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DELAYED RESPONSE IN BIVARIATE SET UP USING GIBBS SAMPLER

Uttam Bandyopadhyay
Department of Statistics
University of Calcutta
35 Ballygunge Circular Road
Calcutta - 700 019, India

Atanu Biswas
Applied Statistics Unit
Indian Statistical Institute
203 B.T. Road
Calcutta - 700 035, India
atanu@isical.ac.in

Summary
In a clinical trial, when there is a continuous flow of patients, response by a patient may be delayed. Our object in the present investigation is to use the Bayesian technique for studying the nature of such delay is the case of bivariate response. Taking suitable priors, several posterior distributions of the probabilities of response are obtained by using Gibbs sampler technique. The applicability of the present approach for grouped data at different stages is also discussed.

Key Words: Delayed response; Gibbs sampler; grouped data; probability model.

1 Introduction

For sequential monitoring of study subjects in a system one by one or batch by batch a lot of works, under unvariate set-up, are available in the literature (see for example, in group sequential framework, Pocock (1977), Lan and DeMets (1983), Jennison and Turnbull (1989), Bandyopadhyay and Biswas (1995)). Bivariate responses are also very common in practice. For example, in clinical trials, a particular drug may have effects simultaneously on two diseases. Bivariate treatment responses are also considered by some authors (see for example, Jennison and Turnbull (1993) and Bandyopadhyay and Biswas (1996a)). All these works are based on the assumption that the response of each subject is either immediate or it is obtained before the entrance of the next subject. But this may not occur in practice. For example, in bivariate set-up, suppose a treatment is applied sequentially to n subjects and, at the $(n+1)st$ stage, we see that some of the n subjects fail to respond completely or partially. That is, there is a delay in response. The phrase 'delayed response' is used to indicate that a subject's complete response is not obtained before the entrance of the next subject. The paper by Tamura, Faires, Andersen and Heiligenstein (1994) describes an actual clinical trial with delayed response.

Bandyopadhyay and Biswas (1996b), under univariate set up, incorporated the possibility of delayed response in their model for comparing two treatments using a randomized play-the-winner rule for allocation. Some decision rules were suggested there from the frequentist's view point. Several performance characteristics and properties of the proposed decision rules were also studied. This problem was also revisited by Biswas (1999).

Here we investigate the nature of bivariate delayed response and the associated probability model in the Bayesian framework. Taking different priors the posterior distributions of the probabilities of getting complete response and no response at different time lags are studied. The popular and powerful Gibbs sampler technique (see Geman and Geman (1984), Gelfand and Smith (1990), Gelfand et al. (1990)), whenever it is required, is used to estimate the marginal posterior distributions of the response probabilities. In the next section, taking general probability model, we consider the probability of obtaining different kinds of responses at different time lags. In Section 3, we concentrate on two different models for the response probabilities. Numerical computations of posterior means, standard deviation (s.d.'s) and modes are also given in different situations. Finally, in Section 4, we indicate a possible extension for grouped data.

2 General probability model

Suppose a treatment is applied sequentially to each of the n subjects under study. At each stage only one subject is allocated and after the n-th patient allocation there is a possibility that some of the n subjects will fail to respond (in one component or both the components). This is called delayed response. To study the behaviour of such delays in response we first introduce the following indicator variables: $\in_{jn+1}= 1$ or 0 according as the response of the first component of the j-th subject is obtained or not after the allocation of the n-th subject (and before the allocation of the $(n+1)st$ subject), $j = 1(1)n$, and η_{jn+1} is a similar indicator variable for the second component of the j-th subject. Then our observed data is $x \equiv (\in_{1n+1}, \in_{2n+1}, \ldots, \in_{nn+1}\,;\ \eta_{1n+1}, \eta_{2n+1}, \ldots, \eta_{nn+1})$. We write,

$$\pi_{n+1-j} = P\{\in_{jn+1}= 1, \eta_{jn+1} = 1\}, \quad \omega_{n+1-j} = P\{\in_{jn+1}= 0,\ \eta_{jn+1} = 0\},$$

