Global Stabilization for Feedforward Nonlinear System Based on Nested Saturated Control

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Abstract In this paper, we discuss global stabilization procedure to make some classes of feedforward nonlinear systems be convergent. Our stabilizer consists of a nested saturation function, which is a nonlinear combination of saturation functions. We extend the existent stabilization results and give exponential convergence of the stabilizer.

Key words Feedforward nonlinear system, nested saturated control, global stabilization, Exponential convergence.

The problem of stabilizing feedforward system has been studied extensively[1][2][3][4]. In 1992, Teel[5][6] presented some bounded control algorithms for feedforward linear system

$$\begin{cases} \dot{x}_1 = x_2, \\ \quad\vdots \\ \dot{x}_{n-1} = x_n, \\ \dot{x}_n = u, \end{cases}$$

where the control u is the following nested saturation functions [6]:

$$u = -\sigma_n(x_n + \sigma_{n-1}(x_{n-1} + \cdots + \sigma_1(x_1)\cdots)).$$

Since Teel's work, there appeared many results on global stabilization of feedforward systems, such as Jankovic, Sepulchre, and Kokotovic[1][7], Mazenc and Praly[8], Teel[9], Liu[10], Zhong[11] etc. In [12], the author has studied the following feedforward nonlinear system

$$\begin{cases} \dot{x}_1 = x_2 + \varphi_1(x_3, \cdots, x_n, u), \\ \dot{x}_2 = x_3 + \varphi_2(x_4, \cdots, x_n, u), \\ \quad\vdots \\ \dot{x}_{n-1} = x_n + \varphi_{n-1}(u), \\ \dot{x}_n = u, \end{cases}$$

where φ_i is vanishing and locally Lipschitz continuous at zero for $i = 1, \cdots, n-1$. In this paper, we study on more general class of feedforward nonlinear systems

$$\begin{cases} \dot{x}_1 = f_1(x_2, x_3, \cdots, x_n, u), \\ \dot{x}_2 = f_2(x_3, x_4, \cdots, x_n, u), \\ \quad\vdots \\ \dot{x}_{n-1} = f_{n-1}(x_n, u), \\ \dot{x}_n = f_n(u), \end{cases} \tag{1}$$

where $x = [x_1, \cdots, x_n]^{\mathrm{T}} \in \mathbf{R}^n$ is the state, f_i is continuously differential for its variable, vanishing at zero.

We'll use nested saturation stabilizer to make above nonlinear system globally stable at the equilibrium $x = 0$. The stabilizer we will use is basically similar to that proposed in [6]. The saturation levels are determined by properties of f_i.

The rest of this paper is organized as follows. In Section 2, we give our main result about global stability and exponential convergence of nonlinear system (1). An extended results is given in Section 3. In Section 4, we give a simulation example. The paper is concluded in Section 5.

1 Global stability and exponential convergence of saturated control

In this section,we always suppose that (1) satisfy

$$\left.\frac{\partial f_i}{\partial x_{i+1}}\right|_0 = c_i \neq 0 (i = 1, \cdots, n-1),\ \left.\frac{\partial f_n}{\partial u}\right|_0 = c_n \neq 0.$$

Then we have

$$f_i(x_{i+1}, \cdots, x_n, u) = c_i x_{i+1} + \varphi_i(x_{i+1}, \cdots, x_n, u), \tag{2}$$

$(i = 1, \cdots, n-1)$ and

$$f_n(u) = c_n u + \varphi_n(u), \tag{3}$$

At the same time, there exist a_i and r_i such that

$$|\varphi_i(x_{i+1}, \cdots, x_n, u)| \leq r_i(|x_{i+1}|^2 + |x_{i+2}| + \cdots + |u|) \tag{4}$$

$(i = 1, \cdots, n-1)$ and

$$|\varphi_n(u)| \leq r_n |u|^2 \tag{5}$$

whenever $\max\{|x_{i+1}|, |x_{i+2}|, \cdots, |x_n|, |u|\} \leq a_i$.

Using (2)(3), system (1) can be written as

$$\begin{cases} \dot{x}_1 = c_1 x_2 + \varphi_1(x_2, x_3, \cdots, x_n, u), \\ \dot{x}_2 = c_2 x_3 + \varphi_2(x_3, x_4, \cdots, x_n, u), \\ \quad\vdots \\ \dot{x}_{n-1} = c_{n-1} x_n + \varphi_{n-1}(x_n, u), \\ \dot{x}_n = c_n u + \varphi_n(u). \end{cases} \tag{6}$$

