Hindawi Publishing Corporation
International Journal of Quality, Statistics, and Reliability
Volume 2008, Article ID 594753, 10 pages
doi:10.1155/2008/594753

Research Article

A Unified Approach for Predicting Long- and Short-Term Capability Indices with Dependence on Manufacturing Target Bias

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Received 15 April 2008; Revised 25 September 2008; Accepted 14 December 2008

Recommended by Satish Bukkapatnam

It is shown that the exact solution for the capability index (CPI) for Gaussian-distributed process with target bias can be expressed in terms of an unbiased CPI and a normalized target bias. The principal advantage of this specific formulation is that it facilitates evaluation of the degradation of the capability of the process due to bias between process mean and the process target. It is shown how this formalism, initially developed for the short-term process, is readily extended to long-term process for which the distribution is Gaussian. Readily isolated in the latter case are the two long-term CPI degrading effects, namely, process instability and target bias. Sufficient conditions to guarantee that long-term processes are distributed as Gaussian are discussed. Within the context of these assumed conditions, a new paradigm for a long-term locator "k" is proposed. For a three sigma process the results indicate that the exact CPI model is a less pessimistic predictor than both of the industry CPI models tested.

1. Introduction

In 1979, Taguchi and Wu [1] introduced a viewpoint on estimating the loss (in monetary units, i.e., cost) associated with lack of precision and accuracy in a manufacturing process. The preeminent manufacturing precursor to the viewpoint introduced by Taguchi is the classical "goal-post model" in which the only consideration to production cost is whether the product parameters fall within the process specification limits. Consistent with the goal-post philosophy, the level of process control is typically characterized in terms of what are known as capability indices (CPIs) [2]. Capability indices provide a numerical assessment of the ability of a process in attaining the predefined specifications [3, 4].

A manufacturing process would commonly be described in terms of three parameters: the finite target value (τ), an upper specification limit (USL), and a lower specification limit (LSL). All the parts for which the measured value "x" for a certain specification exceeds the USL or falls below the LSL are rejected. If the process target value for the product characteristic is centered between the USL and LSL, then the tolerances are said to be symmetric. The capability index is of interest to the manufacturing community because it consolidates the details in a complicated multifaceted manufacturing process down to one quantity which can be used to predict the fraction of parts rejected. Typical capability index values can range from 0.7 to 2.0. In the jargon of the community, three sigma process would correspond to a capability index of 1.0 while a much improved six sigma process suggested originally by Motorola would correspond to a capability index of 2.0 [2]. The standard deviation of the process is indicative of the level of precision. The absolute value of the difference between the distribution mean and the process target (i.e., target bias) is indicative of the process accuracy. According to the Taguchi guide for improved manufacturing quality, it is much easier to adjust the manufacturing process to improve accuracy than to adjust the process to improve the precision [2]. The most commonly assumed probabilistic distribution for a product characteristic with measured value "x" is the normal distribution which can be defined in terms of mean μ, and a standard deviation σ.

The primary situation for the application of asymmetric tolerances [5] occurs when the product parameter of interest exhibits a skewed distribution [6]. Historically, capability indices were first applied under the assumption that the mean of the process is on the target [2]. Target bias is at best approximately zero. In some practical cases, it maybe necessary to consider the impact of the distribution mean of the product parameter being off target. There have been a variety of target-bias-dependent capability index models introduced. A noncomprehensive but high profile list of such models has been assembled for purposes of this paper.

What follows first is a brief qualitative review of capability indices that are assuming zero target bias. The short-term capability index, C_p, is gauged within a relatively narrow window of time. The long-term capability index [2], C_{pk} can be found in the literature to be applied in two ways. It could be applied to extend the short-term capability concept (e.g., measured over days or even hours) to long-term (e.g., measured over weeks or months). It is assumed, in this case, that process mean shifts around the target but on the average is "on target." The concept is that a time-wise shifting around in the short-term process is accounted for with a probability density function (PDF) averaging leading to a higher standard deviation. The long-term precision in the manufacturing process is degraded relative to short-term and, therefore, the long-term capability index is lower than the short-term capability index.

On the other hand, C_{pk} has also found utility as a capability index that can include the impact of target bias [2, 7]. However, as pointed out in [6], this type of usage of C_{pk} to account for the target bias is questionable.Lastly, a third-measured paradigm for a capability index, C_{pm} paradigm [2, 7, 8] is also commonly invoked in the community to account for target bias. Because C_{pm} can be related to Taguchi loss functions [2], it is sometimes referred to as the Taguchi index. One advantage of the C_{pm} approach is that it is nonparametric, that is, makes no a priori assumptions on the underlying distribution of the specified product parameter distribution.

It has been shown [5] that a probabilistic description of the manufacturing process can be used to predict the exact dependence for the fraction of rejected components and related to a CPI. This has been done under assumptions of a normal (Gaussian) distribution for the process product and symmetric specification limits. The derived CPI with target bias was shown to be expressed in terms of four parameters, process mean, process target, upper specification limit, and lower specification limit. Additionally, it has been demonstrated that this exact solution is equivalent to a reparameterized solution expressed in terms of appropriately defined upper and lower capability indices [5].

A target bias-dependent capability index (CPI) for the symmetric-specification limit Gaussian-distributed process is proposed and tested. It is shown that various exact expressions reported in the literature are equivalent to a proposed short-term CPI model dependent on only two parameters, unbiased short-term CPI and a normalized target bias. One advantage of this particular formulation is that it facilitates the evaluation of the degradation of the capability of the process due to an offset between the mean and target of a process parameter. The second advantage of this parameterization is that it allows for a convenient comparison of the exact CPI model with two other commonly used industry models which also estimate the CPI with target offset. A third advantage is that the proposed formalization facilitates setting up a CPI model for the Gaussian-distributed long-term process with a methodology unified in approach with that of the proposed short-term CPI model. Readily isolated are the two long-term CPI degrading effects, namely, long-term process instability and target bias. Sufficient conditions to guarantee that the long-term processes are distributed as Gaussian are discussed. Within the context of these assumed conditions, a new paradigm for a long-term locator "k" is proposed.

