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JACKKNIFING A GENERAL CLASS OF ESTIMATORS

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Summary
The technique of jackknife is applied to a general class of estimators. Considering a natural population, the performance of the jackknifed estimators are compaired with their unjackknifed counterparts.

Key Words: Auxiliary variable; average relative efficacies and variance estimator; bias; inclusion probability; mean square error; ratio and regression estimator.

1 Introduction

The idea of jackknife was introduced by Quenouille (1956) in connection with reduction of bias of nonlinear estimator. The possibility of using this technique for the purpose of estimation of variance or mean square error was brought into light by Tukey (1958). Durbin (1959) perhaps had been the first to use it in the context of finite population. Rao (1965) and Rao and Webster (1966) considered jackknifing classical ratio estimator. Srivastava (1967) defined a variant of the classical ratio estimator, called the Srivastava's modified ratio estimator (SMRE), that reduces in particular situation to classical ratio estimator. Rao (1991) studied SMRE, for four natural populations, along with other estimators and found no merit in considering it. However, Nandi and Aich (1994) observed that a jackknifed version of SMRE induced an improvement and hence was worth considering. Besides classical and modified ratio estimators, many other estimators, which make use of auxiliary information, are available in the literature. Some of them which are worth mentioning are generalized regression estimator due to Särndal (1980), asymptotically design unbiased

(ADU) linear estimator due to Brewer (1979) and generalized ratio estimator due to Hajék (1971). Wright (1983) and Särndal and Wright (1984) brought all these estimators under the same umbrella of a general class of estimators, called QR - class of estimators. In this article we consider a slight modification of QR - class, let it be called modified QR - class (MQR), that includes, among others, SMRE and modified forms of estimators due to Särndal (1980), Brewer (1979) and Hajék (1971). Jackknifing the estimators of MQR - class is considered to yield a jackknifed class of estimators, called the JMQR - class and it is observed, using a natural population, that the estimators in JMQR - class improve their respective counterparts in MQR - class.

The contents of this article may be divided into five parts. In Section 2 we introduced MQR - class of estimators and then in Section 3 we jackknife the MQR - class, observe the bias reduction by jackknifing and find the mean square error (MSE) of jackknifed class of estimators. In Section 4 we take up the problem of estimation of the variance of the estimators in the jackknifed class. We have explored three alternative methods for estimating the variance namely the ordinary delta method, the bootstrap resampling technique and jackknifed variance estimation method. In Section 5 we consider a natural population of Swedish municipality named MU284 in the book by Särndal, Swensson and Wretman (1992) and compare the MSE's of the estimators in the JMQR - class with the MSE's of their counterparts in the MQR - class to observe an improvement.

The same natural population is used also to compare different variance estimators to note that in the region of best performance of the jackknifed estimators, the jackknifed variance estimator seems to have an edge over its competitors, though the bootstrap method has a wider scope of application and appears to be more dependable in general.

2 Notation and modified QR - class (MQR) of estimators

Let U be the finite population on which are defined two real variables y and x taking values y_i and $x_i(> 0$, know) with totals Y and X respectively. To estimate Y a sample s of size n is taken with probability $p(s)$. The design p is assumed to admit positive inclusion probabilities π_i for unit i and π_{ij} for pair of units (i, j) of U. By $\Sigma_U, \Sigma\Sigma_U$ let us denote sums over i in U and $i, j(i < j)$ in U and by $\Sigma_s, \Sigma\Sigma_s$ those in s respectively. Let $Q_i(> 0)$, $R_i(\geq 0)$ and α be arbitrary constants. On noting that a simple linear estimator of the population total Y can be written as

$$\widehat{Y}_s = \frac{1}{n}\Sigma_s R_i y_i$$

which with $R_i = N$, reduces to $N\overline{y}$, we define MQR - class of estimators for the finite population total Y as

$$T_{MQR} = \widehat{Y}_s + B_Q\left[\left(\frac{X}{\widehat{X}_s}\right)^{\alpha} - 1\right]\widehat{X}_s$$

$$= \left(\frac{1}{n}\Sigma_s R_i y_i\right) + B_Q\left[\left(\frac{X}{\frac{1}{n}\Sigma_s R_i x_i}\right)^{\alpha} - 1\right]\left(\frac{1}{n}\Sigma_s R_i x_i\right),$$

where $B_Q = \dfrac{\Sigma_s Q_i y_i}{\Sigma_s Q_i x_i}$.