Next,we will use the linear coordinate transformation

$$y_{n-i} = \sum_{j=0}^{i} \frac{C_i^j \varepsilon^{\frac{j}{n}}}{c_{n-j} \cdots c_n} x_{n-j}, i = 1, \cdots, n,$$

where $C_i^j = \frac{i!}{j!(i-j)!}$ and $\varepsilon \leq 1$ is a scaling factor. Noting that $\varepsilon^{\frac{i}{n}} x_{n-i}$ depends only on $y_{n-i}, \cdots, y_n$, it can be concluded that there exist positive numbers $K_{n-i} \geq 1$ (independent of ε) such that

$$\varepsilon^{\frac{i}{n}} |x_{n-i}| \leq K_{n-i}(|y_{n-i}| + \cdots + |y_n|), i = 0, \cdots, n-1. \tag{7}$$

Since

$$\begin{aligned} y_{n-i+1} + \cdots + y_n &= \sum_{j=0}^{i-1} \frac{\sum_{k=j}^{i-1} C_k^j}{c_{n-j} \cdots c_n} \varepsilon^{\frac{j}{n}} x_{n-j} \\ &= \sum_{j=0}^{i-1} \frac{C_i^{j+1}}{c_{n-j} \cdots c_n} \varepsilon^{\frac{j}{n}} x_{n-j} \\ &= \sum_{j=1}^{i} \frac{C_i^j}{c_{n-j+1} \cdots c_n} \varepsilon^{\frac{j-1}{n}} x_{n-j+1}, \end{aligned}$$

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we have

$$\begin{aligned}\dot{y}_{n-i} &= \sum_{j=0}^{i} \frac{C_i^j \varepsilon^{\frac{j}{n}}}{c_{n-j}\cdots c_n}\dot{x}_{n-j} \\ &= \varepsilon^{\frac{1}{n}}(y_{n-i+1}+\cdots+y_n)+u+\Phi_{n-i},\end{aligned}$$

where

$$\begin{aligned}\Phi_{n-i} = \frac{C_i^0}{c_n}\varphi_n(u) + \frac{C_i^1}{c_{n-1}c_n}\varepsilon^{\frac{1}{n}}\varphi_{n-1}(x_n,u)+\cdots \\ +\frac{C_i^i}{c_{n-i}\cdots c_n}\varepsilon^{\frac{i}{n}}\varphi_{n-i}(x_{n-i+1},x_{n-i+2},\cdots,x_n,u).\end{aligned} \tag{8}$$

Define the nested saturation control

$$u = -\varepsilon^{\frac{1}{n}}\sigma_n(y_n+\sigma_{n-1}(y_{n-1}+\cdots+\sigma_1(y_1))), \tag{9}$$

where

$$\sigma_i(x) = \begin{cases} x, & \text{if } |x|\le\varepsilon_i, \\ \varepsilon_i \mathrm{sign}(x), & \text{if } |x|>\varepsilon_i,\end{cases} \tag{10}$$

and

$$\varepsilon_1 = \frac{1}{4}\varepsilon_2 = \cdots = \left(\frac{1}{4}\right)^{n-1}\varepsilon_n, \quad \varepsilon_n=\varepsilon.$$

By using (9),the system is transformed to the following closed loop system

$$\begin{cases} \dot{y}_1 &= \varepsilon^{\frac{1}{n}}[y_2+\cdots+y_n-\sigma_n(y_n+\sigma_{n-1} \\ & \quad y_{n-1}(+\cdots+\sigma_1(y_1)))]+\Phi_1, \\ \vdots & \\ \dot{y}_{n-1} &= \varepsilon^{\frac{1}{n}}[y_n-\sigma_n(y_n+\sigma_{n-1}(y_{n-1} \\ & \quad +\cdots+\sigma_1(y_1)))]+\Phi_{n-1}, \\ \dot{y}_n &= -\varepsilon^{\frac{1}{n}}\sigma_n(y_n+\sigma_{n-1}(y_{n-1} \\ & \quad +\cdots+\sigma_1(y_1)))+\Phi_n. \end{cases} \tag{11}$$

where$\Phi_1,\cdots,\Phi_n$ are defined by (8).

In the following, we will try to choose one $\varepsilon \le 1$ to see that the system (11) is globally stable and exponential convergent.

Lemma 1. For $i=0,\cdots,n-1$, assume that

$$\max\{|x_{n-i+1}|,\cdots,|x_n|,|u|\}\le a_{n-i},$$

and for $j=0,\cdots,i-1$

$$y_{n-j}\in P_{n-j}=\{y_{n-j}:|y_{n-j}|<\frac{3}{4}\varepsilon_{n-j}\}, \tag{12}$$