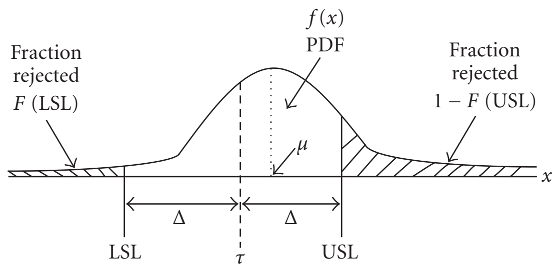

Figure 1: PDF showing manufacturing specifications and rejected fraction of parts.

Two implementation schemes for the proposed model are discussed. One method is based on the availability of numerical built-in mathematical routines for the error function and its inverse. The second scheme supplants the built-in functions used in the first scheme with recently reported analytical approximations [9, 10]. For a three sigma process, the results indicate that the exact CPI model is a less pessimistic predictor than both of the industry CPI models tested.

2. Background

2.1. Background on CPI Model

In general, the measurements for the process parameter to meet desired specifications are characterized by a distribution having a mean μ and a standard deviation σ [2]. The process has an upper specification limit (USL) and a lower specification limit (LSL). The distance between the USL and LSL, as represented on Figure 1, is 2Δ. The specifications are considered to be symmetric if the target satisfies the condition $\tau = (\text{USL} + \text{LSL})/2$ [2]. The defining recipe for the capability index intended for situations for which the measured parameter distribution is normal and has symmetric limits is

$$C_{p_o} = \frac{\text{USL} - \text{LSL}}{6\sigma} = \frac{\Delta}{3\sigma}. \quad (1)$$

The subscript "o" in (1) indicates that it does not account for any target bias. Generalizations of (1) to cover asymmetry in tolerances and nonnormal distributed parameters can be found in the literature [6]. The capability index is a direct measure of the process control and relates C_{p_o} to a fraction of rejection,

$$p_o = 2\Phi(-3C_{p_o}), \tag{2}$$

where $\Phi(z)$ is the standard normal cumulative density function (CDF). However, (2) can be applied to a process with normal distribution, no target bias, and symmetric limits.

2.2. Distribution Independent Observation

Independent of whether the distribution is normal with symmetric limits or not, the fraction rejected, also known as the "component of nonconformity" [6] can be computationally predicted by evaluating the CDF of the process distribution at selected points. As suggested by Figure 1, this prediction rule is given by [5]

$$p = F(\text{LSL}) + (1 - F(\text{USL})). \tag{3}$$

This combines the parts that do not meet the specifications, that is, the parts that have a measured product characteristic "x" which is either lower than the LSL or higher than the USL. In the spirit of (2), the generalized process capability index should be consistent with the rule

$$C_p = -\frac{1}{3}\Phi^{-1}\left(\frac{p}{2}\right), \tag{4}$$

where for the normal distribution with symmetric limits, $p_o = p$, and (4) reduces to special case described by (2). From Figure 1, it follows that

$$\begin{aligned} \text{USL} &= \tau + \Delta, \\ \text{LSL} &= \tau + \Delta. \end{aligned} \tag{5}$$

Consistent with (3) and (4), it can be shown that [5]

$$p = \Phi\left(\frac{\text{LSL} - \mu}{\sigma}\right) + \Phi\left(-\left(\frac{\text{USL} - \mu}{\sigma}\right)\right). \tag{6}$$

However, (6) conjunction with (4) produces an exact short-term capability index model which agrees with the Boyles [5] yield index model. Demonstration details are provided in Appendix A.

3. Model-A: Standard Normal Version and Computer Implementation

3.1. Model-A Analysis

The PDF represented in Figure 1 can be transformed to the standard normal version as shown in Figure 2 [11]. From (5), it can be stated that

$$\begin{aligned} \frac{\text{LSL} - \mu}{\sigma} &= \frac{(\tau - \mu) - \Delta}{\sigma} = \frac{-\Delta}{\sigma} + \frac{\tau - \mu}{\sigma}, \\ \frac{\text{USL} - \mu}{\sigma} &= \frac{(\tau + \Delta) - \mu}{\sigma} = \frac{\Delta}{\sigma} + \frac{\tau - \mu}{\sigma}. \end{aligned} \tag{7}$$

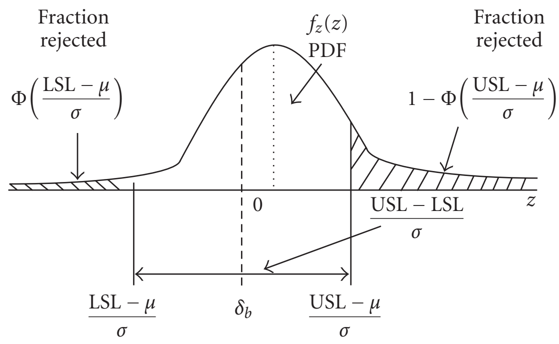

Figure 2: Transformed PDF showing the specification limits rejected fractions of parts.