Different estimators of this class obtained for different choices of $R_i(\geq 0)$ and $Q_i(> 0)$ are given in Table 1.

Table 1

No.	R_i	Q_i	Estimator	Remark
1.	$\frac{n}{\pi_i}$	$\frac{n}{\pi_i}$	$\left[\frac{X}{\Sigma_s (x_i/\pi_i)}\right]^{\alpha} \Sigma_s\left(\frac{y_i}{\pi_i}\right)$	with $\alpha = 1$, it reduces to the estimator proposed by Hajék (1971).
2.	N	$\frac{1-\pi_i}{\pi_i}$	$N\left[\overline{y} + \overline{x}\left\{\left(\frac{\overline{X}}{\overline{x}}\right)^{\alpha} - 1\right\}\frac{\Sigma_s\left(\frac{1-\pi_i}{\pi_i}\right)y_i}{\Sigma_s\left(\frac{1-\pi_i}{\pi_i}\right)x_i}\right]$	with $\alpha = 1$, it reduces to the ADU estimator of Brewer (1979)
3.	$\frac{n}{\pi_i}$	$w_i x_i$ $(w_i \geq 0)$	$\Sigma_s \frac{y_i}{\pi_i} + \Sigma_s \frac{x_i}{\pi_i}\left[\left(\frac{X}{\Sigma_s (x_i/\pi_i)}\right)^{\alpha} - 1\right]B$ where $B = \frac{\Sigma_s w_i x_i y_i}{\Sigma_s w_i x_i^2}$	with $\alpha = 1$, it reduces to generalized regression (GREG) estimator due to Särndal (1980).

An alternative choice for Q_i is often taken as $\dfrac{1-\pi_i}{\pi_i x_i}$ for the estimator in serial number 2 and for this choice it reduces to :

$$N\left[\overline{y} + \overline{x}\left\{\left(\frac{\overline{X}}{\overline{x}}\right)^{\alpha} - 1\right\}\frac{\Sigma_s\left(\frac{1-\pi_i}{\pi_i}\right)\frac{y_i}{x_i}}{\Sigma_s\left(\frac{1-\pi_i}{\pi_i}\right)}\right].$$

For the estimator in serial number 3, the natural choices for w_i are $1/\pi_i$, $1/x_i^g (0 < g < 2)$ etc.

All the estimators under the class T_{MQR} are asymptotically design unbiased (ADU) and by Taylor's expansion one can get the asymptotic MSE or approximate variance of T_{MQR} as

$$V_{MQR} = \Sigma\Sigma_U \Delta_{ij} \left(\frac{A_i y_i - B_i x_i}{\pi_i} - \frac{A_j y_j - B_j x_j}{\pi_j} \right)^2,$$

where

$$\Delta_{ij} = \pi_i \pi_j - \pi_{ij},$$

$$A_i = \frac{\pi_i R_i}{n} + \frac{\pi_i Q_i}{n} \frac{\Sigma_U \pi_i R_i x_i}{\Sigma_U \pi_i Q_i x_i} \left[\left(\frac{nX}{\Sigma_U \pi_i R_i x_i} \right)^\alpha - 1 \right]$$

and

$$B_i = A_i \frac{\Sigma_U \pi_i Q_i y_i}{\Sigma_U \pi_i Q_i x_i} - (1-\alpha) \frac{\pi_i R_i}{n} \left(\frac{nX}{\Sigma_U \pi_i R_i x_i} \right)^\alpha \frac{\Sigma_U \pi_i Q_i y_i}{\Sigma_U \pi_i Q_i x_i}.$$

For $R_i = \dfrac{n}{\pi_i}$, $Q_i = w_i x_i$ and $\alpha = 1, T_{MQR}$ reduces to GREG and for such R_i, Q_i and α

$$V_{MQR} = \Sigma\Sigma_U \Delta_{ij} \left(\frac{y_i - \beta_w x_i}{\pi_i} - \frac{y_j - \beta_w x_j}{\pi_j} \right)^2,$$

where

$$\beta_w = \frac{\Sigma_U \pi_i w_i x_i y_i}{\Sigma_U \pi_i w_i x_i^2}$$

This variance expression for GREG is same as derived by Särndal (1982).

