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Numerical Solution of Grad-Shafranov Equation for the Distribution of Magnetic Flux in Nuclear Fusion Devices

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Abstract

The magnetohydrodynamic equilibrium in an axisymmetric plasma is described by the Grad-Shafranov (GS) equation in terms of the magnetic flux. Boundary element method (BEM) is suitable for plasma equilibrium since it requires the discretization of only the plasma boundary which changes shape during the operation of an actual fusion device. In this paper, numerical solutions of the GS equation are obtained by using the boundary element method, the finite element method (FEM) and the differential quadrature method (DQM) for a rectangular plasma when the source term (current density function) on the right hand side is assumed to be a monomial. Our aim is also to find the most applicable numerical procedure between those three methods for different plasma profiles. We transform the equation to the homogeneous one with a particular solution eliminating the domain integral in the BEM formulation. For the source term containing the magnetic flux (nonlinear right hand side) an iterative procedure is made use of in the BEM and FEM formulations. It is found that the FEM gives better accuracy for a D-shape tokamak plasma whereas the BEM is more suitable for a Solov'ev tokamak plasma. The solutions agree very well with the previously published numerical solutions for a rectangular plasma.

Key Words: Grad-Shafranov equation, BEM, FEM, DQM

Introduction

A plasma is an electrically conducting fluid or gas consisting totally or partially of charged particles. At high temperatures the highly ionized plasma is an excellent electrical conductor, and can be confined and shaped by strong magnetic fields. Particular plasma configurations are described in terms of solutions of the Grad-Shafranov (GS) equation. An analytical solution is not available for the most general form of the GS equation. Therefore, for a given current density function $\mu_0 r J_\varphi = J(\Psi, r)$, Ψ and r being the unknown magnetic flux function and the radial distance from the magnetic axis, respectively, the GS equation can be solved by iterative numerical methods.

One approximation for the right hand side source function is in terms of monomials, especially for the fixed boundary rectangular plasma problems, (Itagaki and Fukunaga, 2006; Itagaki et al., 2004; Itagaki et al., 2005). In this case, since the right hand side function depends only on the space variables r and z, any numerical method can directly be applied for the numerical solution of the GS equation. When the current density function is nonlinear, the GS equation can be solved iteratively as given in (Sawai et al., 1990).

In the free boundary tokamak-type device problems which are encountered in actual fusion devices, plasma changes its boundary (Leuer et al., 2001). To get accurate numerical results, the discretization of the domain should be fine or adaptive mesh techniques should be applied. Since the boundary element method requires only the discretization of the boundary, it gives accurate results with less number of discretization points. This decreases the size of the matrices and therefore the computational cost in the final system of algebraic equations although the coefficient matrix is dense and the zeros scattered arbitrarily.

In this paper, boundary element method, finite element method and the differential quadrature method have all been used to solve the GS equation to give a distribution of magnetic flux function in a rectangular plasma. Both FEM and BEM have been used in D-shape tokamak plasma and the BEM has been preferred in tokamak-like plasma because of its varying boundary. Itagaki et al. (2004), have solved GS equation for a rectangular plasma by using dual reciprocity boundary element method (DRBEM) with a fundamental solution containing elliptic integrals. We use the fundamental solution in a simple definite integral form given in (Tsuchimoto et al., 1990) and (Tezer-Sezgin and Dost, 1993), which can be calculated numerically by any standard numerical quadrature.

We transform the equation to the homogeneous form using a particular solution when the source term is a linear combination of monomials and when a particular solution is available. Then BEM is applied to the homogeneous equation with the new arranged boundary conditions. The magnetic flux function is obtained as the sum of the particular solution and the solution of the homogeneous equation.

Jardin (2004), has shown that the reduced quintic 2D triangular finite element is well-suited for many problems arising in fusion MHD applications (e.g. 2D axisymmetric toroidal equilibrium problem). We use the finite element method with linear triangular and bilinear rectangular elements in solving the GS equation for linear and non-linear right hand side expansions. Although the discretization complexity for the free boundary plasma and high computational cost, the FEM gives more accurate results, especially for non-linear cases (as in D-shape tokamak plasma).

The numerical solutions of GS equation is obtained for the rectangular plasma problem also by using the differential quadrature method which is a simple, easy to implement and computationally inexpensive domain discretization method. We have experienced that the DQM is the best method for the solution of a rectangular plasma because of its simplicity. FEM with an iterative procedure gives very good accuracy for a D-shape plasma and BEM is suitable for a Solov'ev tokamak plasma because of its boundary-only nature.

BEM Formalation of the Grad-Shafranov Equation

Grad-Shafranov equation which describes the ideal magnetohydrodynamics equilibrium of an axisymmetric toroidal plasma confined by a magnetic field is written in axisymmetric coordinate system (r, z) as (Itagaki and Fukunaga, 2006; Itagaki et al., 2004; Itagaki et al., 2005)

$$\begin{aligned} -\Delta^*\Psi &\equiv -\left\{ r\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial}{\partial r}\right) + \frac{\partial^2}{\partial z^2}\right\}\Psi \\ &= \mu_0 r^2 \frac{dP}{d\Psi} + \frac{d}{d\Psi}\left(\frac{F^2}{2}\right) \equiv \mu_0 r J_\varphi, \end{aligned} \tag{1}$$

where Ψ is the magnetic flux function, J_φ is the toroidal component of the plasma current, P is the plasma pressure, F is the poloidal current function and μ_0 is related to plasma current profile as being the magnetic permeability in the vacuum.

The boundary element method is suitable for plasma equilibrium since it discretizes only the boundary of plasma which may change its shape during the operation of an actual fusion device.