After defining a normalized target bias,

$$\delta_b = \left(\frac{\tau - \mu}{\sigma}\right). \tag{8}$$

Therefore, for normal process PDFs with symmetric limits and target bias, it follows that the proposed exact Model-A [11] has the following rejection fraction after substituting from (1), (7), and (8) in (6),

$$p_A = \Phi(-3C_{p_o} + \delta_b) + \Phi(-3C_{p_o} - \delta_b). \tag{9}$$

Consistent with the general approach (4),

$$C_{p_A} = -\frac{1}{3}\Phi^{-1}\left(\frac{p_A}{2}\right). \tag{10}$$

A check with $\delta_b = 0$ from (9) yields $p_A = p_o$, and consistent with (10), $C_{p_A} = C_{p_o}$.

3.2. Model-A Implementation Using Built-In Error Function

Noting that the standard normal arguments "z" needed in (9) are expected to be negative for reasonably limited target bias, the following conversion rule valid for $z \le 0$ is useful with MATLAB [12],

$$\Phi(z) = 0.5 - 0.5\,\text{erf}\left(\frac{|z|}{\sqrt{2}}\right), \tag{11}$$

and for inspection of (11), it follows that

$$z = \sqrt{2}\,\text{erf}^{-1}\left(\frac{0.5 - \Phi(z)}{0.5}\right). \tag{12}$$

Appendix B describes the definitions and the approximations considered for the error function (erf) and the inverse error function (erf^{-1}) in (11) and (12), respectively. However, (11) and (12) can make use of built-in error and inverse error functions of MATLAB. An alternative to using built-in routines for the error function and its inverse is to employ approximate analytic expressions described in the following subsection.

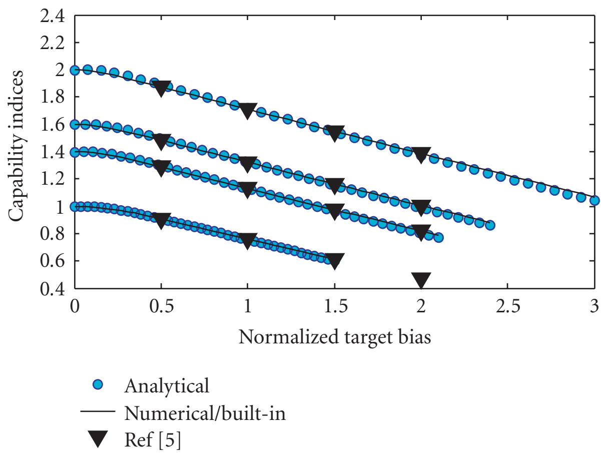

Figure 3: Model-A comparisons for numerical and analytic versions.

3.3. Model-A Implementation Using Analytic Approximation for Error Function

The built-in error function routines, available in MATLAB [12], can be replaced with analytic approximations in predicting the bias-dependent capability indices. After defining $a = 0.14$, the error function can be approximated as [9, 10]

$$\operatorname{erf}(x) \cong \sqrt{1-\exp\left(-x^2\left(\frac{(4/\pi)+ax^2}{1+ax^2}\right)\right)}. \tag{13}$$

Moreover, after taking $r = 2/(\pi a)$ and $t(x) = \ln(1-x^2)$, the inverse of the error function can be approximated as [9, 10]

$$\operatorname{erf}^{-1}(x) \approx \left(-r-\frac{t(x)}{2}+\sqrt{\left(r+\frac{t(x)}{2}\right)^2-\frac{t(x)}{a}}\right)^{0.5} \tag{14}$$

A comparison of (13) and (14) with the MATLAB built-in routines showed a maximum percentage error of 0.58% for the error function and 0.004% for its inverse. Hence, (13) and (14) are applicable to (11) and (12), respectively.

To demonstrate this approach, the built-in error function-based Model-A predictions are compared with the analytic Model-A predictions for $C_{p_o} = 1.0$, $C_{p_o} = 1.4$, $C_{p_o} = 1.6$, and $C_{p_o} = 2.0$. It can be seen from Figure 3 that both the numerical and the analytic approaches are serving as approximately equivalent predictors. *The accuracy of the analytic approximation approach compared to that of the built-in numerical implementation proved to be very good with the maximum percentage error of 0.91% at* $C_{p_o} = 1$.

Figure 3 also shows the comparison of Model-A with a well-established Boyles model [5] from the community. The predictions at various target bias values have been considered to establish a concise comparison of the behavioral pattern of the proposed model with the already existing industry model.

The comparison of Model A with Boyle's exact model [5] requires specification of the USL and the LSL values (e.g., USL = 58 and LSL = 26) [5]. For symmetrical specification limits, this implicitly determines the target value. The USL and LSL values when taken in combination with the selected values, for the unbiased short-term CPI (1) and the normalized target bias (8), lead to the process mean and standard deviation target values. With the four quantities {USL, LSL, μ, and σ}, numerically determined application of (6) and (4) will predict the exact value for the target-bias-dependent CPI which for comparison purposes has been included with the datasets plotted on Figure 3.

4. Alternative Popular Methods

4.1. Model-X: The AMT Model

Model-X is based on incorporating the target bias with capability index by first defining a location index [2, 7, 8],

$$k_b = \frac{|\mu-\tau|}{\Delta}. \tag{15}$$

The subscript b indicates that this model includes the target bias. However, (15) can be combined with the short-term capability index C_{p_o} to define the Model-X capability index rule as

$$C_{pX}^b \equiv C_{p_o}(1-k_b). \tag{16}$$

From (15), it follows that

$$C_{pX}^b = C_{p_o}\left(1-\left(\frac{\mu-\tau}{\sigma}\right)\frac{\sigma}{\Delta}\right). \tag{17}$$

