Simultaneous Stabilization for a Collection of Multi-input Nonlinear Systems with Uncertain Parameters

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Abstract Simultaneous stabilization for a collection of multi-input nonlinear systems with uncertain parameters is dealt with in this paper. A systematic method for obtaining a control Lyapunov function (CLF) is presented by solving the Lyapunov equation. A sufficient condition that a quadratic CLF is a common CLF for these systems is acquired. A continuous state feedback is designed to simultaneously stabilize these systems. Finally, the effectiveness of the proposed scheme is illustrated by a simulation example.

Key words Nonlinear systems, simultaneous stabilization, control Lyapunov functions (CLF), state feedback

The stability analysis and controller design for uncertain systems have been one of the main focuses in the field of control. The simultaneous stabilization problem is important in practice. This problem is concerned with determining a single controller which simultaneously stabilizes a finite collection of systems. Petersen[1] obtained a nonlinear state feedback controller that quadratically stabilized a set of single-input linear systems simultaneously. Schmitendorf[2] acquired a sufficient condition for the existence of a stabilizing linear state feedback for a collection of single-input linear systems. Miller[3−4] employed linear periodically time-varying controllers for the simultaneous stabilization and disturbance rejection for a set of linear systems. For nonlinear systems, the simultaneous stabilization problem is more difficult to solve. Ho-Mock-Qai[5] presented some relevant results for nonlinear systems. Wu[6] provided a method for designing a controller that simultaneously stabilizes a collection of single-input nonlinear systems based on the control Lyapunov function (CLF). The concept of the CLF introduced by Artstein[7] and Sontag[8], made tremendous impact on stabilization theory. It converted stability descriptions into tools for solving stabilization tasks. Sontag[9] presented a universal feedback scheme by using the CLF. The CLF has been widely adopted in various design problems[6−12].

Simultaneous stabilization for a collection of multi-input nonlinear systems with uncertain parameters is dealt with in this paper. A systematic method for obtaining a common CLF is presented. A continuous state feedback is designed to simultaneously stabilize these systems based on the common CLF. Finally, the effectiveness of the proposed scheme is illustrated by a simulation example.

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1 System description and preliminaries

Let us consider a collection of q nonlinear systems with uncertain parameters described by

$$\dot{\boldsymbol{x}} = A\boldsymbol{x} + B[\boldsymbol{F}_s(\boldsymbol{x},\boldsymbol{d}) + G_s(\boldsymbol{x},\boldsymbol{d})\boldsymbol{u}], \quad s = 1,2,\cdots,q \tag{1}$$

where $\boldsymbol{x} \in \mathbf{R}^n$ and $\boldsymbol{u} \in \mathbf{R}^m$ are the states and the inputs, respectively, and $\boldsymbol{d}$ is the vector of uncertain parameters. We assume that $\boldsymbol{d}$ varies in a compact set Ω. For $s = 1,2,\cdots,q$, denote $\boldsymbol{F}_s(\boldsymbol{x},\boldsymbol{d}) = [\ f_{s1}(\boldsymbol{x},\boldsymbol{d}) \quad f_{s2}(\boldsymbol{x},\boldsymbol{d}) \quad \cdots \quad f_{sl}(\boldsymbol{x},\boldsymbol{d})\ ]^{\mathrm{T}}$ and $G_s(\boldsymbol{x},\boldsymbol{d}) = (g_{sij}(\boldsymbol{x},\boldsymbol{d}))_{l\times m}$. The functions $f_{si}(\boldsymbol{x},\boldsymbol{d})$ and $g_{sij}(\boldsymbol{x},\boldsymbol{d})$ are sufficiently differentiable for their variables $\boldsymbol{x}$ and $\boldsymbol{d}$. Moreover, we assume that $f_{si}(\boldsymbol{0},\boldsymbol{d}) = 0$, $\boldsymbol{d} \in \Omega$, and $\mathrm{rank}G_s(\boldsymbol{x},\boldsymbol{d}) = l$ for each $\boldsymbol{x} \in \mathbf{R}^n$ and $\boldsymbol{d} \in \Omega$. The matrices in (1) take the following canonical forms as

$$A = \begin{bmatrix} A_1 & 0 & \cdots & 0 \\ 0 & A_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & A_l \end{bmatrix}, \ A_i = \begin{bmatrix} 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \\ 0 & 0 & \cdots & 0 \end{bmatrix}_{r_i\times r_i}$$

$$B = \begin{bmatrix} B_1 & 0 & \cdots & 0 \\ 0 & B_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & B_l \end{bmatrix}, \ \boldsymbol{B}_i = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}_{r_i\times 1}$$

where $r_1 + r_2 + \cdots + r_l = n$.