The fundamental solution Ψ^* of (1), which satisfies

$$-\Delta^*\Psi^* = r\delta_i, \tag{2}$$

δ_i being the Dirac's delta function, is given in a simple integral form which is given in (Tsuchimoto et al., 1990) and (Tezer-Sezgin and Dost, 1993) as

$$\Psi^* = \frac{ar}{4\pi}\int_0^{2\pi} \frac{\cos(\theta)}{R} d\theta, \tag{3}$$

where $R^2 = r^2 + a^2 - 2ar\cos\theta + (z-b)^2$ and the Dirac's delta function δ_i is defined as

$$\delta(x) = \begin{cases} +\infty, & x = 0 \\ 0, & x \neq 0 \end{cases}$$

and satisfy the identity

$$\int_{-\infty}^{+\infty} \delta(x)dx = 1\,.$$

The points (a,b) and (r,z) are the source (fixed) and variable (field) points, respectively. The differential operator $\Delta^*\Psi^*$ can be written as (Itagaki and Fukunaga, 2006)

$$\Delta^*\Psi = r\left(\nabla^2 - \frac{1}{r}\right)\left(\frac{\Psi}{r}\right) \tag{4}$$

in order to apply the Green's second identity more conveniently.

Multiplying equation (1) with $\frac{\Psi^*}{r^2}$ and equation (2) with $\frac{\Psi}{r^2}$, subtracting, and integrating over the domain Ω

$$\begin{aligned}&\int_\Omega \frac{1}{r}(\Psi^*\Delta^*\Psi - \Psi\Delta^*\Psi^*)d\Omega \\ &\qquad = -\int_\Omega \frac{\Psi^*}{r}\left(\mu_0 rJ_\varphi\right)d\Omega + \int_\Omega \frac{\Psi}{r}\delta_i d\Omega,\end{aligned} \tag{5}$$

and using Green's second identity as is done in (Brebbia and Dominguez, 1992), we obtain

$$\begin{aligned}&\int_\Gamma \left\{\frac{\Psi^*}{r^2}\frac{\partial\Psi}{\partial n} - \frac{\Psi}{r^2}\frac{\partial\Psi^*}{\partial n}\right\}d\Gamma \\ &\qquad = -\int_\Omega \frac{\Psi^*}{r^2}\left(\mu_0 rJ_\varphi\right)d\Omega + \int_\Omega \frac{\Psi}{r}\delta_i d\Omega,\end{aligned} \tag{6}$$

which can be reduced to

$$\begin{aligned} c_i\Psi_i &= \int_\Gamma \left\{\frac{\Psi^*}{r}\frac{\partial\Psi}{\partial n} - \frac{\Psi}{r}\frac{\partial\Psi^*}{\partial n}\right\}d\Gamma \\ &+ \int_\Omega \frac{\Psi^*}{r}\left(\mu_0 rJ_\varphi\right)d\Omega,\end{aligned} \tag{7}$$

where the constant c_i is 1 for the internal points, and 1/2 for the smooth boundary, $\partial/\partial n$ is the derivative with respect to outward normal on the boundary.

When the boundary of the region is discretized using N constant boundary elements Γ_j, the corresponding discretized matrix-vector form of equation (7) is

$$\sum_{j=1}^{N} H_{i,j}\Psi_j - \sum_{j=1}^{N} G_{i,j}q_j = Q_i \qquad i = 1,...,N, \tag{8}$$

where $q_j = \left(\frac{\partial\Psi}{\partial n}\right)_j$ and the entries of matrices and vectors are given by

$$\begin{aligned} H_{i,j} &= c_i + \int_{\Gamma_j} \frac{1}{r}\frac{\partial\Psi^*}{\partial n}d\Gamma = c_i + H_{ij}^{'} \\ G_{i,j} &= \int_{\Gamma_j} \frac{\Psi^*}{r}d\Gamma \\ Q_i &= \int_\Omega \frac{\Psi^*}{r}(\mu_0 rJ_\varphi)d\Omega \qquad i,j = 1,...,N\,. \end{aligned} \tag{9}$$

Assembly procedure for all the boundary elements and the insertion of Dirichlet boundary conditions will result in a linear system of equations

$$[G]\{\frac{\partial\Psi}{\partial n}\} = [H]\{\Psi\} - \{Q\} \tag{10}$$

to be solved for the unknown normal derivative values on the boundary nodes(mid-nodes of the boundary elements). Then the magnetic flux can be calculated at any interior point $i = (a,b)$ by taking $c_i = 1$ and using

$$\Psi_i = -\sum_{j=1}^{N} H_{ij}^{'}\Psi_j + \sum_{j=1}^{N} G_{ij}q_j + Q_i. \tag{11}$$

The domain integral Q_i in the equation (9) spoils the boundary-only nature of the boundary element matrix-vector equations in (8). There are several ways for the elimination of the domain integral when the right hand side of the Grad-Shafranov equation is assumed to be a monomial in terms of independent variables (r, z) or a polynomial in the magnetic flux function Ψ. The DRBEM application is given by using constant elements in (Itagaki et al., 2004; Itagaki et al., 2005) and by using quadratic elements in (Itagaki and Fukunaga, 2006) for monomial sources. In their study, it brings some more matrix-vector multiplications in the right hand side of the equation (8) instead of the vector Q_i. These new vectors now contain particular solution values on the boundary and interior points. Instead, we transform the GS equation to a homogeneous equation by using a particular solution for the case of monomial source.