then there exist constants $\tilde{A}_i$ independent of ε such that

$$|\Phi_{n-i}|\le \tilde{A}_i\varepsilon^{1+\frac{2}{n}}.$$

Proof. First, we bound $|\Phi_{n-i}|$. In fact, by (4), (5) and (8), we can get

$$\begin{aligned}|\Phi_{n-i}| &\le \left|\frac{C_i^0}{c_n}\varphi_n(u)\right| + \left|\frac{C_i^1\varepsilon^{\frac{1}{n}}}{c_{n-1}c_n}\varphi_{n-1}(x_n,u)\right|+\cdots \\ &\quad +\left|\frac{C_i^i\varepsilon^{\frac{i}{n}}}{c_{n-i+1}\cdots c_n}\varphi_{n-i}(x_{n-i+1},\cdots,x_n,u)\right| \\ &\le \frac{C_i^0}{c_n}r_n|u|^2+\frac{C_i^1\varepsilon^{\frac{1}{n}}r_{n-1}}{c_{n-1}c_n}(|x_n|^2+|u|)+\cdots \\ &\quad +\frac{C_i^i\varepsilon^{\frac{i}{n}}r_{n-i}}{c_{n-i+1}\cdots c_n}(|x_{n-i+1}|^2+\cdots+|x_n|+|u|) \\ &= N_n^{(i)}|u|^2+N_{n-1}^{(i)}\varepsilon^{\frac{1}{n}}(|x_n|^2+|u|)+\cdots \\ &\quad +N_{n-i}^{(i)}\varepsilon^{\frac{i}{n}}(|x_{n-i+1}|^2+\cdots+|x_n|+|u|),\end{aligned}$$

where $N_{n-j}^{(i)}=\frac{C_i^j r_{n-j}}{c_{n-j+1}\cdots c_n}$. Obviously, the constants $N_{n-i}^{(i)}$, a_{n-i} depend only on the functions f_i (independent of ε).

Moreover, by (7) and (12), we have

$$\begin{aligned}|x_{n-j}| &\le \varepsilon^{-\frac{j}{n}}K_{n-j}(|y_n|+\cdots+|y_{n-j}|) \\ &\le \varepsilon^{-\frac{j}{n}}K_{n-j}\left(\tfrac{3}{4}\varepsilon+\cdots+\tfrac{3}{4}\tfrac{1}{4^j}\varepsilon\right)\le\varepsilon^{\frac{n-j}{n}}K_{n-j}.\end{aligned}$$

Thus

$$\begin{aligned}|\Phi_{n-i}| &\le \varepsilon^{1+\frac{2}{n}}[N_n^{(i)}+N_{n-1}^{(i)}(K_n^2+1)+\cdots \\ &\quad +N_{n-i}^{(i)}(K_{n-i+1}^2+\cdots+K_n+1)] \\ &= \varepsilon^{1+\frac{2}{n}}\tilde{A}_i,\end{aligned}$$

where $\tilde{A}_i=N_n^{(i)}+\cdots+N_{n-i}^{(i)}(K_{n-i+1}^2+\cdots+K_n+1)$. □

Lemma 2. If the scaling factor ε satisfies

$$\varepsilon\le\min\left\{1,\delta_n,\delta_{n-1}\cdots,\delta_1,\left(\frac{\min a_i}{\max K_i}\right)^n\right\} \tag{13}$$

where

$$\delta_i=\left(\frac{1}{4^{i+1}\tilde{A}_i}\right)^n, i=0,\cdots,n-1,$$

then in finite time, $y_{n-i}\in P_{n-i}=\{y_{n-i}:|y_{n-i}|<\frac{3}{4}\varepsilon_{n-i}\}$, where $i=0,\cdots,n-1$.

Proof. First, consider the evolution of the state y_n

$$\frac{\mathrm{d}y_n}{\mathrm{d}t}=-\varepsilon^{\frac{1}{n}}\sigma_n(y_n+\sigma_{n-1}(y_{n-1}+\cdots+\sigma_1(y_1)))+\Phi_n. \tag{14}$$

If $y_n\bar{\in}P_n=\{y_n:|y_n|<\frac{3}{4}\varepsilon_n\}$, then $|y_n|\ge\frac{3}{4}\varepsilon_n$, and

$$|y_n+\sigma_{n-1}(y_{n-1}+\cdots)|\ge|y_n|-\varepsilon_{n-1}\ge\frac{1}{2}\varepsilon_n.$$

Let ε be small enough such that (13) holds, then $|u|\le\varepsilon^{1+\frac{1}{n}}\le a_n$. By Lemma 1, there holds $|\Phi_n(u)|\le\tilde{A}_0\varepsilon^{1+\frac{2}{n}}$. Now it can be seen from (14) that if $y_n\bar{\in}P_n$, then

$$\begin{aligned}\left|\tfrac{\mathrm{d}y_n}{\mathrm{d}t}\right| &\ge \varepsilon^{\frac{1}{n}}|\sigma_n(y_n+\sigma_{n-1}(y_{n-1}+\cdots))|-|\Phi_n| \\ &\ge \tfrac{1}{2}\varepsilon^{1+\frac{1}{n}}-\tilde{A}_0\varepsilon^{1+\frac{1}{n}}\delta_n^{\frac{1}{n}}=\tfrac{1}{4}\varepsilon^{1+\frac{1}{n}},\end{aligned}$$

and

$$\mathrm{sign}\left(\frac{\mathrm{d}y_n}{\mathrm{d}t}\right)=-\mathrm{sign}(y_n).$$

Therefore, when $|y_n|\ge\frac{3}{4}\varepsilon_n$, $|y_n|$ decreases at the rate $\frac{1}{4}\varepsilon^{1+\frac{1}{n}}$. Consequently, y_n enters P_n in finite time and remains in P_n thereafter.