This can be simplified via (1) and (8) [11]:

$$C_{pX}^b = C_p - \frac{|\delta_b|}{3}, \tag{18}$$

and hence the corresponding fraction of rejection is

$$p_x = 2\Phi(-3C_{px}^b). \tag{19}$$

4.2. Model-Y: C_{pm} Model

The C_{pm} model was modeled to include the impact of the bias of the mean from the target and the variance of the process parameter. As in the similarly defined Taguchi loss function, it is not assumed that the PDF is normal [2, 11]. The capability index in this model is defined using the variance of the process as

$$C_{pm} \equiv \frac{\Delta}{3\sigma_m}. \tag{20}$$

In (20), the variance is given by $\sigma_m = \sqrt{\sigma^2+(\mu-\tau)^2}$. Hence, the Model-Y [11] with bias can be defined as [2, 7]

$$C_{pY}^b \equiv \frac{\Delta}{3\sigma_m} = \left(\frac{\Delta}{3\sigma}\right)\frac{1}{\sqrt{1+\delta_b^2}} = \frac{C_{p_o}}{\sqrt{1+\delta_b^2}}, \tag{21}$$

and the corresponding fraction of rejection predicted by

$$p_Y = 2\Phi(-3C_{pY}^b). \tag{22}$$

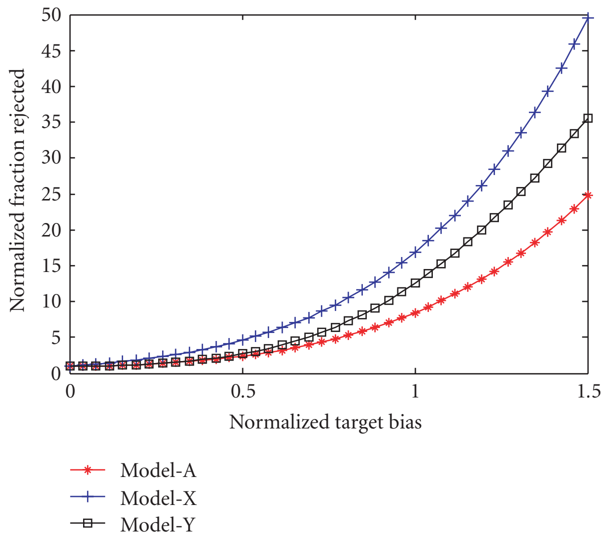

Figure 4: Plot showing fraction rejected for models A, X and Y ($C_p = 1$) versus normalized bias.

4.3. Computer Tests for Various Capability Index Models

The normalized fraction rejected for models A, X, and Y are defined in terms of the fraction rejected under zero bias conditions given in (2) as

$$(p_{A_N}, p_{X_N}, p_{Y_N}) \equiv \frac{p_A, p_X, p_Y}{p_o}. \tag{23}$$

In Figure 4, the fraction rejected obtained from (23) is plotted versus normalized target bias (8) as derived from (9), (19), and (22) for models A, X, and Y, respectively. It should be noted that Model-X and Model-Y are both more pessimistic (i.e., higher fraction is rejected) than that predicted from Model-A.

The dependence of bias-inclusive capability indices on short-term capability index and normalized target bias for the models A, X, and Y were taken from (10), (18), and (21), respectively. These were plotted versus the normalized target bias as shown in Figure 5.

These final results indicate that the Model-X and Model-Y capability index rules are consistently overpessimistic (i.e., lower in value). Of the two industry standard models Model-Y, using C_{pm}, should be a better choice than Model-X in that it is closer in prediction to Model-A.

5. Extension to Long-Term Process

The standard approach [2, 13] for predicting the long-term capability index is given by

$$C_{pk_o} \equiv \frac{\Delta}{3\sigma_k}. \tag{24}$$

The suffix "o" in (24) indicates that there is no target bias and is now expressed in terms of a long-term standard deviation

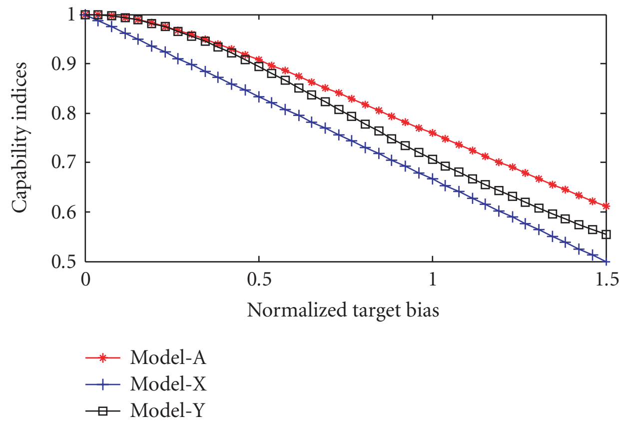

Figure 5: Plot showing capability indices for models A, X, and Y ($C_{p_o} = 1$) versus normalized target bias.

σ_k. Consistent with the qualitative description in Section 1 for the long-term capability index as required by (24)

$$\sigma_k \geq \sigma. \tag{25}$$

This is consistent with a long-term process being less precise, that is, higher standard deviation than that associated with the corresponding short-term processes. By assumption, the short-term processes have the same standard deviation. The claim here is that under certain restrictions, to be described, the long-term process will be Gaussian, with standard deviation σ_k, and mean $\overline{\mu}$, the latter is given by

$$\overline{\mu} = \text{avg}\{\mu_i\}, \quad i = 1, 2, \ldots, M, \tag{26}$$

where avg{} is the average operator and i indexes the M short-term processes to be averaged. If there is a target bias, $\overline{\mu} \neq \tau$, it can be accounted for with Model-A type analysis by defining a normalized target bias [11]:

$$\delta_{bk} = \frac{\tau - \overline{\mu}}{\sigma_k}. \tag{27}$$

The long-term capability index representation equivalent to (24) is

$$C_{pk_o} = C_{p_o}(1 - k) \leq C_{p_o}, \quad 0 \leq k < 1. \tag{28}$$

Unlike the locator index defined by (15), this representation for CPI in (28) assumes a location index k which only accounts for the long-term effective spread. The target bias is accounted for via (27). A revised form for the locator index which excludes target bias effect is [11]

$$k_{\text{revised}} \equiv \frac{\text{avg}\{|\overline{\mu} - \mu_i|\}}{\Delta}, \quad 1 > k \geq 0. \tag{29}$$

It should be noted that in this revised form of locator (29), the average of the short-term process means $\overline{\mu}$ supplants τ in (15). If the long-term process is "on target" (i.e., if $\overline{\mu} = \tau$), then the definition (29) reduces to the commonly used (15) [2]. As discussed in what follows, the restrictive mathematical conditions for reproducing an exact long-term Gaussian process from the superposition of short-term Gaussian processes will lead to a different locator model than commonly seen model in (15) or (29).