The particular solution $\varphi^{(l,m)}$ that satisfies the GS equation with a monomial source $-\Delta^\star\varphi^{(l,m)} = r^l z^m$ is given by an infinite series (Itagaki et al., 2004):

$$\varphi^{(l,m)} = -\frac{r^l z^{m+2}}{(m+1)(m+2)}\{1+ \sum_{k=1}^{\infty}\prod_{s=1}^{k}\left[-\frac{(l-2s+2)(l-2s)}{(m+2s+1)(m+2s+2)}\frac{z^2}{r^2}\right]\} . \tag{12}$$

We define a new variable $\hat{\Psi}$

$$\hat{\Psi} = \Psi - \varphi^{(l,m)} \tag{13}$$

which satisfies the homogeneous equation

$$-\Delta^*\hat{\Psi} = 0 \tag{14}$$

with the boundary conditions

$$\hat{\Psi}_{BC} = \Psi_{BC} - \varphi^{(l,m)}_{BC}. \tag{15}$$

BEM formulation of the equation (14) with the boundary conditions (15) will result in a system of discretized equations (8) with a zero vector on the right hand side. $\hat{\Psi}$ is the solution of this homogeneous system and then the magnetic flux function Ψ will be calculated as a sum of homogeneous solutions and a particular solution:

$$\Psi = \hat{\Psi} + \varphi^{(l,m)}. \tag{16}$$

This computation is much cheaper than the computation of domain integrals in (8).

When the right hand side $\mu_0 r J_\varphi$ of the GS equation is assumed to be a polynomial F in Ψ(non-linear function of Ψ) an iteration can be imposed as

$$-\Delta^*\Psi^{(n+1)} = F(r,z,\Psi^{(n)}). \tag{17}$$

Starting value $\Psi^{(0)}$ will be guessed with the related boundary conditions and the solution is obtained after a finite number of iterations with a prescribed tolerance.

FEM Formulation of the Grad-Shafranov Equation

Rewriting the Grad-Shafranov equation as

$$\begin{aligned} -\left\{\frac{\partial^2\Psi}{\partial r^2} + \frac{\partial^2\Psi}{\partial z^2} - \frac{1}{r}\frac{\partial\Psi}{\partial r}\right\} &= F(\Psi,r,z) \\ &= \mu_0 r J_\varphi, \end{aligned} \tag{18}$$

we can apply Galerkin FEM procedure (Reddy, 1993) with the approximation for Ψ:

$$\Psi(r,z) = \sum_{j=1}^{n_{\mathrm{dof}}} \Psi_j N_j(r,z) \tag{19}$$

where n_{dof} is the number of degree of freedom for an element and N_j's are the shape functions. Then, we multiply the equation (18) with the shape functions N_i's and integrate over the domain (elements) Ω^e:

$$\begin{aligned} &-\sum_{j=1}^{n_{\mathrm{dof}}}\int_{\partial\Omega^e}\left(\frac{\partial^2 N_j}{\partial r^2}N_i + \frac{\partial^2 N_j}{\partial z^2}N_i\right)d\Omega \\ &+\sum_{j=1}^{n_{\mathrm{dof}}}\int_{\Omega^e}\frac{1}{r}\frac{\partial N_j}{\partial r}N_i d\Omega \\ &= \int_{\Omega^e} F(\Psi,r,z)N_i d\Omega, \end{aligned} \tag{20}$$

where $i = 1,..,n_{\mathrm{dof}}$. Applying Green's identity to the first term we get

$$\begin{aligned} &-\sum_{j=1}^{n_{\mathrm{dof}}}\int_{\partial\Omega^e}\frac{\partial N_j}{\partial n}N_i ds \\ &+\sum_{j=1}^{n_{\mathrm{dof}}}\int_{\Omega^e}\left(\frac{\partial N_j}{\partial r}\frac{\partial N_i}{\partial r} + \frac{\partial N_j}{\partial z}\frac{\partial N_i}{\partial z}\right)d\Omega \\ &+\sum_{j=1}^{n_{\mathrm{dof}}}\int_{\Omega^e}\frac{1}{r}\frac{\partial N_j}{\partial r}N_i d\Omega \\ &= \int_{\Omega^e} F(\Psi,r,z)N_i d\Omega. \end{aligned} \tag{21}$$

Then the following element matrix equation is obtained:

$$\sum_{i=1}^{n_{\mathrm{dof}}}\sum_{j=1}^{n_{\mathrm{dof}}} K_{i,j}\Psi_j = \sum_{i=1}^{n_{\mathrm{dof}}} F_i,$$

where

$$\begin{aligned} K_{i,j} = &-\int_{\partial\Omega^e}\frac{\partial N_j}{\partial n}N_i ds \\ &+\int_{\Omega^e}\left\{\frac{\partial N_j}{\partial r}\frac{\partial N_i}{\partial r} + \frac{\partial N_j}{\partial z}\frac{\partial N_i}{\partial z} + \frac{1}{r}\frac{\partial N_j}{\partial r}N_i\right\}d\Omega \end{aligned}$$

$$F_i = \int_{\Omega^e} F(\Psi,r,z)N_i d\Omega = \int_{\Omega^e}\mu_0 r J_\varphi d\Omega,$$

and the boundary integral in K_{ij} takes the value zero for Dirichlet type boundary conditions.

Assembly procedure and the insertion of Dirichlet boundary conditions will give the final system of equations to be solved for Ψ as

$$[K]\{\Psi\} = \{F\}. \tag{22}$$

If the function $F(\Psi,r,z)$ is nonlinear, an iterative procedure similar to equation (17) can be used. As in the BEM procedure for the non-linear case, initial values of Ψ are guessed, for the iteration process, from the given boundary conditions.