3 Jackknifing the MQR - class of estimators

The estimator T_{MQR}, introduced in the earlier section is nonlinear and hence, as pointed out at the beginning of this article, the technique of jackknife was resorted to with the objective of reducing its bias and estimating its variance.

To jackknife the estimator $T_{MQR} = T_n$ (say), we denote by $T_{n-1}^{(-i)}$ the value of T_n based on a sample of size $(n-1)$, when the ith pair (x_i, y_i) is deleted from the sample. Next we note from the expression of bias of T_{MQR}, given in the appendix, that at least for our choices of R_i and Q_i

and for a number of commonly used sampling designs, it is of the order of $1/n$. This motivates us to choose the pseudo values as

$$T_i^* = nT_n - (n-1)T_{n-1}^{(-i)}$$

and propose

$$T_{MQR}(jk) = \frac{1}{n}\Sigma_s T_i^*$$

as the jackknifed estimator of the population total.

To further justify the choice of T_i^*, we may have a quick look at the generalized jackknife (GJK) technique due to Schucany et. al. (1971). As pointed out by Miller (1974) the GJK - technique was developed to handle more general form of bias. If T and T^* are two estimators of a parameter $\theta(y)$, then for any real $R \neq 1$, the GJK - estimator is defined by

$$G(T, T^*) = \frac{T - RT^*}{1 - R},$$

where $R = R(n)$ is a function of the sample size n.

If T and T^* are consistent estimators of $\theta(y)$, then so is $G(T, T^*)$; provided $\lim_{n\to\infty} R(n)$ exists and $\neq 1$. If the biases of T and T^* factorize in the following manner

$$\begin{aligned} E_p(T) &= \theta(y) + f(n)b(\theta) \\ E_p(T^*) &= \theta(y) + f^*(n)b(\theta), \end{aligned}$$

then

$$E_p\left[G(T, T^*)\right] = \theta(y) \Rightarrow R = \frac{f(n)}{f^*(n)} = \frac{Bias(T)}{Bias(T^*)}.$$

Thus the above choice of R, renders $G(T, T^*)$ exactly unbiased. The usual jackknife estimator fits into $G(T, T^*)$ with $T = T_n, T^* = \frac{1}{n}\Sigma_s T_{n-1}^{(-i)}$. Since, as pointed out earlier, bias of T_n is of order $1/n$, the choice of R in our case is given by the bias ratio of T_n and $\frac{1}{n}\Sigma_s T_{n-1}^{(-i)}$ which is precisely $1 - \frac{1}{n}$. Although for the present choice of $R(n)$, $\lim_{n\to\infty} R(n) = 1$, there is nothing much to worry about if one notes that T_{MQR}, as pointed out earlier, is asymptotically design unbiased (ADU) and hence need not be jackknifed at all when n is very large. Thus for finite n, with $T = T_n$, $T^* = \frac{1}{n}\Sigma_s T_{n-1}^{(-i)}$ and $R = 1 - \frac{1}{n}$,

$$G\left(T, T^*\right) = \frac{1}{n}\Sigma_s T_i^*$$

is our proposed jackknife estimator. For details of generalized jackknife technique, the reader is referred to Gray and Schucany (1972) and Wolter (1985).