We complete the proof by induction. Suppose $y_{n-j}\in P_{n-j}=\{y_{n-j}:|y_{n-j}|<\frac{3}{4}\varepsilon_{n-j}\}$, $j=0,\cdots,i-1$, consider the evolution of the state y_{n-i}. The argument of $\sigma_{n-j}, j=0,\cdots,i-1$, is bounded as

$$|y_{n-j}+\sigma_{n-j-1}(y_{n-j-1}+\cdots)|\le\frac{3}{4}\varepsilon_{n-j}+\varepsilon_{n-j-1}=\varepsilon_{n-j}.$$

By the definition of σ_i (see (10)), the evolution of y_{n-i} is given by

$$\frac{\mathrm{d}y_{n-i}}{\mathrm{d}t}=-\varepsilon^{\frac{1}{n}}\sigma_{n-i}(y_{n-i}+\cdots+\sigma_1(y_1))+\Phi_{n-i}.$$

Similarly, if $y_{n-i}\bar{\in}P_{n-i}$, then $|y_{n-i}|\ge\frac{3}{4}\varepsilon_{n-i}$, and

$$|\sigma_{n-i}(y_{n-i}+\sigma_{n-i-1}(y_{n-i-1}+\cdots+\sigma_1(y_1)\cdots))|\ge\frac{1}{2}\varepsilon_{n-i}.$$

If ε satisfies (13), then $\varepsilon\le\left(\frac{\min a_i}{\max K_i}\right)^n$, and

$$|x_{n-j}|\le\varepsilon^{-\frac{j}{n}}K_{n-j}(|y_n|+\cdots+|y_{n-j}|)\le\varepsilon^{\frac{1}{n}}K_{n-j}\le\min\{a_i\},$$

where $j = 0, \cdots, i-1$. By Lemma 1, we have

$$\begin{aligned}\left|\frac{dy_{n-i}}{dt}\right| &\geq \varepsilon^{\frac{1}{n}}|\sigma_{n-i}(y_{n-i}+\cdots)| - |\Phi_{n-i}| \\ &\geq \varepsilon^{\frac{1}{n}}\left(\tfrac{1}{2}\varepsilon_{n-i} - \varepsilon^{1+\frac{1}{n}}\tilde{A}_i\right) \\ &\geq \varepsilon^{\frac{1}{n}}\left(\tfrac{1}{2}\varepsilon_{n-i} - \varepsilon\delta_{n-i}^{\frac{1}{n}}\tilde{A}_i\right) \\ &= \varepsilon^{\frac{1}{n}}\left(\tfrac{1}{2}\varepsilon_{n-i} - \tfrac{1}{4}\tfrac{\varepsilon}{4^i}\right) = \tfrac{1}{4}\varepsilon^{\frac{1}{n}}\varepsilon_{n-i},\end{aligned}$$

and

$$\operatorname{sign}\left(\frac{dy_{n-i}}{dt}\right) = -\operatorname{sign}(y_{n-i}).$$

Therefore, when $|y_{n-i}| \geq \frac{3}{4}\varepsilon_{n-i}$, $|y_{n-i}|$ decreases at rate $\frac{1}{4}\varepsilon^{\frac{1}{n}}\varepsilon_{n-i}$. Consequently, y_{n-i} enters P_{n-i} in finite time and remains in P_{n-i} thereafter. □

Since the right terms of (11) are all bounded, we know that the solution of (11) exists globally. In the following, we will study exponential convergence of the system (1).

If ε satisfies (13), then

$$|y_{n-i} + \sigma_{n-i-1}(y_{n-i-1}+\cdots)| \leq \varepsilon_{n-i}, \quad i = 0, \cdots, n-1.$$

By the definition of σ_i(10), we have

$$\sigma_{n-i}(y_{n-i} + \sigma_{n-i-1}(\cdots)) = y_{n-i} + \sigma_{n-i-1}(\cdots).$$

The closed loop system (11) becomes

$$\left\{\begin{aligned} \tfrac{dy_1}{dt} &= -\varepsilon^{\frac{1}{n}}y_1 + \Phi_1, \\ \tfrac{dy_2}{dt} &= -\varepsilon^{\frac{1}{n}}(y_1+y_2) + \Phi_2, \\ &\vdots \\ \tfrac{dy_{n-1}}{dt} &= -\varepsilon^{\frac{1}{n}}(y_1+y_2+\cdots+y_n) + \Phi_{n-1}, \\ \tfrac{dy_n}{dt} &= -\varepsilon^{\frac{1}{n}}(y_1+y_2+\cdots+y_n) + \Phi_n,\end{aligned}\right. \tag{15}$$