Then the iteration is repeated until the condition $\|\Psi^{n+1} - \Psi^n\| < TOL$ is satisfied for a preassigned tolerance TOL. For the solution of the resulting linear system of equations a solver which is suitable for sparse systems is used.

DQM Formulation of the Grad-Shafranov Equation

The polynomial based differential quadrature method, (Shu, 2000), which is simple to apply and uses considerably small number of grid points for a prescribed accuracy, is also applied to solve GS equation for obtaining the magnetic flux function Ψ when the right hand side is assumed to be monomials.

The differential quadrature method approximates the derivative of a smooth function at a grid point by a linear weighted summation of all the functional values in the whole computational domain. Suppose the degree of the polynomials approximating the unknown Ψ are M_r and M_z in r and z directions, respectively, then the second order derivatives can be approximated at a grid point (r_i, z_j) by the polynomial based differential quadrature approach as

$$\Psi_{rr}(r_i, z_j) = \sum_{k=1}^{M_r} b_r(i,k)\Psi(r_k, z_j),$$
$$\Psi_{zz}(r_i, z_j) = \sum_{k=1}^{M_z} b_z(j,k)\Psi(r_i, z_k).$$

Therefore, the DQM formulation of the GS equations (18) becomes for $\mu_0 r J_\varphi = r^l z^m$

$$\begin{aligned}\sum_{k=1}^{M_r} b_r(i,k)\Psi_{kj} + \sum_{k=1}^{M_z} b_z(j,k)\Psi_{ik} \\ -\frac{1}{r_i}\sum_{k=1}^{M_r} a_r(i,k)\Psi_{kj} = {r_i}^l {z_j}^m \\ i = 1, ..., M_r \ , \ j = 1, ..., M_z,\end{aligned} \tag{23}$$

where $\Psi_{ik} = \Psi(r_i, z_k)$, M_r and M_z are the number of grid points, $b_r(i,k)$ and $b_z(j,k)$ are the weighting coefficients for the second order derivatives, and $a_r(i,k)$ and $a_z(j,k)$ are the weighting coefficients for the first order derivatives of Ψ in r and z-directions, respectively. Explicit forms of these coefficients are given in (Shu, 2000) as

$$a_r(i,k) = \frac{M(r_i)}{(r_i - r_k)M(r_k)} \qquad i \neq k$$

$$a_r(i,i) = -\sum_{\substack{k=1\\k\neq i}}^{M_r} a_r(i,k) \qquad i,k = 1, ..., M_r$$

$$a_z(j,k) = \frac{M(z_j)}{(z_j - z_k)M(z_k)} \qquad j \neq k;$$

$$a_z(j,j) = -\sum_{\substack{k=1\\k\neq j}}^{M_z} a_z(j,k) \qquad j,k = 1, ..., M_z$$

$$b_r(i,k) = 2a_r(i,k)\left[a_r(i,i) - \frac{1}{r_i - r_k}\right], \ i \neq k$$

$$b_r(i,i) = -\sum_{\substack{k=1\\k\neq i}}^{M_r} b_r(i,k) \qquad i,k = 1, ..., M_r$$

$$b_z(j,k) = 2a_z(j,k)\left[a_z(j,j) - \frac{1}{z_j - z_k}\right], \ j \neq k$$

$$b_z(j,j) = -\sum_{\substack{k=1\\k\neq j}}^{M_z} b_z(j,k) \qquad j,k = 1, ..., M_z$$

where

$$M(r_k) = \prod_{\substack{j=1\\j\neq k}}^{M_r} (r_k - r_j)$$

$$M(z_k) = \prod_{\substack{j=1\\j\neq k}}^{M_z} (z_k - z_j).$$

The linear system of algebraic equations resulting from the discretized equations (23) can be written in matrix-vector form for the unknown vector Ψ after the insertion of the given boundary conditions as

$$[A]\{\Psi\} = \{B\},$$

where the entries of the matrix $[A]$ are formed from the weighting coefficients shown on the equation (23) and the vector $\{B\}$ has coefficients $B_{ij} = r_i^l z_j^m$.

A natural choice for the grid points is that of equally spaced points but a nonuniform grid with

Chebyshev-Gauss-Lobatto points delivers more accurate solutions, (Shu and Richards, 1992). The matrix $[A]$ contains many zero elements which are irregularly distributed. Thus, it is still considered as a full matrix and may be stored as a whole. There is a storage problem when the number of mesh points is increased but the DQM has the advantage of using considerably small number of mesh points for giving very good accuracy. Also, the ordering of the unknown vector is important and one should carefully select the order of elements of $\{\Psi\}$ to get a well-conditioned coefficients matrix $[A]$, (Tezer-Sezgin, 2004).

Numerical Results

In this section, solutions of the GS equation for the fixed boundary (rectangular plasma) and the free boundary (tokamak-type devices) cases are presented in terms of magnetic flux function contours. The variables r and z, which denote distances, are taken in meters (m) and the magnetic flux unit is in Webers (Wb). BEM, FEM and DQM are all applied in the case of rectangular plasma where the nonhomogeneity is assumed to be monomials. BEM and FEM results are given for D-shape tokamak plasma where the non-homogeneity is nonlinear in the unknown magnetic flux function. In this case the iteration procedure is made use of assuming the solution is known at the previous step. Either the right hand side vector is calculated in the resulting linear system of equations (equations (8) or (22)) or with the help of particular solution (whenever available) the homogeneous system is solved. Only BEM solution of tokamak plasma given by Solov'ev is presented because of the advantages of the method which is suitable for the irregular boundary.