and hence $P\{\in_{jn+1}=1, \eta_{jn+1}=0\} + P\{\in_{jn+1}=0, \eta_{jn+1}=1\} = 1 - \pi_{n+1-j} - \omega_{n+1-j}$, which are, respectively, the probabilities of getting the complete response, no response and the partial response of the j-th subject with a time lag $(n+1-j)$ before the entrance of the $(n+1)st$ subject. Clearly, π_t is non-decreasing and ω_t is non-incresing in t, and we can safely assume that as $t \to \infty$, (i) $\pi_t \to 1$ and (ii) $\omega_t \to 0$.

Denoting $\theta \equiv (\pi_1, \pi_2, \ldots \pi_n; \omega_1, \omega_2, \ldots \omega_n)$, the conditional probability of x given θ is

$$\begin{aligned} f(x|\theta) &= \prod_{j=1}^{n} \Big\{ \pi_{n+1-j}^{\in_{jn+1}\eta_{jn+1}} \omega_{n+1-j}^{(1-\in_{jn+1})(1-\eta_{jn_1})} \\ &\quad (1-\pi_{n+1-j}-\omega_{n+1-j})^{\in_{jn+1}(1-\eta_{jn+1})+(1-\in_{jn+1})\eta_{jn+1}} \Big\} \\ &= \prod_{j=1}^{n} [\in_{jn+1} \eta_{jn+1}\pi_{n+1-j} + (1-\in_{jn+1})(1-\eta_{jn+1})\omega_{n+1-j} \\ &\quad + \{\in_{jn+1}(1-\eta_{jn+1}) + (1-\in_{jn+1})\eta_{jn+1}\}(1-\pi_{n+1-j}-\omega_{n+1-j})] \\ &= \prod_{j=1}^{n} [a_j + b_j\pi_j + c_j\omega_j], \end{aligned} \tag{2.1}$$

where

$$\begin{aligned} a_j &= \in_{n+1-j,n+1}(1-\eta_{n+1-j,n+1}) + (1-\in_{n+1-j,n+1})\pi_{n+1-j,n+1}, \\ b_j &= 3\in_{n+1-j,n+1}\eta_{n+1-j,n+1} - \in_{n+1-j,n+1} - \eta_{n+1-j,n+1}, \\ c_j &= 1-2\in_{n+1-j,n+1} - 2\eta_{n+1-j,n+1} + 3\in_{n+1-j,n+1}\eta_{n+1-j,n+1}. \end{aligned}$$

At this stage, for the sake of simplicity, we consider the uniform prior:

$$g(\theta) = \text{ constant,} \quad \left.\begin{array}{l} 0 < \pi_1 < \pi_2 < \ldots < \pi_n < 1 \\ 0 < \omega_n < \omega_{n-1} < \ldots < \omega_1 < 1 \\ 0 < \pi_j + \omega_j < 1 \ \forall \ j \end{array}\right\} \tag{2.2}$$

Obviously, this prior being defined over a bounded domain is proper. Then the joint distribution of x and θ is given by:

$$p(x,\theta) = f(x|\theta)g(\theta) \propto \prod_{j=1}^{n} [a_j + b_j\pi_j + c_j\omega_j]. \tag{2.3}$$

Then, integrating π_j out, we get

$$p(x, \pi_j, \ldots, \pi_{j-1}, \pi_{j+1}, \ldots, \pi_n, \omega_1, \ldots \omega_n)$$

$$\propto \left(\prod_{\substack{j'=1 \\ j' \neq j}}^{n} [a_{j'} + b_{j'}\pi_{j'} + c_{j'}\omega_{j'}] \right) . \int_{\pi_{j-1}}^{A_j} (a_j + b_j\pi_j + c_j\omega_j)\, d\pi_j$$

$$= \left(\prod_{\substack{j'=1 \\ j' \neq j}}^{n} \left[a_{j'} + b_{j'}\pi_{j'} + c_{j'}\omega_{j'} \right] \right) . \left\{ \left(a_j + c_j\omega_j\right)\left(A_j - \pi_{j-1}\right) + \frac{b_j}{2}\left(A_j^2 - \pi_{j-1}^2\right) \right\}, \quad (2.4)$$