It can also be written as the compact form

$$\frac{d\boldsymbol{y}}{dt} = \varepsilon^{\frac{1}{n}}A\boldsymbol{y} + \boldsymbol{\Phi}, \tag{16}$$

where

$$A = \begin{bmatrix} -1 & & \\ \vdots & \ddots & \\ -1 & \cdots & -1\end{bmatrix}, \quad \boldsymbol{\Phi} = \begin{bmatrix}\Phi_1 \\ \vdots \\ \Phi_n\end{bmatrix}. \tag{17}$$

In order to prove exponential convergence, we must bound the norm of $\boldsymbol{\Phi}$.

Lemma 3. Suppose ε satisfies (13), then there exists a constant $\tilde{C}$, which is independent of ε, such that

$$\|\boldsymbol{\Phi}\| \leq \varepsilon^{\frac{2}{n}}\tilde{C}\|\boldsymbol{y}\|. \tag{18}$$

Proof. Since for every $0 \leq i \leq n-1$, $|y_{n-i}| \leq \frac{3}{4}\varepsilon_{n-i}$, we have

$$\|\boldsymbol{y}\| = (|y_1|^2 + \cdots + |y_n|^2)^{\frac{1}{2}} \leq \left(\sum_{i=0}^{n-1}(\frac{3}{4}\frac{\varepsilon}{4^i})^2\right)^{\frac{1}{2}} \leq \varepsilon. \tag{19}$$

If (13) holds, we have

$$\begin{aligned}|\Phi_{n-i}| \leq\ & N_n|u|^2 + \varepsilon^{\frac{1}{n}}N_{n-1}(|x_n|^2 + |u|) + \cdots + \\ & \varepsilon^{\frac{i}{n}}N_{n-i}(|x_{n-i+1}|^2 + |x_{n-i+2}| + \cdots + |u|) \\ \leq\ & \varepsilon^{\frac{2}{n}}[N_n n\|\boldsymbol{y}\| + N_{n-1}(K_n^2+n)\|\boldsymbol{y}\| + \cdots + \\ & N_{n-i}(iK_{n-i+1}^2 + \cdots + K_n + n)\|\boldsymbol{y}\|],\end{aligned}$$

and

$$\|\boldsymbol{\Phi}\| = (|\Phi_1|^2 + \cdots + |\Phi_n|^2)^{\frac{1}{2}} \leq \varepsilon^{\frac{2}{n}}\tilde{C}\|\boldsymbol{y}\|,$$

where

$$\begin{aligned}\tilde{C} = \ & (\sum_{i=1}^{n}[N_n n + N_{n-1}(K_n^2+n) + \cdots + N_{n-i} \\ & (iK_{n-i+1}^2 + (i-1)K_{n-i+2} + \cdots + K_n + n)]^2)^{\frac{1}{2}}.\end{aligned}$$

□

Theorem 1. If the scaling factor ε satisfies (13) and

$$\varepsilon \leq \tilde{\delta} = \left(\frac{1}{4\|P\|}\right)^n\tilde{C}^{-n}, \tag{20}$$

where P satisfies $PA + A^\top P = -I$, then the system (1) is exponentially convergent with the control (9).

Proof. It is obvious that the exponential convergence of $\boldsymbol{x}(t)$ is equivalent to the exponential convergence of $\boldsymbol{y}(t)$. Therefore, we only need to prove that $\boldsymbol{y}(t)$ is exponentially convergent.

Since the matrix A has only one eigenvalue -1, the Lyapunov equation $PA + A^\top P = -I$ has a solution[13]

$$P = \int_0^{+\infty} e^{A^{\mathrm{T}}t}e^{At}dt.$$

It is obvious that P is positive definite.

Let $\alpha = \frac{\varepsilon^{\frac{1}{n}}}{4\|P\|}$ and $\boldsymbol{z}(t) = e^{\alpha t}\boldsymbol{y}(t)$, we can show that $\boldsymbol{z}(t)$ is bounded. Define $V(t) = \boldsymbol{z}^{\mathrm{T}}(t)P\boldsymbol{z}(t)$, we have

$$\begin{aligned}\frac{dV(t)}{dt} &= \left(\frac{d(e^{\alpha t}\boldsymbol{y}(t))}{dt}\right)^\top P\boldsymbol{z}(t) + \boldsymbol{z}(t)^\top P\frac{d(e^{\alpha t}\boldsymbol{y}(t))}{dt} \\ &= e^{2\alpha t}(\varepsilon^{\frac{1}{n}}\boldsymbol{y}^{\mathrm{T}}(A^{\mathrm{T}}P + PA)\boldsymbol{y} + 2\alpha\boldsymbol{y}^{\mathrm{T}}P\boldsymbol{y} + 2\boldsymbol{\Phi}^{\mathrm{T}}P\boldsymbol{y}) \\ &\leq e^{2\alpha t}[-\varepsilon^{\frac{1}{n}}\|\boldsymbol{y}\|^2 + 2\alpha\|P\|\|\boldsymbol{y}\|^2 + 2\|\boldsymbol{\Phi}\|\|P\|\|\boldsymbol{y}\|] \\ &\leq e^{2\alpha t}\varepsilon^{\frac{1}{n}}\|\boldsymbol{y}\|^2\left(-1 + \tfrac{1}{2} + 2\tilde{\delta}^{\frac{1}{n}}\tilde{C}\|P\|\right) \\ &\leq 0.\end{aligned}$$