Rectangular plasma

In a hypothetical rectangular plasma with $0.5\,\mathrm{m} \leq r \leq 1.5\,\mathrm{m}$, $-0.5\,\mathrm{m} \leq z \leq 0.5\,\mathrm{m}$, the GS-equation is solved with a monomial source term 1, r^2z^3 and r^3z^2 for $\mu_0 r J_\varphi$. The boundary condition $\Psi = 0$ is imposed along each side of the rectangle. This problem with the source r^3z^2 has been solved with DRBEM in (Itagaki et al., 2004) using the fundamental solution in the elliptic integrals form. In this study, the problem is solved by transforming the GS equation to homogeneous equation using a particular solution and applying BEM with fundamental solution in definite integral form, (Tezer-Sezgin and Dost, 1993). This fundamental solution can be computed by any numerical quadrature (e.g. Gauss-Legendre). Thus, the complexity of computations is eliminated and the computational time is decreased. In the BEM application, $N = 80$ constant boundary elements are used. The resulting linear system of equations has the size 80×80, only. The coefficient matrix contains scattered zeros and does not show a sparse behavior but still the computational cost is small since the size is small. Figures 1, 2, 3 represent contours of magnetic flux function Ψ obtained by using BEM.

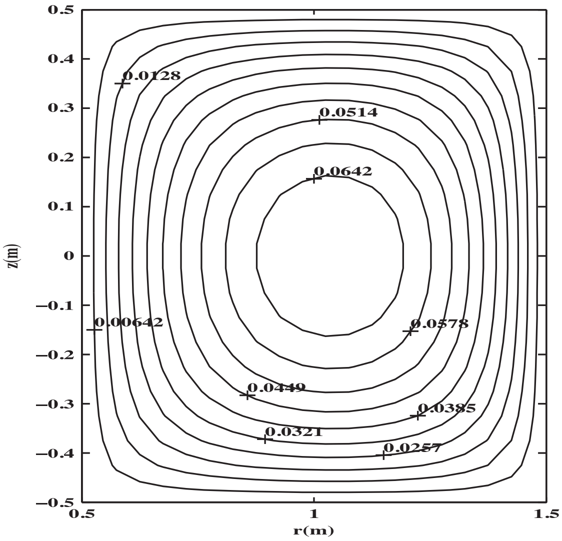

Figure 1. $\Psi(Wb)$ for rectangular plasma with BEM for $\mu_0 r J_\varphi = 1$

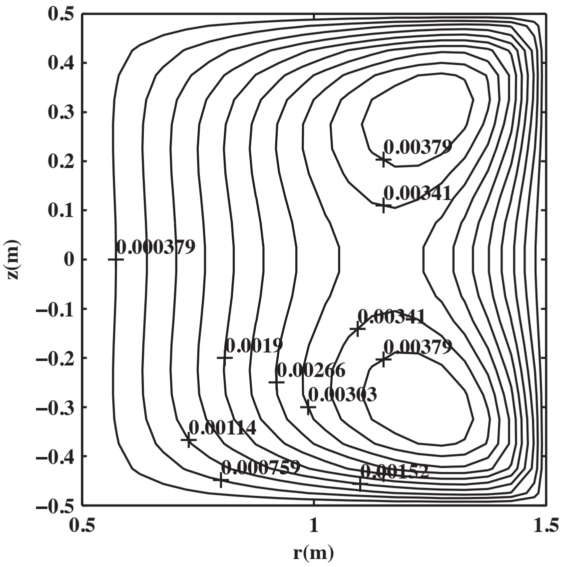

Figure 2. $\Psi(Wb)$ for rectangular plasma with BEM for $\mu_0 r J_\varphi = r^3z^2$

The same problem has been solved with finite element method by using both linear triangular and bilinear rectangular elements with 21×21 points which corresponds to 800 triangular elements and 400 rectangular elements, respectively. Figures 4, 5, 6 represents the FEM results in terms of Ψ contours for triangular elements since the same accuracy is obtained for both type of the elements.

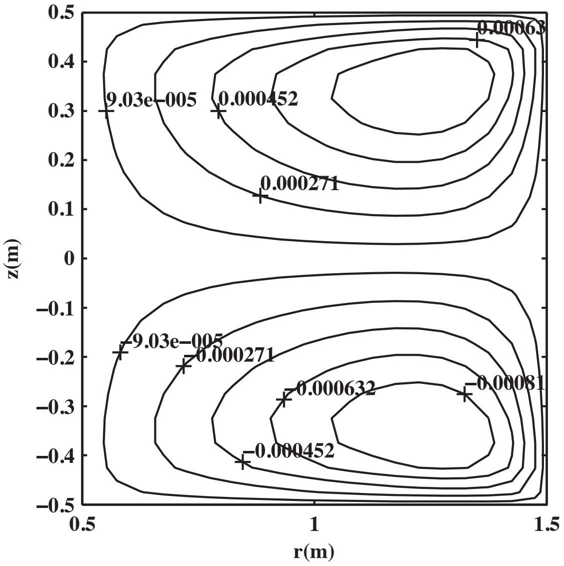

Figure 3. $\Psi(Wb)$ for rectangular plasma with BEM for $\mu_0 r J_\varphi = r^2 z^3$