where $A_j = \min(\pi_{j+1}, 1 - \omega_j)$. Then the conditional distribution of π_j given other π_i's, $i \neq j$, ω_j's and x can be easily obtained as:

$$\begin{aligned} & p(\pi_j|\pi_1, \ldots, \pi_{j-1}, \pi_{j+1}, \ldots, \pi_n, \omega_1, \ldots, \omega_n, x) \\ & = \frac{a_j + b_j\pi_j + c_j\omega_j}{(a_j + c_j\omega_j)(A_j - \pi_{j-1}) + \frac{b_j}{2}(A_j^2 - \pi_{j-1}^2)}, \\ & \pi_{j-1} < \pi_j < A_j, \ j = 1, 2, \ldots, n, \end{aligned} \quad (2.5)$$

with $\pi_0 = 0, \pi_{n+1} = 1$. Similarly, the conditional distribution of ω_j given other ω_i's, $i \neq j$, π_i's and x can be easily obtained as

$$\begin{aligned} & p(\omega_j|\pi_1, \ldots, \pi_n, \omega_1, \ldots, \omega_{j-1}, \omega_{j+1}, \ldots, \omega_n, x) \\ & = \frac{a_j + b_j\pi_j + c_j\omega_j}{(a_j + b_j\pi_j)(B_j - \omega_{j+1}) + \frac{c_j}{2}(B_j^2 - \omega_{j+1}^2)}, \\ & \omega_{j+1} < \omega_j < B_j, \ j = 1, 2, \ldots, n, \end{aligned} \quad (2.6)$$

with $\omega_0 = 1$, $\omega_{n+1} = 0$ and $B_j = \min(\omega_{j-1}, 1 - \pi_j)$. Using (2.5) and (2.6) and taking help of the popular Gibbs sampler approach we can estimate the marginal posterior distributions of π_j's and ω_j's.

In the above analysis, depending upon the situation, some other priors can easily be used over the appropriate domain. Besides uniform prior one can consider prior proportional to

(i) π_1, the probability of complete response in a lag of unity,

(ii) ω_1, the probability of no response in lag of unity,

and so on. In a different context, Chen (1994), in the case of only one set of ordered parameters $\beta_1 \leq \ldots \leq \beta_n, (n = 10)$, used the prior proportional to $1_s(\underset{\sim}{\beta}).p(\beta_n)$, where $1_s(\underset{\sim}{\beta}) = 1$ if $\underset{\sim}{\beta} \epsilon S = \left\{(\beta_1, \ldots, \beta_n)' : 0 \leq \beta_1 \leq \beta_2 \leq \ldots \leq \beta_n, \underset{\sim}{\beta} \epsilon R^n\right\}$, and 0 otherwise, and $p(\beta_n)$ is a normal distribution with hyperparameters μ_n and σ_n^2. See also Chen and Deely (1996) in this connection. In our case, as π_i's are probabilities, it would be logical to consider a beta prior over any such π_i instead of normal prior. Beta priors for probabilities are widely used in literature (see, for example, Berry (1972), Jones and Kandeel (1984)). Again, in Chen (1994) or in Chen and Deely (1996), the most important parameter was β_n, but in our case π_1 (or ω_1) is the most important one (see model 1 of Section 3). So it would be logical to assume a beta prior on π_1 (or ω_1). The uniform prior is, of course, a special case of this. In the present context we take beta (2,1) on π_1 or beta (2,1) on ω_1. In Table 1, the posterior means, standard deviations (s.d.'s) and modes of different π_j's and ω_j's are also given. These computations are done using 30000 simulations and we have considered $n = 5$ and $\underset{\sim 1}{\in} = (1, 0, 1, 0, 0)$, $\underset{\sim 1}{\eta} = (1, 0, 1, 0, 0)$, $\underset{\sim 2}{\eta} = (1, 1, 1, 1, 0)$. The three entries in each cell corresponds to posterior mean, s.d. and mode respectively.