The long-term (LT) process distribution PDF can be viewed as being constructed from the mathematical average of multiple short-term PDFs as represented in the classic Harris and Lawson text on six sigma methods [13]. In Appendix C, it is formally shown that a long-term Gaussian PDF can be constructed from an average of short-term PDFs,

$$f_{\mathrm{LT}}(y) = \frac{1}{M}\sum_{i=1}^{M} f_{\mathrm{ST}}(y, \mu_i). \tag{30}$$

In this case, the short-term process PDF is given by

$$f_{\mathrm{ST}}(y, \mu_i) = \frac{1}{\sqrt{2\pi}\sigma} e^{-(y-\mu_i)^2/2\sigma^2}, \qquad -\infty \le y \le \infty, \quad i = 1, 2, \ldots, M. \tag{31}$$

The often-cited mean-finding rule [2], taking discrete-term expectation operator E{} [2], with respect to long-term PDF for y on both sides of (30), would lead to

$$\overline{\mu} = \mathrm{E}\{y\} = \frac{1}{M}\sum_{i=1}^{M} \mathrm{E}\{y f_{\mathrm{ST}}(y, \mu_i)\} = \frac{1}{M}\sum_{i=1}^{M} \mu_i, \tag{32}$$

which confirms (26). No claim is made that $f_{\mathrm{LT}}(y)$ is guaranteed to be Gaussian. For example, if two short-term Gaussians with same standard deviation σ but with different means were averaged via (30), the resultant process would not be a Gaussian. Nonetheless, the assumption which is often made and is implicit in the applicability of (24) is that the overriding long-term distribution in (30) is approximately normal with mean $\overline{\mu}$ and standard deviation σ_k [2, 13]. This assumption is shown to be justified in Appendix D with the restriction that the random variable associated with the (short-term) mean, μ, is approximately distributed Gaussian:

$$f_\mu(\mu) = \frac{1}{\sqrt{2\pi}\sigma_\mu} e^{-(\mu-\overline{\mu})^2/2\sigma_\mu^2}, \quad -\infty \le \mu \le \infty, \tag{33}$$

with an average mean $\overline{\mu}$ and a standard deviation σ_μ. Consider first a special case that the short-term distributions are identical (i.e., same constant mean $\mu = \overline{\mu}$). Then, the mean μ ceases to be a random variable in the usual sense. This is accounted for in (33) by taking a limit $\sigma_\mu \to 0$ and in that limit (33) reduces to the Dirac delta function [2]:

$$\lim_{\sigma_\mu \to 0} f_\mu(\mu) \longrightarrow \delta(\mu - \overline{\mu}). \tag{34}$$

For large-enough sampling of the short-term PDF (see (30)) (i.e., $M \to \infty$), discrete averaging with μ_i can be switched to averaging over the continuous random variable, μ, with an integration rule,

$$f_{\mathrm{LT}}(y) = \mathrm{E}\{f_{\mathrm{ST}}(y, \mu)\} = \int_{-\infty}^{\infty} f_{\mathrm{ST}}(y, \mu) f_\mu(\mu) d\mu. \tag{35}$$

In the special case where all the short-term Gaussian distributions are the same (i.e., same mean), the PDF for random variable μ reduces to a Dirac delta function (34) and subsequent substitution into (35) yields, as expected, a long-term PDF that is identical to the time-wise stable short-term processes. It is demonstrated in Appendix D that the integrated (35) is distributed Gaussian with mean $\overline{\mu}$ and standard deviation

$$\sigma_k = \sqrt{\sigma^2 + \sigma_\mu^2} = \sigma\sqrt{1 + \left(\frac{\sigma_\mu}{\sigma}\right)^2}. \tag{36}$$

In Appendix D, it is shown that

$$\sigma_\mu = \sqrt{\frac{\pi}{2}} \mathrm{avg}\{|\overline{\mu} - \mu_i|\}. \tag{37}$$

It follows from (24) and (36) that the capability index is given by

$$C_{pk_o} = \frac{C_{p_o}}{\sqrt{1 + (\sigma_\mu/\sigma)^2}} = \frac{C_{p_o}}{\sqrt{1 + (\sqrt{\pi/2}(\mathrm{avg}\{|\overline{\mu}\text{-}\mu_i|\}/\sigma))^2}}, \tag{38}$$

and as expected C_{pk_o} is reduced with increasing temporal instability in the short-term process as gauged by σ_μ. Consistent with (28), the effective locator, k, would be given by

$$k_{\mathrm{eff}} = 1 - \frac{1}{\sqrt{1 + (\sigma_\mu/\sigma)^2}} = 1 - \frac{1}{\sqrt{1 + (\sqrt{\pi/2}(\mathrm{avg}\{|\overline{\mu} - \mu_i|\}/\sigma))^2}}. \tag{39}$$