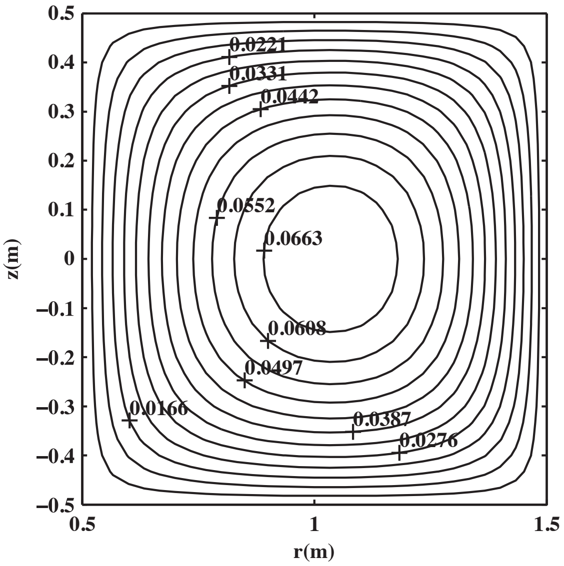

Figure 4. $\Psi(Wb)$ for rectangular plasma with FEM for $\mu_0 r J_\varphi = 1$

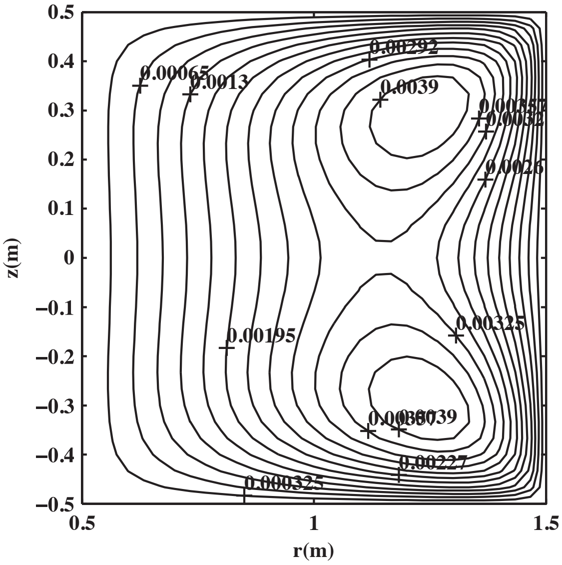

Figure 5. $\Psi(Wb)$ for rectangular plasma with FEM for $\mu_0 r J_\varphi = r^3 z^2$

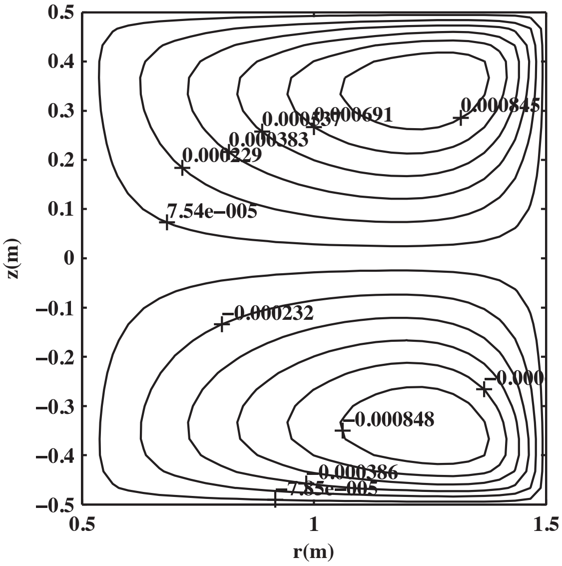

Figure 6. $\Psi(Wb)$ for rectangular plasma with FEM for $\mu_0 r J_\varphi = r^2 z^3$

Magnetic flux function is obtained also by using DQM, which is simple, easy to implement and computationally inexpensive domain discretization method (Shu, 2000). The aim is to compare the results obtained by BEM and FEM. The magnetic flux function Ψ for the same sources 1, $r^3 z^2$ and $r^2 z^3$ is obtained with almost the same accuracy by using DQM with only $M_r = M_z = 20$ grid points and the contours are presented in Figures 7, 8, 9. One may notice that the DQM results in a 441×441

linear system of equations whereas the BEM requires only a 80×80 and the FEM 441×441 linear system of equations to be solved with the dicretizations used in DQM, BEM and FEM, respectively. Since in the BEM formulation the evaluation of integrals are required, the DQM is still preferred in terms of computational cost.