Therefore, $V(t)$ is bounded and and thus

$$\boldsymbol{y}(t) = O(e^{-\epsilon t}).$$

□

2 Further results for system (1)

In this section, we study the nonlinear system (1) without the assumptions $\left.\frac{\partial f_i}{\partial x_{i+1}}\right|_0 \neq 0 (i = 1, \cdots, n-1)$. In this case, the system (1) cannot be linearized to the form (6). In fact, if $\left.\frac{\partial^j f_i}{\partial (x_{i+1})^j}\right|_0 = 0 (j = 1, \cdots, p_{i+1}-1)$, $\left.\frac{\partial^{p_{i+1}} f_i}{\partial (x_{i+1})^{p_{i+1}}}\right|_0 \neq 0 (i = 1, \cdots, n-1)$ and $\left.\frac{\partial f_n}{\partial u}\right|_0 \neq 0$, we can obtain the following formulation

$$\left\{\begin{aligned}\dot{x}_1 &= x_2^{p_2} + \varphi_1(x_3, \cdots, x_n, u), \\ \dot{x}_2 &= x_3^{p_3} + \varphi_2(x_4, \cdots, x_n, u), \\ &\vdots \\ \dot{x}_{n-1} &= x_n^{p_n} + \varphi_{n-1}(u), \\ \dot{x}_n &= u,\end{aligned}\right. \tag{21}$$

where φ_i is continuously differential for its variable. In this section, we assume φ_i satisfies

$$|\varphi_i(x_{i+1}, \cdots, x_n, u)| \leq a_i(x_{i+1}^{p_{i+1}+1} + \cdots + x_n^{p_n+1} + u^2),$$

where $i = 1, \cdots, n-1$, and

$$|\varphi_n(u)| \le a_n |u|^2.$$

Consider linear coordinate transformation:

$$y_{n-i} = \sum_{j=0}^{i} C_i^j x_{n-j}, \quad z_{n-i} = \sum_{j=0}^{i} C_i^j x_{n-j}^{p_{n-j}},$$

or in detail,

$$\begin{bmatrix} y_1 \\ \vdots \\ y_n \end{bmatrix} = \begin{bmatrix} C_{n-1}^{n-1} & \cdots & C_{n-1}^0 \\ & \ddots & \vdots \\ & & C_0^0 \end{bmatrix} \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix},$$

$$\begin{bmatrix} z_1 \\ \vdots \\ z_n \end{bmatrix} = \begin{bmatrix} C_{n-1}^{n-1} & \cdots & C_{n-1}^0 \\ & \ddots & \vdots \\ & & C_0^0 \end{bmatrix} \begin{bmatrix} x_1 \\ \vdots \\ x_n^{p_n} \end{bmatrix},$$

where $C_i^j = \frac{i!}{j!(i-j)!}$.

Since

$$\begin{aligned} \sum_{j=n-i+1}^{n} z_j &= \sum_{j=0}^{i-1} \sum_{k=j}^{i-1} C_k^j x_{n-j}^{p_{n-j}} = \sum_{j=0}^{i-1} C_i^{j+1} x_{n-j}^{p_{n-j}} \\ &= \sum_{j=1}^{i} C_i^j x_{n-j+1}^{p_{n-j+1}}, \end{aligned}$$

we have

$$\begin{aligned} \dot{y}_{n-i} &= \sum_{j=0}^{i} C_i^j \dot{x}_{n-j} \\ &= \sum_{j=1}^{i} C_i^j x_{n-j+1}^{p_{n-j+1}} + u + \sum_{j=1}^{i} C_i^j \varphi_{n-j+1} \\ &= (z_{n-i+1} + \cdots + z_n) + u + \Phi_{n-i}, \end{aligned}$$

where

$$\begin{aligned} \Phi_{n-i} = \; & \varphi_n(u) + C_i^1 \varphi_{n-1}(x_n, u) \\ & + \cdots + C_i^i \varphi_{n-i}(x_{n-i+1}, x_{n-i+2}, \cdots, x_n, u). \end{aligned}$$

satisfies

$$|\Phi_i(x_{i+1}, \cdots, x_n, u)| \le r_i (x_{i+1}^{p_{i+1}+1} + \cdots + x_n^{p_n+1} + u^2),$$