Assuming

$$\left|\frac{R_i x_i}{\Sigma_s R_i x_i}\right| < 1, \ \left|\frac{R_i y_i}{\Sigma_s R_i y_i}\right| < 1, \ \left|\frac{Q_i x_i}{\Sigma_s Q_i x_i}\right| < 1 \text{ and } \left|\frac{Q_i y_i}{\Sigma_s Q_i y_i}\right| < 1,$$

we get

$$\begin{aligned} T_{MQR}(jk) &= \frac{1}{n}\Sigma_s R_i y_i + B_Q\left[\left(\frac{X}{\frac{1}{n}\Sigma_s R_i x_i}\right)^{\alpha} n_\alpha - 1\right]\frac{1}{n}\Sigma_s R_i x_i \\ &= \widehat{Y}_s + B_Q\left[\left(\frac{X}{\widehat{X}_s}\right)^{\alpha} n_\alpha - 1\right]\widehat{X}_s \end{aligned}$$

where $n_\alpha = n - \left(1 - \frac{1}{n}\right)^{\alpha}(n + \alpha - 1)$.

Using linearization technique, MSE of $T_{MQR}(jk)$ may be derived and can be put in the form

$$V_{MQR}(jk) = \Sigma\Sigma_U \Delta_{ij}\left(\frac{g_i y_i - f_i x_i}{\pi_i} - \frac{g_j y_j - f_j x_j}{\pi_j}\right)^2,$$

where

$$g_i = \frac{\pi_i R_i}{n} + \frac{\pi_i Q_i}{n}\frac{\Sigma_U \pi_i R_i x_i}{\Sigma_U \pi_i Q_i x_i}\left[n_\alpha\left(\frac{nX}{\Sigma_U \pi_i R_i x_i}\right)^{\alpha} - 1\right]$$

and

$$f_i = g_i - n_\alpha(1-\alpha)\frac{\pi_i R_i}{n}\left(\frac{nX}{\Sigma_U \pi_i R_i x_i}\right)^{\alpha}\frac{\Sigma_U \pi_i Q_i y_i}{\Sigma_U \pi_i Q_i x_i}.$$

Let us now enumerate different estimators of the jackknifed class for different choices of $R_i(\geq 0)$ and $Q_i(> 0)$ as observed in Table 2.

Table 2

No.	R_i	Q_i	Jackknifed Estimator	Remark
1.	$\frac{n}{\pi_i}$	$\frac{n}{\pi_i}$	$n_\alpha \left[\frac{X}{\Sigma_s\,(x_i/\pi_i)}\right]^\alpha \Sigma_s \left(\frac{y_i}{\pi_i}\right)$	This was derived by Nandi and Aich (1994) in equiprobability situation.
2a.	N	$\frac{1-\pi_i}{\pi_i}$	$N\left[\overline{y}+\overline{x}\left\{\left(\frac{\overline{X}}{\overline{x}}\right)^\alpha n_\alpha - 1\right\}\frac{\Sigma_s\left(\frac{1-\pi_i}{\pi_i}\right)y_i}{\Sigma_s\left(\frac{1-\pi_i}{\pi_i}\right)x_i}\right]$	
2b.	N	$\frac{1-\pi_i}{\pi_i x_i}$	$N\left[\overline{y}+\overline{x}\left\{\left(\frac{\overline{X}}{\overline{x}}\right)^\alpha n_\alpha - 1\right\}\frac{\Sigma_s\left(\frac{1-\pi_i}{\pi_i}\right)\frac{y_i}{x_i}}{\Sigma_s\left(\frac{1-\pi_i}{\pi_i}\right)}\right]$	
3.	$\frac{n}{\pi_i}$	$w_i x_i$ $(w_i \geq 0)$	$\Sigma_s \frac{y_i}{\pi_i} + \Sigma_s \frac{x_i}{\pi_i}\left[\left(\frac{X}{\Sigma_s\,(x_i/\pi_i)}\right)^\alpha n_\alpha - 1\right] B$ where $B = \frac{\Sigma_s w_i x_i y_i}{\Sigma_s w_i x_i^2}$	

In Section 5, we demonstrate by computer simulation using real data that the MSE's of the estimators of the JMQR - class are much less than those of the estimators of MQR - class specially when the sample is small.