For relatively small ratios of $x = \sigma_\mu/\sigma$, a two-term expansion can be approximated as

$$\frac{1}{\sqrt{1 + x^2}} \approx \frac{1}{(1 + x^2/2)} \approx \left(1 - \frac{x^2}{2}\right) |x| \ll 1. \tag{40}$$

Hence, it facilitates for setting up an approximation for (39),

$$k_{\mathrm{eff}} \approx \frac{1}{2}\left(\frac{\sigma_\mu}{\sigma}\right)^2 = \frac{1}{2}\left(\sqrt{\frac{\pi}{2}}\frac{\mathrm{avg}\{|\overline{\mu} - \mu_i|\}}{\sigma}\right)^2, \quad \frac{\sigma_\mu}{\sigma} \ll 1. \tag{41}$$

As expected, $k_{\mathrm{eff}} = 0$ for stable short-term processes (i.e., when $\sigma_\mu = 0$); and the locator k_{eff} increases with increasing instability in the short-term process. If there is no target bias, then $\overline{\mu} = \tau$ and according to (27), $\delta_{bk} = 0$.

Under the restriction (33), a unified approach, which includes impact of target bias, is possible with the identical mathematical thread-of-logic (Section 3). A long-term Model-A-type capability index with nonzero target bias is then given by

$$p_{kA} = \Phi(-3C_{pk_o} + \delta_{bk}) + \Phi(-3C_{pk_o} - \delta_{bk}). \tag{42}$$

Following (8), (9), and (10) after the substitutions,

$$C_{p_o} \longrightarrow C_{pk_o}, \quad \delta_{\mathrm{b}} \longrightarrow \delta_{bk}, \quad C_{pA} \longrightarrow Cp_{kA}, \quad p_A \longrightarrow p_{kA}, \quad \sigma \longrightarrow \sigma_k. \tag{43}$$

6. Conclusion

It is has been shown that a short-term CPI model dependent on only two parameters, unbiased short-term CPI and a normalized target bias is equivalent to various exact CPI expressions reported in the literature. The demonstrated principal advantage of this specific formulation is that it facilitates the evaluation of the degradation of the capability of the process due to an offset between the mean and the target of a process parameter. The unified methodology for predicting short-term CPI is applicable to the long-term CPI, pending a condition that the long term process is distributed Gaussian. Sufficient conditions to guarantee that the long-term processes are distributed as Gaussian were discussed. Within the context of these assumed conditions, a new paradigm for a long-term locator "k" is proposed.

Two implementation schemes for the proposed reformulation for the exact solution were discussed. One method is dependent on the availability of a built-in error function and its inverse while the other method uses an analytic approximation for the error function and its inverse.

The second scheme supplants the built-in functions used in the first scheme with recently reported analytical approximations. For a three sigma process, the results indicate that the exact CPI model is a less pessimistic predictor than both of the industry CPI models tested. Our results indicate that the C_{pm} model (Model-Y) and the AMT model (Model-X) were more pessimistic than the exact model (Model-A) in estimating manufacturing loss.

In the literature, methods have been reported to account for nonnormality in process distribution. For future work, it would be interesting to demonstrate that any distribution can be converted to an equivalent Gaussian. In such case, the Model-A approach would again be applicable as long as a combination of equivalent process specifications and equivalent Gaussian parameters are appropriately defined.

Appendices

A. Comparison of C_{pA} to Yield Index S_{pk}

In 1994, R. A. Boyles proposed a yield-based capability index [5] which agrees with the model proposed in this paper. The current Model-A approach can be compared to that of Boyles' by initially considering the yield index S_{pk} [5]:

$$S_{pk} = S(3C_{pl}, 3C_{pu}), \tag{A.1}$$

where C_{pl} and C_{pu} are the capability indices with mean μ, standard deviation σ, lower specification limit (LSL), and an upper specification limit (USL) and are given by [5]

$$C_{pl} = \frac{\mu - \text{LSL}}{3\sigma}, \qquad C_{pu} = \frac{\text{USL} - \mu}{3\sigma}. \tag{A.2}$$

The operator S in (A.1) for a standard normal cumulative distribution function is [5]

$$S(x, y) = \frac{1}{3}\Phi^{-1}\left(\frac{\Phi(x) + \Phi(y)}{2}\right). \tag{A.3}$$

Hence, the yield index in (A.1) can be represented as,

$$S_{pk} = \frac{1}{3}\Phi^{-1}\left[\frac{1}{2}\{\Phi(3C_{pl}) + \Phi(3C_{pu})\}\right]. \tag{A.4}$$

Substituting C_{pl} and C_{pu} from (A.2) (into (A.4)) gives the yield index proposed by Boyles in terms of the mean, the standard deviation, and the specification limits:

$$S_{pk} = \frac{1}{3}\Phi^{-1}\left[\frac{1}{2}\left\{\Phi\left(\frac{\mu - \text{LSL}}{\sigma}\right) + \Phi\left(\frac{\text{USL} - \mu}{\sigma}\right)\right\}\right]. \tag{A.5}$$

From the basic properties of normalized distribution functions

$$\Phi(x) = 1 - \Phi(-x), \tag{A.6}$$

and after noting that

$$\Phi^{-1}(x) = \Phi^{-1}[1 - \Phi\{\Phi^{-1}(1 - x)\}]. \tag{A.7}$$

The use of (A.6) leads to

$$\Phi^{-1}(x) = -\Phi^{-1}(1 - x). \tag{A.8}$$

Applying (A.8) on (A.5) leads to

$$S_{pk} = -\frac{1}{3}\Phi^{-1}\left[1 - \frac{1}{2}\Phi\left(\frac{\mu - \text{LSL}}{\sigma}\right) - \frac{1}{2}\Phi\left(\frac{\text{USL} - \mu}{\sigma}\right)\right]; \tag{A.9}$$

hence,

$$S_{pk} = -\frac{1}{3}\Phi^{-1}\left[\frac{1}{2}\left(1-\Phi\left(\frac{\mu-\text{LSL}}{\sigma}\right)\right)+\frac{1}{2}\left(1-\Phi\left(\frac{\text{USL}-\mu}{\sigma}\right)\right)\right]. \tag{A.10}$$