where $i = 1, \cdots, n-1$, and

$$|\Phi_n(u)| \le r_n |u|^2.$$

Using the following nested saturation control

$$u = -\sigma_n(z_n + \sigma_{n-1}(z_{n-1} + \cdots + \sigma_1(z_1) \cdots)),$$

where

$$\sigma_i(x) = \begin{cases} x, & \text{if } |x| \le \varepsilon_i, \\ \varepsilon_i \text{sign}(x), & \text{if } |x| > \varepsilon_i, \end{cases}$$

$$\varepsilon_1 = \frac{1}{4}\varepsilon_2 = \cdots = \left(\frac{1}{4}\right)^{n-1} \varepsilon_n, \quad \varepsilon_n = \varepsilon.$$

System (21) is then transformed to the following closed loop system

$$\begin{cases} \dot{y}_1 = z_2 + \cdots + z_n - \sigma_n(z_n + \cdots + \sigma_1(z_1)) + \Phi_1, \\ \dot{y}_2 = z_3 + \cdots + z_n - \sigma_n(z_n + \cdots + \sigma_1(z_1)) + \Phi_2, \\ \quad \vdots \\ \dot{y}_{n-1} = z_n - \sigma_n(z_n + \cdots + \sigma_1(z_1))] + \Phi_{n-1}, \\ \dot{y}_n = -\sigma_n(z_n + \cdots + \sigma_1(z_1)) + \Phi_n. \end{cases}$$

Now, we will prove that system (21) is globally stable with small enough ε.

Theorem 2. Choose ε to be small enough, then in finite time, $z_{n-i} \in P_{n-i} = \{z_{n-i} : |z_{n-i}| < \frac{3}{4}\varepsilon_{n-i}\}$, where $i = 0, \cdots, n-1$.

Proof. Since all $x_i \to 0$ if and only if all $z_i \to 0$, this theorem implies the state of control system (1) is globally stable.

We begin by considering the evolution of the state y_n

$$\frac{\mathrm{d}y_n}{\mathrm{d}t} = -\sigma_n(z_n + \sigma_{n-1}(z_{n-1} + \cdots + \sigma_1(z_1) \cdots)) + \Phi_n. \quad (22)$$

If $z_n \bar{\in} P_n = \{z_n : |z_n| < \frac{3}{4}\varepsilon_n\}$, then $|z_n| \ge \frac{3}{4}\varepsilon_n$, and

$$|z_n + \sigma_{n-1}(z_{n-1} + \cdots)| \ge |z_n| - \varepsilon_{n-1} \ge \frac{1}{2}\varepsilon_n.$$

Now it can be seen from (22) that if $z_n \bar{\in} P_n$, then

$$\begin{aligned} \left|\tfrac{\mathrm{d}y_n}{\mathrm{d}t}\right| &\ge |\sigma_n(z_n + \sigma_{n-1}(z_{n-1} + \cdots))| - |\Phi_n| \\ &\ge \tfrac{1}{2}\varepsilon_n - r_n \varepsilon_n^2 \ge \tfrac{1}{4}\varepsilon_n, (\text{when } \varepsilon \le \tfrac{1}{4r_n}), \end{aligned}$$

and

$$\text{sign}\left(\frac{\mathrm{d}y_n}{\mathrm{d}t}\right) = -\text{sign}(z_n).$$

Note that all p_i are odd numbers and all $y_i \to 0$ if and only if all $z_i \to 0$. Therefore, when $z_n \ge \frac{3}{4}\varepsilon_n$, y_n decreases at the rate $\frac{1}{4}\varepsilon_n$ until $z_n < \frac{3}{4}\varepsilon_n$. Similarly, when $z_n \le -\frac{3}{4}\varepsilon_n$, y_n increases at the rate $\frac{1}{4}\varepsilon_n$ until $z_n > -\frac{3}{4}\varepsilon_n$. Consequently, z_n enters P_n in finite time and remains in P_n thereafter.

Now, we proceed by induction. Suppose $z_{n-j} \in P_{n-j} = \{z_{n-j} : |z_{n-j}| < \frac{3}{4}\varepsilon_{n-j}\}$, $j = 0, \cdots, i-1$, we consider the evolution of the state z_{n-i}. The argument of $\sigma_{n-j}, j = 0, \cdots, i-1$, is bounded as

$$|z_{n-j} + \sigma_{n-j-1}(z_{n-j-1} + \cdots)| \le \frac{3}{4}\varepsilon_{n-j} + \varepsilon_{n-j-1} = \varepsilon_{n-j}.$$

By the definition of σ_i (see (10)), the evolution of z_{n-i} is given by

$$\frac{\mathrm{d}y_{n-i}}{\mathrm{d}t} = -\sigma_{n-i}(z_{n-i} + \cdots + \sigma_1(z_1)) + \Phi_{n-i}.$$