3 Modelling of π_j's and ω_j's: two different models

3.1 Model 1

Here we take

$$\pi_j = 1 - (1 - \pi_1)^j \text{ and } \omega_j = \omega_1^j. \tag{3.1}$$

Thus θ reduces to (π_1, ω_1), and it is enough to concentrate on this θ only. Under this model the number of parameters is reduced to 2 form $2n$, and the amount of computation is reduced considerably. Again the model is logical in the sense that as time lag increases the probability of getting responses from both the components increases and tends to 1 and, on the other hand, the probability of getting no response decreases geometrically and tends to 0. Also, it has some Markov Chain property. For example, the probability of getting both responses after $(j + 1)$ lags given that at least one of the responses were not available after j lags is π_1, same for every j. Now $f(x|\theta)$ simplifies to

$$\begin{aligned}
f(x|\theta) &= \prod_{j=1}^{n} \Big[\Big\{1-(1-\pi_1)^{n+1-j}\Big\}^{\in_{jn+1}\eta_{jn+1}} \Big\{\omega_1^{n+1-j}\Big\}^{(1-\in_{jn+1})(1-\eta_{jn+1})} \\
&\quad \cdot \Big\{(1-\pi_1)^{n+1-j}\omega_1^{n+1-j}\Big\}^{\in_{jn+1}(1-\eta_{jn+1})+(1-\in_{jn+1})\eta_{jn+1}}\Big]. \\
&= \prod_{j=1}^{n} \Big[\in_{jn+1}\, \eta_{jn+1} \Big\{1-(1-\pi_1)^{n+1-j}\Big\} \\
&\quad +(1-\in_{jn+1})(1-\eta_{jn+1}) \Big\{\omega_1^{n+1-j}\Big\} + \{\in_{jn+1}\,(1-\eta_{jn+1}) \\
&\quad +(1-\in_{jn+1})\eta_{jn+1}\} \Big\{(1-\pi_1)^{n+1-j} - \omega_1^{n+1-j}\Big\}\Big] \\
&= \prod_{j=1}^{n} (e_j + d_j u_j + c_j v_j)\,, \qquad (3.2)
\end{aligned}$$

Table 1 *Posterior mean, s.d. and mode for the general model.*

Uniform Prior	$(\underset{\sim 1}{\in}, \underset{\sim 1}{\eta})$	π_1	π_2	π_3	π_4	π_5
		0.0976	0.2241	0.3933	0.5391	0.7413
		0.0782	0.1198	0.1407	0.1489	0.1422
		0.0100	0.1800	0.3600	0.5750	0.7950
		ω_1	ω_2	ω_3	ω_4	ω_5
		0.7715	0.5906	0.4132	0.2790	0.1162
		0.1342	0.1469	0.1418	0.1235	0.0880
		0.8500	0.5600	0.3750	0.2400	0.0100
	$(\underset{\sim 1}{\in}, \underset{\sim 2}{\eta})$	π_1	π_2	π_3	π_4	π_5
		0.0969	0.2095	0.3889	0.5203	0.7569
		0.0771	0.1149	0.1442	0.1509	0.1472
		0.0050	0.1650	0.3500	0.5200	0.8350
		ω_1	ω_2	ω_3	ω_4	ω_5
		0.7156	0.4410	0.2949	0.1618	0.0771
		0.1628	0.1567	0.1358	0.0964	0.0623
		0.8450	0.4400	0.2250	0.1350	0.0200
Prior $\propto \pi_1$	$(\underset{\sim 1}{\in}, \underset{\sim 1}{\eta})$	π_1	π_2	π_3	π_4	π_5
		0.2414	0.3430	0.4758	0.6002	0.7723
		0.1230	0.1291	0.1343	0.1357	0.1265
		0.1200	0.3550	0.4500	0.5850	0.8100
		ω_1	ω_2	ω_3	ω_4	ω_5
		0.6524	0.5043	0.3575	0.2430	0.1041
		0.1368	0.1365	0.1272	0.1103	0.0783
		0.6650	0.5000	0.3450	0.2050	0.0100
	$(\underset{\sim 1}{\in}, \underset{\sim 2}{\eta})$	π_1	π_2	π_3	π_4	π_5
		0.2227	0.3162	0.4598	0.5748	0.7814
		0.1155	0.1250	0.1382	0.1408	0.1335
		0.1200	0.3000	0.4600	0.5950	0.8350
		ω_1	ω_2	ω_3	ω_4	ω_5
		0.6177	0.3846	0.2597	0.1441	0.0703
		0.1553	0.1419	0.1218	0.0869	0.0562
		0.6250	0.4050	0.2200	0.0800	0.0050