Applying (A.6) on (A.10) leads to

$$S_{pk} = -\frac{1}{3}\Phi^{-1}\left[\frac{1}{2}\Phi\left(\frac{\text{LSL} - \mu}{\sigma}\right) + \frac{1}{2}\Phi\left(-\frac{\text{USL} - \mu}{\sigma}\right)\right]. \tag{A.11}$$

Using (6) described in Section 2, it can be shown that (A.11) is equal to the capability index C_{pA} proposed in this paper:

$$S_{pk} = -\frac{1}{3}\Phi^{-1}\left[\frac{p_A}{2}\right] = C_{pA}. \tag{A.12}$$

B. Error Function Approximation

The error function is usually encountered while integrating the normal distribution and is applied as twice the integral of the Gaussian distribution [14]:

$$\text{erf} = \frac{2}{\sqrt{\pi}}\int_0^x e^{-t^2}dt. \tag{B.1}$$

The error function has the values of 0 and 1 for $x = 0$ and $x = \infty$, respectively. Considering the standard normal equation [2]:

$$\Phi(z) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{z} e^{-\xi^2/2}d\xi, \tag{B.2}$$

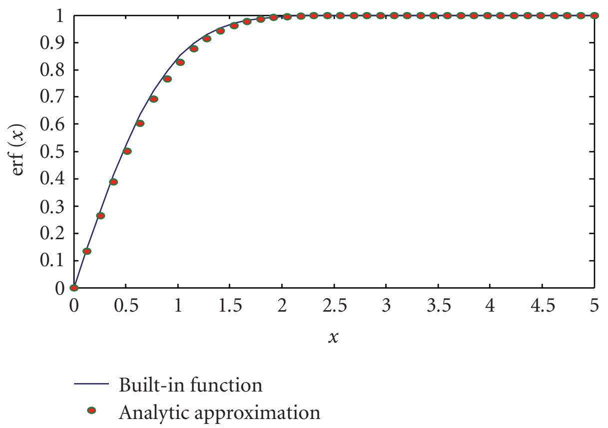

Figure 6: Comparison of built-in error function [12] and analytic approximation [9, 10].

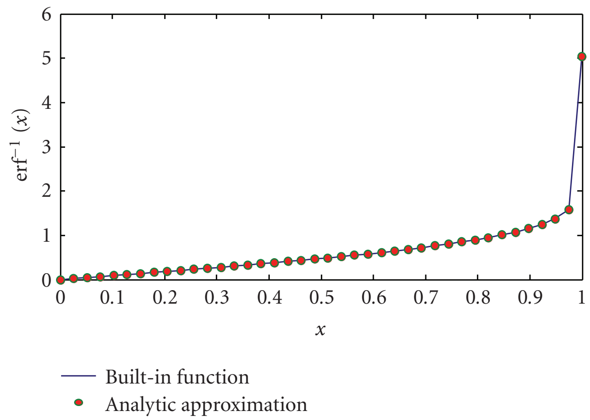

Figure 7: Comparison of built-in inverse error function [12] and analytic approximation [9, 10].

and substituting $y = \xi/2$ and separating the integral for positive and negative domains, the integral leads to

$$\Phi(z) = 0.5 + \left(\frac{1}{\sqrt{\pi}}\int_0^{z/\sqrt{2}} e^{-y^2}dy\right). \tag{B.3}$$

For negative values of z, (B.3) can be written as

$$\Phi(z) = 0.5 - \frac{1}{2}\left(\frac{2}{\sqrt{\pi}}\int_0^{|z|/\sqrt{2}} e^{-t^2}dt\right), \tag{B.4}$$

where $t = -y$. Now, using the definition of the error function in (B.1) will lead to

$$\Phi(z) = 0.5 - 0.5\,\mathrm{erf}\left(\frac{|z|}{\sqrt{2}}\right). \tag{B.5}$$

Solving z in (B.5) gives

$$z = \sqrt{2}\,\mathrm{erf}^{-1}\left(\frac{0.5-\Phi(z)}{0.5}\right), \tag{B.6}$$

where erf^{-1} is the inverse error function. Figure 6 shows the plot of the error function in (B.1) using a built-in MATLAB function. The plot also depicts a comparison of the built-in MATLAB function with the analytic approximation (13) [9, 10] described in Section 4. It can be observed from the plots that the analytic formula is, at all points, in close proximity to the built-in error function. This confirms that the analytic approximation can be substituted for the built-in function as necessary.

The inverse error function can also be implemented using the built-in MATLAB routines and can be approximated by (14). Figure 7 confirms the compatibility of the analytic formula for the inverse error function in (14) with the MATLAB built-in routine.

C. Histogram Approach to Long Term PDF Process

The heuristic gateway to the probabilistic approach for analysis is to interpret the probability density function (PDF) as being generated from the limit case of a histogram of measured process parameter values $\{y_1, y_2, \ldots, y_N\}$. Specifically, the limit is for large number of measurements, N, and infinitesimal bin size Δy. The histogram can be converted to an approximate PDF $f(y)$ at the value of process parameter y_i taken to be the center of the ith bin. The conversion rule [2] is

$$f(y_i) \approx \frac{n_i}{(N\Delta y)}, \tag{C.1}$$

where n_i is the bin count in the ith bin.