4 Variance estimation

For variance estimation we have adopted three alternative techniques. The first one is delta method or Taylor's series method in which one can estimate V_{MQR} by

$$V_T = \Sigma\Sigma_s \frac{\Delta_{ij}}{\pi_{ij}}\left(\frac{a_i y_i - b_i x_i}{\pi_i} - \frac{a_j y_j - b_j x_j}{\pi_j}\right)^2,$$

where a_i, and b_i are obtained from A_i and B_i (see Section 2) after replacing the population sums Σ_U by their corresponding Horvitz - Thompson's (1952)(HT) estimators.

Secondly, we estimate V_{MQR} by jackknife variance estimator given by

$$V_J = \frac{1}{n(n-1)}\Sigma_s \left[T_i^* - T_{MQR}(jk)\right]^2.$$

Lastly to estimate V_{MQR} by using bootstrap resampling technique, we take a SRSWOR sample of size n, where $n = N/k, k$ being an integer, take k copies of these sampled units to generate an artificial population U^* of size N. From this population, generated artificially, we draw samples, called the bootstrap samples, of size n, each by Midzuno-Sen's scheme of sampling (see Midzuno, 1952). Denoting by $T_{MQR}(r)$, the estimator based on the rth bootstrap sample, the bootstrap variance estimator is calculated as

$$V_B = \frac{1}{(M-1)} \sum_{r=1}^{M} \left[T_{MQR}(r) - \frac{1}{M} \sum_{r=1}^{M} T_{MQR}(r) \right]^2 .$$

We compare performance of these three variance estimators in the following section using a natural population.

5 A simulation study

Let us now consider a natural population of Swedish municipality, named MU284 in the book by Särndal, Swensson and Wretman (SSW) (1992). Sweden is divided into 284 municipalities having considerable variation in size and other characteristics. The data on a few variables selected by SSW include P85 and P75, the population (in thousands) of 1985 and 1975 respectively for all the 284 municipalities shown separately. We take P85 as the study variable y and P75 as the auxiliary variable x. From the finite population of size $N = 284$ we select a sample of size n following Midzuno - Sen scheme of sampling and observe the Average Relative Efficacies (ARE) of $T_{MQR}(jk)$ with respect to T_{MQR} for different choices of $R_i(\geq 0)$, $Q_i(> 0)$ and α. Let us define

$$ARE = \frac{V_{MQR}}{V_{MQR}(jk)} \times 100$$

and observe the improvement due to jackknifing in the following :

1. Putting $R_i = Q_i = \dfrac{n}{\pi_i}, T_{MQR}$ reduces to Srivastava's modified ratio estimator t_{SMRE}, and its jackknifed version is $t_{SMRE}(jk) = n_\alpha t_{SMRE}$. Now we construct the following table to demonstrate that $t_{SMRE}(jk)$ is more efficient.

Table 3 *ARE table with* $R_i = Q_i = \dfrac{n}{\pi_i}$

n / α	5	10	20	30
0.1	101.93	100.93	100.45	100.30
0.2	103.42	101.65	100.81	100.54
0.3	104.46	102.16	101.06	100.71
0.4	105.04	102.46	101.21	100.81
0.5	105.18	102.55	101.26	100.84
0.6	104.89	102.43	101.21	100.80
0.7	104.19	102.10	101.05	100.70
0.8	103.12	101.59	100.80	100.53
0.9	101.72	100.88	100.45	100.30

Observation: Thus for small sample size t_{SMRE} is best when α is around 0.5. This was also observed by Nandi and Aich (1994) in equiprobability situation for a different finite population.

2. (a) With $R_i = N$ and $Q_i = \left(\frac{1-\pi_i}{\pi_i}\right)$ we get Brewer's estimator as follows :

$$t_B = N\left[\overline{y} + \overline{x}\left\{\left(\frac{\overline{X}}{\overline{x}}\right)^{\alpha} - 1\right\}\frac{\Sigma_s\left(\frac{1-\pi_i}{\pi_i}\right)y_i}{\Sigma_s\left(\frac{1-\pi_i}{\pi_i}\right)x_i}\right]$$

and its jackknifed version is

$$t_B(jk) = N\left[\overline{y} + \overline{x}\left\{\left(\frac{\overline{X}}{\overline{x}}\right)^{\alpha} n_{\alpha} - 1\right\}\frac{\Sigma_s\left(\frac{1-\pi_i}{\pi_i}\right)y_i}{\Sigma_s\left(\frac{1-\pi_i}{\pi_i}\right)x_i}\right].$$

Now we present below the ARE of $t_B(jk)$ with respect to t_B.