Prior $\propto \omega_1$	$(\underset{\sim 1}{\in}, \underset{\sim 1}{\eta})$	π_1	π_2	π_3	π_4	π_5
		0.0742	0.1969	0.3564	0.5002	0.6944
		0.1247	0.1342	0.1291	0.1177	0.1054
		0.0050	0.1550	0.3400	0.5200	0.7450
		ω_1	ω_2	ω_3	ω_4	ω_5
		0.8309	0.6847	0.5156	0.3303	0.1787
		0.1169	0.1294	0.1286	0.1329	0.0921
		0.8950	0.6900	0.5050	0.2900	0.1300
	$(\underset{\sim 1}{\in}, \underset{\sim 2}{\eta})$	π_1	π_2	π_3	π_4	π_5
		0.0594	0.1637	0.3270	0.4776	0.6663
		0.1322	0.1406	0.1501	0.1320	0.1178
		0.0050	0.1150	0.3200	0.4950	0.7000
		ω_1	ω_2	ω_3	ω_4	ω_5
		0.8040	0.6486	0.4738	0.2910	0.1429
		0.1210	0.1314	0.1367	0.1126	0.1008
		0.8700	0.7050	0.4950	0.2700	0.0950

where

$$
\begin{aligned}
e_j &= \in_{jn+1} \eta_{jn+1},\ d_j = \in_{jn+1} + \eta_{jn+1} - 3 \in_{jn+1} \eta_{jn+1}, \\
u_j &= (1-\pi_1)^{n+1-j} \text{ and } v_j = \omega_1^{n+1-j},
\end{aligned}
$$

and hence we can write

$$f(x|\theta) = \sum_{k=1}^{5} A_{kn}, \tag{3.3}$$

where

$$A_{1n} = \prod_{j=1}^{n} e_j + \prod_{j=1}^{n} d_j u_j + \prod_{j=1}^{n} c_j v_j,$$

$$
\begin{aligned}
A_{2n} = \sum_{i=1}^{n} e_i \Bigg[& \prod_{\substack{j=1 \\ j\neq i}}^{n} d_j u_j + \sum_{\substack{j=1 \\ j\neq i}}^{n} c_j v_j \prod_{j'\neq j,i} d_{j'} u_{j'} + \sum_{\substack{j_1=1 \\ j_1\neq}}^{n} \sum_{\substack{j_2=1 \\ j_2\neq i}}^{n} \left(\prod_{j=j_1,j_2} c_j v_j \right) \\
& \left(\prod_{j'\neq j,i} d_{j'} u_{j'} \right) + \ldots + \prod_{\substack{j=1 \\ j\neq i}}^{n} c_j v_j \Bigg]
\end{aligned}
$$

$$+\sum_{i_1\neq}\sum_{i_2=1}^{n}\left(\prod_{k=i_1,i_2} e_k\right)\left[\prod_{\substack{j=1\\ j\neq i_1,i_2}}^{n} d_j u_j + \sum_{\substack{j=1\\ j\neq i_1,i_2}}^{n} c_j v_j \prod_{j'\neq j,k} d_{j'}u_{j'}\right.$$

$$+\sum_{\substack{j_1=1\\ j_1\neq}}^{n}\sum_{\substack{j_2=1\\ j_2\neq(i_1,i_2)}}^{n}\left(\prod_{j=j_1,j_2} c_j v_j\right)\left(\prod_{j'\neq j_1,j_2,i_1,i_2} d_{j'}u_{j'}\right)$$