Definition 1. A differentiable, radially unbounded, and positive definite function V is a common CLF for systems (1), if for any $\boldsymbol{x} \in \mathbf{R}^n/\boldsymbol{0}$, and $s = 1,2,\cdots,q$,

$$\inf_{\boldsymbol{u}} \sup_{\boldsymbol{d}\in\Omega} \frac{\partial V}{\partial \boldsymbol{x}}\{A\boldsymbol{x} + B[\boldsymbol{F}_s(\boldsymbol{x},\boldsymbol{d}) + G_s(\boldsymbol{x},\boldsymbol{d})\boldsymbol{u}]\} < 0 \tag{2}$$

The condition that $V(\boldsymbol{x})$ is a common CLF of (1) is precisely the statement that

$$\frac{\partial V}{\partial \boldsymbol{x}} BG_s(\boldsymbol{x},\boldsymbol{d}) = 0,\ \boldsymbol{x} \neq \boldsymbol{0} \Rightarrow \frac{\partial V}{\partial \boldsymbol{x}}(A\boldsymbol{x} + B\boldsymbol{F}_s(\boldsymbol{x},\boldsymbol{d})) < 0 \\ s = 1,2,\cdots,q \tag{3}$$

The objective of this paper is to show that we can find a common CLF for the collection of systems (1) if $\boldsymbol{F}_s(\boldsymbol{x},\boldsymbol{d})$ and $G_s(\boldsymbol{x},\boldsymbol{d})$ hold some properties. Moreover, we intend to find a determined state feedback which can globally asymptotically stabilize the collection of systems (1) simultaneously.

2 Main results

Consider the collection of systems (1). Divide A_i and $\boldsymbol{B}_i$ into their block forms as

$$A_i = \begin{bmatrix} A_{i-1} & \boldsymbol{A}_{i2} \\ \boldsymbol{0} & 0 \end{bmatrix}, \quad \boldsymbol{B}_i = \begin{bmatrix} \boldsymbol{0} \\ 1 \end{bmatrix}$$

where

$$A_{i-1} = \begin{bmatrix} 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \\ 0 & 0 & \cdots & 0 \end{bmatrix}, \quad \boldsymbol{A}_{i2} = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}$$

Assume that $\beta_{i1}, \beta_{i2}, \cdots$, and β_{i,r_i-1} are the coefficients of the following polynomial

$$\lambda_i(\beta) = \lambda^{r_i-1} + \beta_{i,r_i-1}\lambda^{r_i-2} + \cdots + \beta_{i2}\lambda + \beta_{i1} \tag{4}$$

Let $P_i = \begin{bmatrix} P_{r_i-1} & \boldsymbol{P}_{i2} \\ \boldsymbol{P}_{i2}^{\rm T} & p_{i3} \end{bmatrix}$, $i = 1, 2, \cdots, l$ be symmetric matrices, where $P_{r_i-1} \in \mathbf{R}^{(r_i-1)\times(r_i-1)}$, $\boldsymbol{P}_{i2} \in \mathbf{R}^{r_i-1}$, and $p_{i3} \in \mathbf{R}$. It is required that $p_{i3}^{-1}\boldsymbol{P}_{i2}^{\rm T} = \begin{bmatrix} \beta_{i1} & \beta_{i2} & \cdots & \beta_{i,r_i-1} \end{bmatrix}$.

Then,

$$A_{i-1} - \boldsymbol{A}_{i2}p_{i3}^{-1}\boldsymbol{P}_{i2}^{\rm T} = C_{i\beta} = \begin{bmatrix} 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \\ -\beta_{i1} & -\beta_{i2} & \cdots & -\beta_{i,r_i-1} \end{bmatrix}$$

Let P_i, for each $i = 1, 2, \cdots, l$ satisfy the following hypotheses:

H 1. $p_{i3} > 0$ and $\lambda_i(\beta)$ is a Hurwitz polynomial.

H 2. $(P_{r_i-1} - p_{i3}^{-1}\boldsymbol{P}_{i2}\boldsymbol{P}_{i2}^{\rm T})C_{i\beta} + C_{i\beta}^{\rm T}(P_{r_i-1} - p_{i3}^{-1}\boldsymbol{P}_{i2}\boldsymbol{P}_{i2}^{\rm T})$ is negative definite.