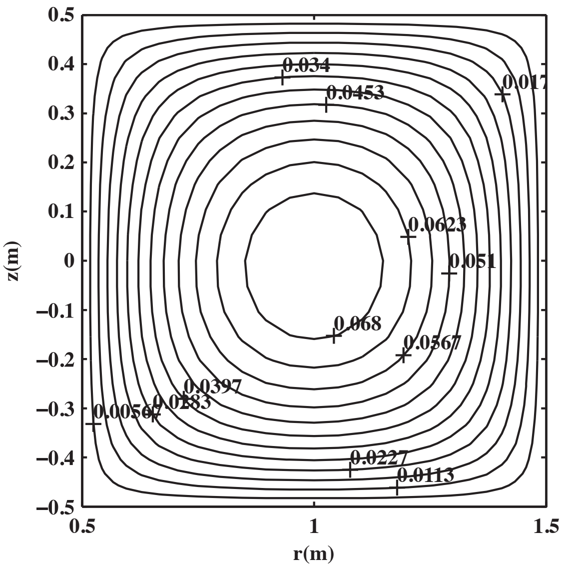

Figure 7. $\Psi(Wb)$ for rectangular plasma with DQM for $\mu_0 r J_\varphi = 1$

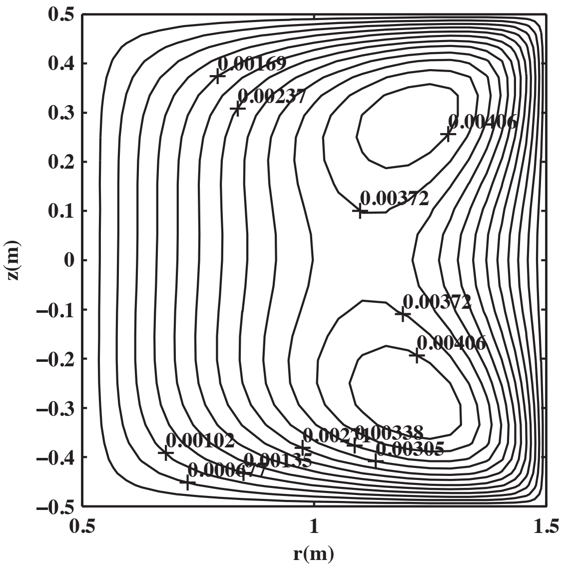

Figure 8. $\Psi(Wb)$ for rectangular plasma with DQM for $\mu_0 r J_\varphi = r^3 z^2$

Although DQM gives quite good accuracy with minimal computational cost compared to BEM and FEM results, it is suitable only for rectangular regions. For tokamak-type devices, D-shape and Solov'ev tokamak plasmas, the FEM and BEM are preferred, respectively.

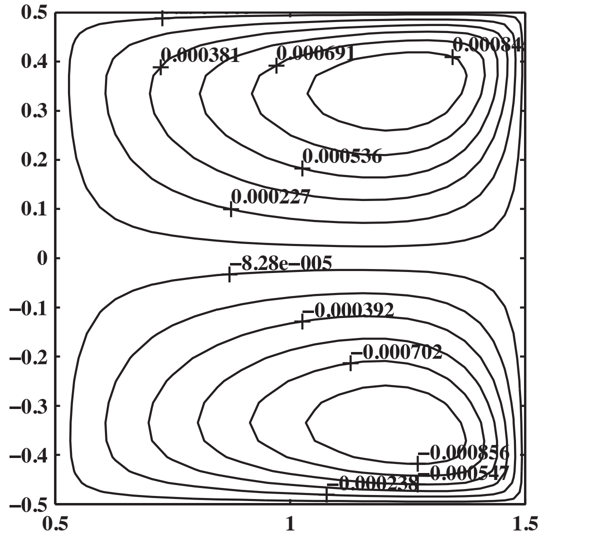

Figure 9. $\Psi(Wb)$ for rectangular plasma with DQM for $\mu_0 r J_\varphi = r^2 z^3$

D-Shape tokamak plasma

Another problem in modeling a tokamak-type plasma in D-shape is also solved iteratively when the right hand side function is taken as $\mu_0 r J_\varphi = 0.1(1-0.70199r^2)(1-\Psi)^{0.6}$ and the constant parameter values are used as in the reference (Itagaki et al., 2004). The problem is solved by applying both BEM and FEM methods in $[0.5\,\mathrm{m}, 1.5\,\mathrm{m}] \times [-1\,\mathrm{m}, 1\,\mathrm{m}]$ with homogeneous boundary conditions. Figure 10 shows the contours of magnetic flux function Ψ for this D-shape tokamak plasma. Since the right hand side is nonlinear an iterative procedure as in (17) is used. The use of FEM is more advantageous than the use of BEM in considering the rate of convergence with a tolerance of 10^{-9}.