$$\left.+\ldots+\prod_{\substack{j=1\\ j\neq i_1,i_2}}^{n} c_j v_j\right]+\ldots+\sum_{i_1\neq\ldots}\cdots\sum_{\neq i_{n-2}=1}^{n}\left(\prod_{k=i_1,\ldots,i_{n-2}} e_k\right)$$

$$\left[\prod_{j\neq(i_1,\ldots,i_{n-2})} d_j u_j + \sum_{j_1,j_2\neq(i_1,\ldots,i_{n-2})} d_{j_1}u_{j_1}c_{j_2}v_{j_2} + \prod_{j\neq(i_1,\ldots,i_{n-2})} c_j v_j\right],$$

$$A_{3n} = \sum_{i=1}^{n} d_i u_i\left[\prod_{\substack{j=1\\ j\neq i}}^{n} e_j + \prod_{\substack{j=1\\ j\neq i}}^{n} c_j v_j,\right],\ A_{4n} = \sum_{i=1}^{n} c_i v_i\left[\prod_{\substack{j=1\\ j\neq i}}^{n} e_j + \prod_{\substack{j=1\\ j\neq i}}^{n} d_j u_j\right],$$

$$A_{5n} = \sum_{i_1\neq}\sum_{i_2}\left(\prod_{j=i_1,i_2} d_j u_j\right)\prod_{j'\neq(i_1,i_2)} c_{j'}v_{j'} + \sum_{i_1\neq}\sum_{i_2\neq}\sum_{i_3}\left(\prod_{j=i_1,i_2,i_3} d_j u_j\right)$$

$$\prod_{j'\neq(i_1,i_2,i_3)} c_{j'}v_{j'} + \ldots + \sum_{i_1\neq\ldots}\cdots\sum_{\neq i_{n-2}}\left(\prod_{j=i_1,\ldots,i_{n-2}} d_j u_j\right)\prod_{j'\neq j} c_{j'}v_{j'}.$$

In the expression of A_{kn}'s we replace

$$\prod_{j\epsilon\Omega_j} d_j u_j = \left(\prod_{j\epsilon\Omega_j} d_j\right) u^{\sum\limits_{j\epsilon\Omega_j}(n+1-j)}$$

and

$$\prod_{j\epsilon\Omega_j} c_j v_j = \left(\prod_{j\epsilon\Omega_j} c_j\right) v^{\sum\limits_{j\epsilon\Omega_j}(n+1-j)}$$

where $u = 1-\pi_1, v = \omega_1$. We consider the uniform prior:

$$g(\theta) = k,\ 0 < v < u < 1, \tag{3.4}$$

and employing the technique of Gibbs sampler we can estimate the posterior marginals. Taking uniform prior and $(\underset{\sim 1}{\in}, \underset{\sim 1}{\eta})$ the posterior mean, s.d. and mode of π_1 are respectively 0.6556, 0.1631, 0.6750 and those of ω_1 are respectively 0.1921, 0.1539, 0.1350.

3.2 Model 2

Here we have two pairs of random components (π_1, ω_1) and (π_n, ω_n). We consider

$$\pi_j = 1 - (1-\pi_1)^{\frac{n-j}{n-1}}(1-\pi_n)^{\frac{j-1}{n-1}} \text{ and } \omega_j = \omega_1^{\frac{n-j}{n-1}}\omega_n^{\frac{j-1}{n-1}}. \quad (3.5)$$

