests of hypotheses were performed in two data sets. Dependent Weibull competing risks model seems to be adequate for both data sets, since the hazards functions are increasing (for the first data), constant or decreasing (for the second data). The results of the applications agree with the previous analyses performed for these data.

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