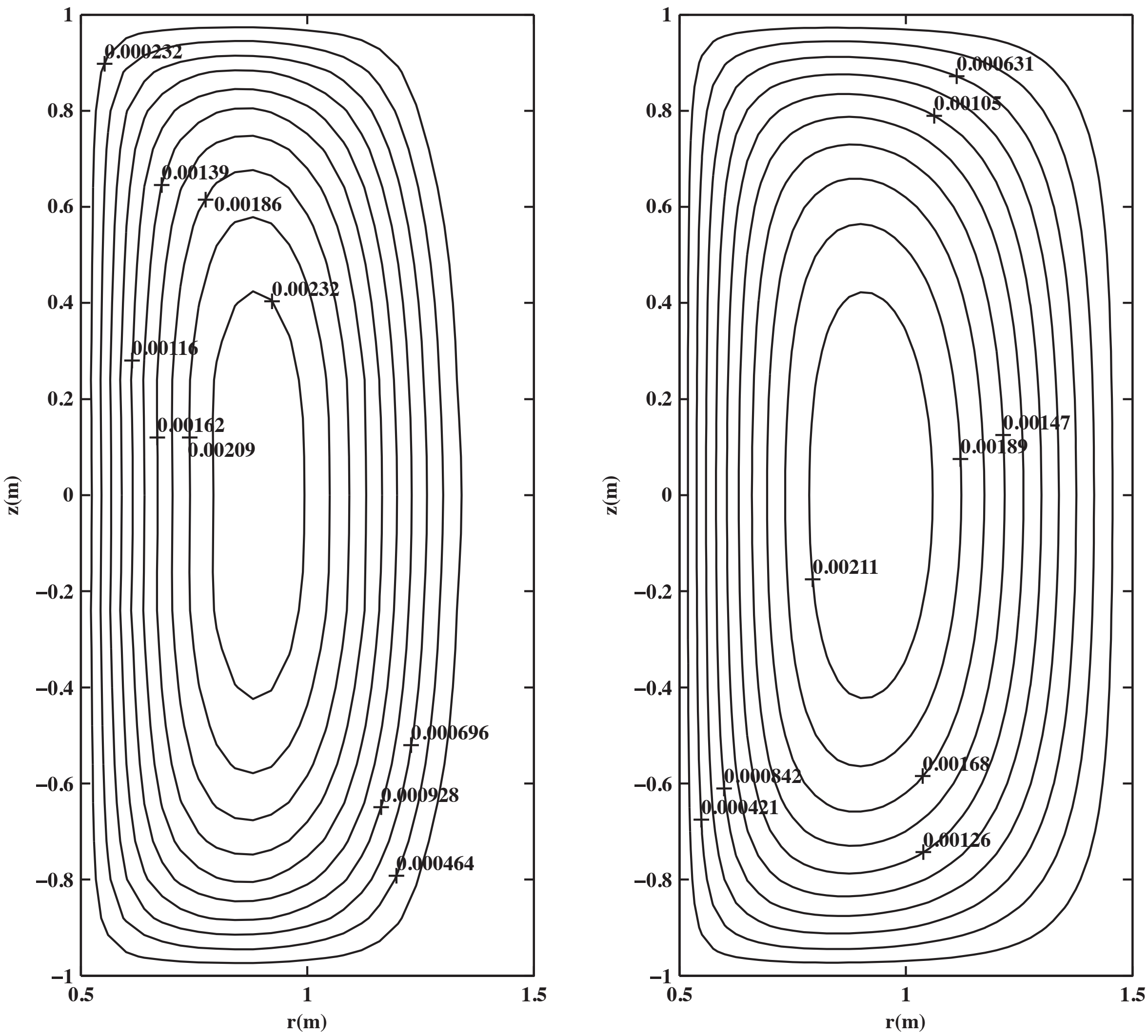

Figure 10. $\Psi(Wb)$ for D-shape tokamak plasma (BEM and FEM)

Solov'ev tokamak plasma

Leuer et al. (Leuer et al., 2001) defined the Solov'ev tokamak plasma with the current density function

$$\mu_0 r J_\varphi = f_0\left(r^2 + {r^2}_0\right)$$

with the boundary values $r_b = r_0\sqrt{1 + \dfrac{2a\cos\alpha}{r_0}}$ and $z_b = ar_0 \sin\alpha$ where $\alpha = 0 : 2\pi$, $f_0 = 1$, $r_0 = 1$, $a = 0.5$. The exact solution is given as (Leuer et al., 2001)

$$\Psi_{\text{exact}} = \frac{f_0 {r_0}^2 a^2}{2}\left[1 - \left(\frac{z}{a}\right)^2 - \left(\frac{r - r_0}{a} + \frac{(r - r_0)^2}{2ar_0}\right)^2\right]$$

with the Dirichlet type boundary condition $\Psi = 0$.

The problem has been solved with FEM in (Leuer et al., 2001) using 3200 adaptive grid points, which is quite complex for the discretization and computations. We have solved the same problem with BEM using only 200 boundary elements and transforming the equation to the homogeneous equation using the particular solution $\varphi_p = \dfrac{r^4}{8} + \dfrac{z^2}{2}$. Results are compared with the exact solution and represented in terms of contours in Figure 11.

It can be seen that the BEM is more advantageous to use in this problem concerning the sizes of the resulting algebraic system of equations.

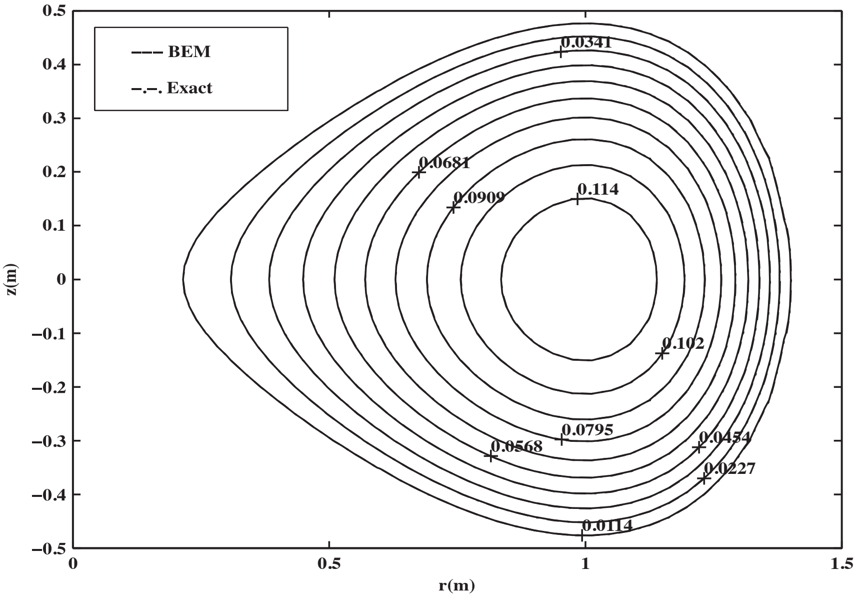

Figure 11. $\Psi(Wb)$ for the tokamak plasma(Solov'ev's GS Equation) with BEM and Exact solution.

Conclusion

In this paper, BEM, FEM and DQM methods have been discussed in solving GS equation numerically. GS equation describes magnetohydrodynamic equilibria of axisymmetric plasmas such as those produced in tokamak experiments. These three numerical methods are compared for different plasma profiles. The results obtained from all the procedures for a rectangular plasma are in good agreement with the results obtained by (Itagaki et al., 2004) . This demonstrates the validity of the presented methods, especially when the monomial approximations are used for the source function $\mu_0 r J_\varphi$. Because of the simplicity and small computational cost, the DQM is preferred for the rectangular plasma. For a non-linear source function, iterative procedure together with the FEM is very efficient in obtaining solutions for D-shape tokamak plasma when the rate of the convergence is concerned whereas iterative procedure with BEM is more suitable for Solov'ev tokamak plasma when the size of the final discretized systems are considered.

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