To demonstrate the linking of short-term (ST) PDFs to long-term (LT) PDFs, assume M short-term production processes are to be combined. Each of the $j = 1, 2, 3, \ldots, M$-associated ST histograms of measured values is taken to have the same number of measurements, N, and bin size Δy. The associated LT histogram is then obtained by simply summing the ST bin count numbers $(n_{ij})_{\mathrm{ST}}$, that is,

$$(n_i)_{\mathrm{LT}} = \sum_{j=1}^{M}(n_{ij})_{\mathrm{ST}}, \tag{C.2}$$

with a total requisite LT number of measurements:

$$N' = N \times M. \tag{C.3}$$

Following the rule (C.1), the long term approximate PDF is given by

$$f_{\mathrm{LT}}(y_i) \approx \frac{(n_i)_{\mathrm{LT}}}{(N'\Delta y)} = \frac{\sum_{j=1}^{M}(n_{ij})_{\mathrm{ST}}}{(NM\Delta y)} = \frac{1}{M}\sum_{j=1}^{M}\left(\frac{(n_{ij})_{\mathrm{ST}}}{N\Delta y}\right). \tag{C.4}$$

The limits of large N and infinitesimal bin size lead to (30). This confirms the intuitive proposition that the LT process parameter PDF can be constructed from the average of the PDFs for ST processes.

D. Random Variable Analysis of Long Term PDF

The steps leading to (36) main text are summarized in (D.1)–(D.9). Considering (34) and substituting (31) and (33) into (35), the PDF for the long term process is

$$f_{\text{LT}}(y) = \left(\frac{1}{\sqrt{2\pi}}\right)^2 \frac{1}{\sigma_\mu \sigma} \int_{-\infty}^{\infty} e^{-[(\mu-\overline{\mu})/2\sigma_\mu^2 - (y-\mu)^2/2\sigma^2]} d\mu. \tag{D.1}$$

The substitution

$$t = \mu - y, \tag{D.2}$$

$$g = \overline{\mu} - y \tag{D.3}$$

will facilitate completing-the-square process for the argument of the exponential in (D.1). It is found that the variance associated with the Gaussian t variable is given by

$$\sigma_t^2 = \frac{\sigma^2 \sigma_\mu^2}{\sigma^2 + \sigma_\mu^2}. \tag{D.4}$$

An additional substitution,

$$g' \equiv g\left(\frac{\sigma_t^2}{\sigma_\mu^2}\right), \tag{D.5}$$

will then lead to a more compact form given by

$$f_{\text{LT}}(g) = \left(\frac{1}{\sqrt{2\pi}}\right) \frac{\sigma_t}{\sigma_\mu \sigma} e^{-g^2/2(\sigma^2+\sigma_\mu^2)} \times \left[\frac{1}{\sqrt{2\pi}\sigma_t} \int_{-\infty}^{\infty} e^{-((t-g')^2/2\sigma_t^2)} dt\right]. \tag{D.6}$$

Additional simplifications are possible after two observations. First, the term in brackets of (D.6) is unity because it is the maximum limit of a Gaussian CDF [2]. Second, it follows from (D.4) that

$$\frac{\sigma_t}{\sigma_\mu \sigma} = \frac{1}{\sqrt{\sigma^2 + \sigma_\mu^2}} \tag{D.7}$$

Combining observations and returning to "y" dependence via (D.3), the long-term PDF can be stated as

$$f_{\text{LT}}(y) = \left(\frac{1}{\sqrt{2\pi}}\right) \frac{1}{\sigma_k} e^{-(y-\overline{\mu})^2/2\sigma_k^2}, \tag{D.8}$$

where the variance of the long-term distribution is

$$\sigma_k^2 = \sigma^2 + \sigma_\mu^2, \tag{D.9}$$

and this confirms (36) in the main text. The steps leading to (36) main text are summarized in (D.10)–(D.14). After using a transformation

$$z = \frac{\mu - \overline{\mu}}{\sigma_\mu}, \tag{D.10}$$

the Gaussian PDF in random variable μ (33) can be converted to the standard normal PDF:

$$f_z(z) = \left(\frac{1}{\sqrt{2\pi}}\right) e^{-z^2/2}, \quad -\infty \le z \le \infty. \tag{D.11}$$

Again, making use of (D.10) the expectation of $|\mu - \overline{\mu}|$ is given by

$$\text{E}\{|\mu - \overline{\mu}|\} = \int_{-\infty}^{\infty} f_\mu(\mu)|\mu - \overline{\mu}| d\mu = \sigma_\mu \int_{-\infty}^{\infty} f_z(z)|z| dz, \tag{D.12}$$

because the PDF $f_z(z)$ is by inspection (D.12) an even function. The initial integration limits in the (D.11) $(-\infty, \infty)$ can be converted to $[0, \infty)$ by including a multiplicative factor of two. After simplification of (D.12) the integration, is then given by

$$\text{E}\{|\mu - \overline{\mu}|\} = \frac{2\sigma_\mu}{\sqrt{2\pi}} \int_0^{\infty} e^{-z^2/2} z \, dz \tag{D.13}$$

because $|z| = z$ on the new domain of integration. Evaluation of (D.13) produces

$$\text{E}\{|\mu - \overline{\mu}|\} = \text{avg}\{|\mu - \overline{\mu}|\} = \sqrt{\frac{2}{\pi}} \sigma_\mu, \tag{D.14}$$

which leads to (37).

Acknowledgments

This work was supported in part by Office of Sponsored Research at The University of Texas at Tyler. The authors appreciate the helpful remarks and suggestions provided by the reviewers. They specially thank the reviewer who pointed the thread-of-logic, summarized in Appendix A, relating important components in this work with that previously appearing in the literature [5]. They are also grateful to S. Winitzki (LMU, Munich) for allowing them to use his approximate analytic models for the error function and its inverse.

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