Denote

$$P = \begin{bmatrix} P_1 & & & \\ & P_2 & & \\ & & \ddots & \\ & & & P_l \end{bmatrix}, \quad V(\boldsymbol{x}) = \boldsymbol{x}^{\rm T}P\boldsymbol{x} \tag{5}$$

The following theorem shows that $V(\boldsymbol{x}) = \boldsymbol{x}^{\rm T}P\boldsymbol{x}$ is a common CLF for the collection of multi-input systems (1).

Theorem 1. $V(\boldsymbol{x}) = \boldsymbol{x}^{\rm T}P\boldsymbol{x}$ is a common CLF for the collection of systems (1) if P_i, for each $i = 1, 2, \cdots, l$ satisfies hypotheses H 1 and H 2.

Proof. If P_i, for each $i = 1, 2, \cdots, l$ satisfies hypotheses H 1 and H 2, then it is easy to deduce that P_i is positive definite. It follows that $V(\boldsymbol{x}) = \boldsymbol{x}^{\rm T}P\boldsymbol{x}$ is positive definite.

Let $\phi_s(\boldsymbol{x}, \boldsymbol{d}) = A\boldsymbol{x} + B\boldsymbol{F}_s(\boldsymbol{x}, \boldsymbol{d})$ and $\varphi_s(\boldsymbol{x}, \boldsymbol{d}) = BG_s(\boldsymbol{x}, \boldsymbol{d})$. Since $(G_s(\boldsymbol{x}, \boldsymbol{d}))$ is of full row rank for each $\boldsymbol{x} \in \mathbf{R}^n$ and $\boldsymbol{d} \in \Omega$, we have

$$\frac{\partial V}{\partial \boldsymbol{x}}\varphi_s(\boldsymbol{x}, \boldsymbol{d}) = 2\boldsymbol{x}^{\rm T}PBG_s(\boldsymbol{x}, \boldsymbol{d}) = \boldsymbol{0} \Leftrightarrow \boldsymbol{x}^{\rm T}PB = \boldsymbol{0} \tag{6}$$

Owing to

$$\begin{aligned} \frac{\partial V}{\partial \boldsymbol{x}}\phi_s(\boldsymbol{x}, \boldsymbol{d}) = &(A\boldsymbol{x} + B\boldsymbol{F}_s(\boldsymbol{x}, \boldsymbol{d}))^{\rm T}P\boldsymbol{x} + \boldsymbol{x}^{\rm T}P(A\boldsymbol{x} + \\ &B\boldsymbol{F}_s(\boldsymbol{x}, \boldsymbol{d})) = \boldsymbol{x}^{\rm T}(A^{\rm T}P + PA)\boldsymbol{x} + \\ &\boldsymbol{F}_s^{\rm T}(\boldsymbol{x}, \boldsymbol{d})B^{\rm T}P\boldsymbol{x} + \boldsymbol{x}^{\rm T}PB\boldsymbol{F}_s(\boldsymbol{x}, \boldsymbol{d}) \end{aligned} \tag{7}$$

by (6) and (7), we have

$$\begin{aligned} &\frac{\partial V}{\partial \boldsymbol{x}}\varphi_s(\boldsymbol{x}, \boldsymbol{d}) = \boldsymbol{0}, \boldsymbol{x} \neq \boldsymbol{0} \Rightarrow \boldsymbol{x}^{\rm T}PB = \boldsymbol{0}, \boldsymbol{x} \neq \boldsymbol{0} \\ &\Rightarrow \frac{\partial V}{\partial \boldsymbol{x}}\phi_s(\boldsymbol{x}, \boldsymbol{d}) = \boldsymbol{x}^{\rm T}(A^{\rm T}P + PA)\boldsymbol{x} \end{aligned} \tag{8}$$

Using block matrix to express $\boldsymbol{x}^{\rm T}$, we have

$$\boldsymbol{x}^{\rm T} = \begin{bmatrix} \boldsymbol{x}_1^{\rm T} & \boldsymbol{x}_2^{\rm T} & \cdots & \boldsymbol{x}_l^{\rm T} \end{bmatrix}, \quad \boldsymbol{x}_i^{\rm T} = \begin{bmatrix} \boldsymbol{X}_{i,r_i-1}^{\rm T} & x_{i,r_i} \end{bmatrix}$$
$$\boldsymbol{X}_{i,r_i-1}^{\rm T} = \begin{bmatrix} x_{i1} & x_{i2} & \cdots & x_{i,r_i-1} \end{bmatrix}, \quad i = 1, 2, \cdots, l$$