Table 4 *ARE table with $R_i = N$ and $Q_i = \left(\frac{1-\pi_i}{\pi_i}\right)$*

n / α	5	10	20	30
0.1	107.68	104.50	102.42	101.60
0.2	107.84	104.52	102.45	101.62
0.3	107.80	104.44	102.35	101.55
0.4	107.51	104.24	102.20	101.45
0.5	106.97	103.92	101.99	101.31
0.6	106.15	103.46	101.72	101.13
0.7	105.05	102.86	101.39	100.92
0.8	103.67	102.80	100.99	100.66
0.9	102.00	101.81	100.53	100.35

Observation : Thus for small sample size, $t_B(jk)$ is best when α is around 0.2.

(b). Putting $R_i = N$ and $Q_i = \left(\frac{1-\pi_i}{\pi_i x_i}\right)$, we get another type of Brewer's estimator as

$$t_B^* = N\left[\overline{y} + \overline{x}\left\{\left(\frac{\overline{X}}{\overline{x}}\right)^{\alpha} - 1\right\}\frac{\Sigma_s\left(\frac{1-\pi_i}{\pi_i}\right)\frac{y_i}{x_i}}{\Sigma_s\left(\frac{1-\pi_i}{\pi_i}\right)}\right]$$

and its jackknifed version is

$$t_B^*(jk) = N\left[\overline{y} + \overline{x}\left\{\left(\frac{\overline{X}}{\overline{x}}\right)^{\alpha} n_\alpha - 1\right\}\frac{\Sigma_s\left(\frac{1-\pi_i}{\pi_i}\right)\frac{y_i}{x_i}}{\Sigma_s\left(\frac{1-\pi_i}{\pi_i}\right)}\right].$$

Now we present below the ARE of $t_B^*(jk)$ with respect to t_B^*.

Table 5 *ARE table with* $R_i = N$ *and* $Q_i = \left(\frac{1-\pi_i}{\pi_i x_i}\right)$

n / α	5	10	20	30
0.1	106.51	103.42	101.77	101.47
0.2	106.86	103.49	101.68	101.12
0.3	106.85	103.48	101.65	101.05
0.4	106.69	103.37	101.61	101.02
0.5	106.26	103.15	101.52	100.98
0.6	105.56	102.80	101.37	100.89
0.7	104.58	102.32	101.14	100.75
0.8	103.32	101.69	100.85	100.56
0.9	101.79	100.92	100.47	100.31

Observation : As in case of 2(a).

3. Putting $R_i = \dfrac{n}{\pi_i}$ and $Q_i = w_i x_i$, we get the modified generalized regression estimator as,

$$t_{MR} = \Sigma_s \frac{y_i}{\pi_i} + \Sigma_s \frac{x_i}{\pi_i}\left[\left(\frac{X}{\Sigma_s\,(x_i/\pi_i)}\right)^{\alpha} - 1\right]\frac{\Sigma_s w_i x_i y_i}{\Sigma_s w_i x_i^2}$$

and its jackknifed version is

$$t_{MR}(jk) = \Sigma_s \frac{y_i}{\pi_i} + \Sigma_s \frac{x_i}{\pi_i}\left[\left(\frac{X}{\Sigma_s\,(x_i/\pi_i)}\right)^{\alpha} n_\alpha - 1\right]\frac{\Sigma_s w_i x_i y_i}{\Sigma_s w_i x_i^2}.$$

Now in particular, here we choose $w_i = 1/x_i^g,\ 0 < g < 2$.

We present below the ARE of $t_{MR}(jk)$ with respect to t_{MR}.