Then the conditional distribution of x given θ is:

$$\begin{aligned} f(x|\theta) &= \prod_{j=1}^{n}\Bigg[\left\{1-(1-\pi_1)^{\frac{n-j}{n-1}}\ (1-\pi_n)^{\frac{j-1}{n-1}}\right\}^{\in_{jn+1}\eta_{jn+1}} \\ &\quad \left\{\omega_1^{\frac{n-j}{n-1}}\omega_n^{\frac{j-1}{n-1}}\right\}^{(1-\in_{jn+1})(1-\eta_{jn+1})} \left\{(1-\pi_1)^{\frac{n-j}{n-1}}(1-\pi_n)^{\frac{j-1}{n-1}}\right. \\ &\quad \left. -\omega_1^{\frac{n-j}{n-1}}\omega_n^{\frac{j-1}{n-1}}\right\}^{\in_{jn+1}(1-\eta_{jn+1})+(1-\in_{jn+1})\eta_{jn-1}}\Bigg] \\ &= \prod_{j=1}^{n}(e_j + d_j u_j + c_j v_j), \end{aligned} \quad (3.6)$$

where

$$u_j = (1-\pi_1)^{\frac{n-j}{n-1}}(1-\pi_n)^{\frac{j-1}{n-1}} \text{ and } v_j = \omega_1^{\frac{n-j}{n-1}}\omega_n^{\frac{j-1}{n-1}},$$

and e_j, d_j, c_j are as in Model 1. Then, making a similar expression as in Model 1 we replace

$$\begin{aligned} \prod_{j\epsilon\Omega_j} d_j u_j &= \left(\prod_{j\epsilon\Omega_j} d_j\right)(1-\pi_1)^{\frac{1}{n-1}\sum_{j\epsilon\Omega_j}(n-j)}(1-\pi_n)^{\frac{1}{n-1}\sum_{j\epsilon\Omega_j}(j-1)}, \\ \prod_{j\epsilon\Omega_j} c_j v_j &= \left(\prod_{j\epsilon\Omega_j} c_j\right)\omega_1^{\frac{1}{n-1}\sum_{j\epsilon\Omega_j}(n-j)}\omega_n^{\frac{1}{n-1}\sum_{j\epsilon\Omega_j}(j-1)}. \end{aligned}$$

Considering uniform prior :

$$g(\theta) = k, \left.\begin{array}{l} 0 < \pi_1 < \pi_n < 1 \\ 0 < \omega_n < \omega_1 < 1 \\ 0 < \pi_j + \omega_j < 1,\ j = 1(1)n \end{array}\right\},$$

we have the different conditional distributions. For computation we use Gibbs sampler technique. Some computations are in the following table.

Table 2 *Posterior mean, s.d. and modes of* (π_1, π_n) *and* (ω_1, ω_n) *for* $n = 5$ *and uniform prior.*

$(\underset{\sim}{\in}, \underset{\sim}{\eta})$	π_1	π_n	ω_1	ω_n
	0.6382	0.7896	0.2104	0.0723
$(\underset{\sim 1}{\in}, \underset{\sim 1}{\eta})$	0.1266	0.1446	0.1524	0.1102
	0.6550	0.8000	0.2050	0.1000
	0.3019	0.4743	0.4976	0.3574
$(\underset{\sim 1}{\in}, \underset{\sim 2}{\eta})$	0.1326	0.1345	0.1406	0.1412
	0.3100	0.4900	0.5050	0.3350

4 Grouped data: a possible extension

Suppose, instead of one by one monitoring, subjects are assigned in gorups at every state. Let s_i samples are allocated at the i-th stage, $i = 1, 2, \ldots, n$. Then,

$$\sum_{i=1}^{n} s_i = N$$

is the total sample size. Let us define as in Section 2, the following indicator variables :

$\in^{(k)}_{in+1} = 1$ (or 0) according as the response (or non-response) of the first component corresponding to the k-th unit of the i-th stage, $k = 1(1)s_i$, $i = 1(1)n$ and $\eta^{(k)}_{in+1}$ is a similar indicator for the second component. Then $\left(\in^{(k)}_{in+1}, \eta^{(k)}_{in+1}\right)$'s are independent pairs for all k. Suppose

$$P\left(\in^{(k)}_{in+1} = 1, \eta^{(k)}_{in+1} = 1\right) = \pi_{n+1-i}, \; P\left(\in^{(k)}_{in+1} = 0, \eta^{(k)}_{in+1} = 0\right) = \omega_{n+1-i},$$