Similarly, if $z_{n-i} \bar{\in} P_{n-i} = \{z_{n-i} : |z_{n-i}| < \frac{3}{4}\varepsilon_{n-i}\}$, then $|z_{n-i}| \ge \frac{3}{4}\varepsilon_{n-i}$, and

$$|\sigma_{n-i}(z_{n-i} + \sigma_{n-i-1}(z_{n-i-1} + \cdots + \sigma_1(z_1) \cdots))| \ge \frac{1}{2}\varepsilon_{n-i}.$$

Hence, we have (for small enough ε)

$$\begin{aligned} \left|\tfrac{\mathrm{d}y_{n-i}}{\mathrm{d}t}\right| &\ge |\sigma_{n-i}(z_{n-i} + \cdots)| - |\Phi_{n-i}| \\ &\ge \tfrac{1}{2}\varepsilon_{n-i} - r_i \left(\varepsilon_{n-i}^{\frac{p_m+1}{p_m}} + \varepsilon^2\right) \ge \tfrac{1}{4}\varepsilon_{n-i}, \end{aligned}$$

(where $p_m = \max\{p_2, \cdots, p_n\}$) and

$$\text{sign}\left(\frac{\mathrm{d}y_{n-i}}{\mathrm{d}t}\right) = -\text{sign}(z_{n-i}).$$

Therefore, when $z_{n-i} \ge \frac{3}{4}\varepsilon_{n-i}$, y_{n-i} decreases at the rate $\frac{1}{4}\varepsilon_{n-i}$ until $z_{n-i} < \frac{3}{4}\varepsilon_{n-i}$. Similarly, when $z_{n-i} \le -\frac{3}{4}\varepsilon_{n-i}$, y_{n-i} increases at the rate $\frac{1}{4}\varepsilon_{n-i}$ until $z_{n-i} > -\frac{3}{4}\varepsilon_{n-i}$. Consequently, z_{n-i} enters P_{n-i} in finite time and remains in P_{n-i} thereafter. □

3 Numerical test

Example. Consider the feedforward nonlinear system:

$$\begin{cases} \dot{x}_1 = -\sin(x_2 + x_3 + u) + \cos(x_3 + u) + \tan(u) - 1 \\ \dot{x}_2 = \tan(x_3) - \sin(u^2 x_3) \\ \dot{x}_3 = -\sin(u - u^2) + \frac{u^2}{4} \end{cases}$$

Let $u = -\sqrt[3]{\varepsilon}(\sigma_3(y_3 + \sigma_2(y_2 + \sigma_1(y_1))))$, where

$$\begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} 1 & -2 & -1 \\ 0 & -1 & -1 \\ 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} \varepsilon^{\frac{2}{3}} x_1 \\ \varepsilon^{\frac{1}{3}} x_2 \\ x_3 \end{bmatrix}.$$

Simulation results are described in Figure 1 (the initial point $x_0 = (2, -3, 1)$, $(-2, 1, -1)$, $(1.5, 1, -1)$ and $(-1, 1.5, 1)$, the parameter $\varepsilon = 0.65$).

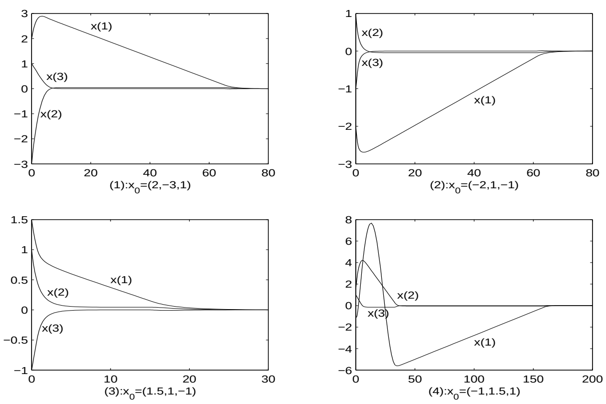

Figure 1 Simulation results

From the simulation test, we observe that the solution trajectory $x_3(t)$, $x_2(t)$ and $x_1(t)$ converge to zero one by one, that is, $x_3(t)$ converges to zero firstly, then $x_2(t)$, and $x_1(t)$ converges to zero finally (especially from figure 1 (4)). In fact, from the proof of Lemma 2, the control (9) makes the solution trajectories of nonlinear system (1) come into the small interval P_{n-i} by the order of y_n, y_{n-1}, $\cdots$, y_1. The numerical simulation verifies that exactly.

4 Conclusion

In this paper, we study the saturated control of a general class of feedforward nonlinear systems. By linear coordinate transformation, we transfer this feedforward nonlinear system to a closed loop system. We use nested saturation stabilizer [6] to make the nonlinear system globally stable and exponential convergent at the equilibrium $\boldsymbol{x} = 0$. The results of this paper enlarge the class of feedforward nonlinear systems investigated before clearly. Numerical example is given, by which efficiency of the result given above is verified.

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