Table 6 *ARE table with* $R_i = \dfrac{n}{\pi_i}$ *and* $Q_i = x_i^{1-g}, g = 1.6$

n α	5	10	20	30
0.1	101.37	100.66	100.32	100.21
0.2	102.37	101.17	100.56	100.38
0.3	103.08	101.53	100.74	100.50
0.4	103.48	101.74	100.84	100.57
0.5	103.58	101.80	100.87	100.59
0.6	103.38	101.71	100.83	100.57
0.7	102.90	101.49	100.72	100.50
0.8	102.17	101.12	100.55	100.38
0.9	101.19	100.62	100.31	100.21

Observation : So we may say that for small sample size, $t_{MR}(jk)$ is best when $\alpha = 0.5$.

Variance Estimation : For comparison of the three variance estimators, we take from the same population MU284 a sample of size $n = 71$ by Midzuno - Sen scheme of sampling, compute the variance estimators for jackknifed modified generalized regression estimator (MGREG) by

(i) delta method or Taylor's series method (V_T),

(ii) jackknife method (V_J) and

(iii) bootstrap method (V_B).

We compare three variance estimators with the actual variance V_{AC} in Table 7.

Table 7 *Variance estimators for* $T_{MQR}(jk)$ *with* $R_i = \dfrac{n}{\pi_i}$ *and* $Q_i = x_i^{1-g},\ g = 1.3.$

α	V_{AC}	V_T	V_J	V_B
0.1	17.695	19.700	20.255	17.964
0.2	13.802	15.316	15.378	13.856
0.3	10.395	11.491	11.245	10.319
0.4	7.474	8.220	7.815	7.331
0.5	5.038	5.501	5.050	4.877
0.6	3.088	3.331	2.921	2.942
0.7	1.623	1.709	1.402	1.516
0.8	0.645	0.632	0.473	0.592
0.9	0.152	0.103	0.115	0.164

Remark :

1. It was seen in Table 6 that performance of $T_{MQR}(jk)$ with present choice of R_i and Q_i was best when α was around 0.5 and in this region of α, V_J is closest to the actual variance V_{AC}. However, considering overall performance, irrespective of the value of α, V_B appears to be better than the other two.

2. While the performance of jackknife estimator is very good in a special case (i.e. for specific value of α), the behaviour of the bootstrap variance estimator is α - independent. This observation supplements to a comment of Efron (1979) that bootstrap methods are more widely applicable than jackknife and also more dependable.

Appendix

Under the same assumption as in Section 3, it can be easily shown that, the bias of MQR-class of estimators can be written as

$$B(T_{MQR}) = \frac{1}{n}\left[\Sigma_U(R_i\pi_i - n)y_i + \frac{T_{Rx}.T_{Qy}}{T_{Qx}}\left\{\left(\frac{nX}{T_{Rx}}\right)^{\alpha}\left(1 + \frac{(1-\alpha)S_{Rx,Qy}}{T_{Rx}.T_{Qy}} - \frac{(1-\alpha)S_{Rx,Qx}}{T_{Rx}.T_{Qx}} - \frac{S_{Qx,Qy}}{T_{Qx}.T_{Qy}} + \frac{S^2_{Qx}}{T^2_{Qx}} + \frac{\alpha(\alpha-1)}{2}\frac{S^2_{Rx}}{T^2_{Rx}}\right) - \left(1 + \frac{S_{Rx,Qy}}{T_{Rx}.T_{Qy}} - \frac{S_{Qx,Qy}}{T_{Qx}.T_{Qy}} - \frac{S_{Rx,Qx}}{T_{Rx}.T_{Qx}} + \frac{S^2_{Qx}}{T^2_{Qx}}\right)\right\}\right],$$

where

$$T_{ab} = \Sigma_U \pi_i a_i b_i$$

and

$$S_{ab} = \Sigma\Sigma_U \delta_{ij} a_i b_i$$

$$\delta_{ij} = \begin{cases} \left(\dfrac{\pi_{ij}}{\pi_i\pi_j} - 1\right) & \text{if } i \neq j \\ \left(\dfrac{1}{\pi_i} - 1\right) & \text{if } i = j \end{cases}$$

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