are independent of k. Then, in the i-th stage, writing

$$e_i = \sum_{k=1}^{s_i} \in^{(k)}_{in+1} \eta^{(k)}_{in+1}, \; r_i = \sum_{i=1}^{s_i} (1 - \in^{(k)}_{in+1})(1 - \eta^{(k)}_{in+1}),$$

as the total number of responses and non-responses, we have (e_i, r_i) follows trinomial $(s_i; \pi_{n+1-i}, \omega_{n+1-i})$ distribution. Our data here is obviously $x \equiv \left(\left(\in^{(k)}_{in+1}, \in^{(k)}_{in+1}\right); k = 1, \ldots, s_i, i = 1, \ldots, n\right)$. Then, with the earlier notations

$$f(x|\theta) = \prod_{j=1}^{n}\left\{\frac{s_j!}{e_j!r_j!(s_j-e_j-r_j)!}\pi_{n+1-j}^{e_j}\omega_{n+1-j}^{r_j}(1-\pi_{n+1-j}-\omega_{n+1-j})^{s_j-e_j-r_j}\right\}, \quad (4.1)$$

and

$$g(\theta) = k, \quad \left.\begin{array}{l} 0 < \pi_1 < \pi_2 < \ldots < \pi_n < 1 \\ 0 < \omega_n < \omega_{n-1} < \ldots < \omega_1 < 1 \\ 0 < \pi_j + \omega_j < 1 \ \forall \ j \end{array}\right\}, \quad (4.2)$$

we get the different conditional p.d.f.'s as

$$\begin{array}{l} p(\pi_j|\pi_i, \ldots, \pi_{j-1}, \pi_{j+1}, \ldots, \pi_n, \omega_1, \ldots, \omega_n, x) \\ = D_A^{-1}(j)\pi_j^{e_{n+1-j}}\omega_j^{r_{n+1-j}}(1-\pi_j-\omega_j)^{s_{n+1-j}-e_{n+1-j}-r_{n+1-j}}, \\ \pi_{j-1} < \pi_j < A_j, \end{array}$$

and

$$\begin{array}{l} p(\omega_j|\pi_i, \ldots, \pi_n, \omega_1, \ldots \omega_{j-1}, \omega_{j+1} \ldots, \omega_n, x) \\ = D_B^{-1}(j)\pi_j^{e_{n+1-j}}\omega_j^{r_{n+1-j}}(1-\pi_j-\omega_j)^{s_{n+1-j}-e_{n+1-j}-r_{n+1-j}}, \\ \omega_{j+1} < \omega_j < B_j, \end{array}$$

where

$$\begin{array}{rcl} D_A(j) & = & \sum_{i=0}^{\tau_j}(-1)^i \binom{\tau_j}{i} \sum_{l=0}^{i} \binom{i}{l} \omega_j^{r_{n+1-j}+i-l} \left(e_{n+1-j}+l+1\right)^{-1} \\ & & \left(A_j^{e_{n+1-j}+l+1} - \pi_{j-1}^{e_{n+1-j}+l+1}\right), \\ D_B(j) & = & \sum_{i=0}^{\tau_j}(-1)^i \binom{\tau_j}{i} \sum_{l=0}^{i} \binom{i}{l} \pi_j^{r_{n+1-j}+l} \left(r_{n+1-j}+i-l+1\right)^{-1} \\ & & \left(B_j^{r_{n+1-j}+l+1} - \omega_{j+1}^{r_{n+1-j}+i-l+1}\right), \end{array}$$

with

$$\begin{array}{rcl} \tau_j & = & s_{n+1-j} - e_{n+1-j} - r_{n+1-j}, \ \pi = 0, \pi_{n+1} = 1, \omega_{n+1} = 0, \omega_0 = 1, \\ A_j & = & \min\left(\pi_{j+1}, 1-\omega_j\right), \ B_j = \min(\omega_{j-1}, 1-\pi_j). \end{array}$$

Then, as in Section 2, a similar analysis can be performed.

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