Then,

$$\boldsymbol{x}^{\rm T}PB = \begin{bmatrix} \boldsymbol{x}_1^{\rm T}P_1\boldsymbol{B}_1 & \boldsymbol{x}_2^{\rm T}P_2\boldsymbol{B}_2 & \cdots & \boldsymbol{x}_l^{\rm T}P_l\boldsymbol{B}_l \end{bmatrix} \tag{9}$$

and

$$\begin{aligned} \boldsymbol{x}_i^{\rm T}P_i\boldsymbol{B}_i = &\begin{bmatrix} \boldsymbol{X}_{i,r_i-1}^{\rm T} & x_{i,r_i} \end{bmatrix} \begin{bmatrix} P_{r_i-1} & \boldsymbol{P}_{i2} \\ \boldsymbol{P}_{i2}^{\rm T} & p_{i3} \end{bmatrix} \begin{bmatrix} \boldsymbol{0} \\ 1 \end{bmatrix} = \\ &\boldsymbol{X}_{i,r_i-1}^{\rm T}\boldsymbol{P}_{i2} + p_{i3}x_{i,r_i}, \quad i = 1, 2, \cdots, l \end{aligned} \tag{10}$$

From (9) and (10), because $\boldsymbol{x}^{\rm T}PB = \boldsymbol{0}, \boldsymbol{x} \neq \boldsymbol{0}$, there exists an $\boldsymbol{x}_i \neq \boldsymbol{0}$ at least, and

$$\boldsymbol{x}_i^{\rm T}P_i\boldsymbol{B}_i = \boldsymbol{X}_{i,r_i-1}^{\rm T}\boldsymbol{P}_{i2} + p_{i3}x_{i,r_i} = 0, \quad i = 1, 2, \cdots, l \tag{11}$$

Owing to

$$\begin{aligned} 2\boldsymbol{x}_i^{\rm T}P_iA_i\boldsymbol{x}_i = &\begin{bmatrix} \boldsymbol{X}_{i,r_i-1}^{\rm T} & x_{i,r_i} \end{bmatrix} \begin{bmatrix} P_{r_i-1} & \boldsymbol{P}_{i2} \\ \boldsymbol{P}_{i2}^{\rm T} & p_{i3} \end{bmatrix} \times \\ &\begin{bmatrix} A_{i-1} & \mathbf{A}_{i2} \\ \boldsymbol{0} & 0 \end{bmatrix} \begin{bmatrix} \boldsymbol{X}_{i,r_i-1} \\ x_{i,r_i} \end{bmatrix} + \\ &\begin{bmatrix} \boldsymbol{X}_{i,r_i-1}^{\rm T} & x_{i,r_i} \end{bmatrix} \begin{bmatrix} A_{i-1} & \boldsymbol{A}_{i2} \\ \boldsymbol{0} & 0 \end{bmatrix}^{\rm T} \times \\ &\begin{bmatrix} P_{r_i-1} & \boldsymbol{P}_{i2} \\ \boldsymbol{P}_{i2}^{\rm T} & p_{i3} \end{bmatrix} \begin{bmatrix} \boldsymbol{X}_{i,r_i-1} \\ x_{i,r_i} \end{bmatrix} \end{aligned} \tag{12}$$

by (11) and (12), we arrive at

$$\begin{aligned} 2\boldsymbol{x}_i^{\rm T}P_iA_i\boldsymbol{x}_i = &\boldsymbol{X}_{i,r_i-1}^{\rm T}[(P_{r_i-1} - p_{i3}^{-1}\boldsymbol{P}_{i2}\boldsymbol{P}_{i2}^{\rm T})(A_{i-1} - \\ &\mathbf{A}_{i2}\boldsymbol{P}_{i2}^{\rm T}p_{i3}^{-1}) + (A_{i-1} - \boldsymbol{A}_{i2}\boldsymbol{P}_{i2}^{\rm T}p_{i3}^{-1})^{\rm T} \times \\ &(P_{r_i-1} - p_{i3}^{-1}\boldsymbol{P}_{i2}\boldsymbol{P}_{i2}^{\rm T})]\boldsymbol{X}_{i,r_i-1} \end{aligned} \tag{13}$$

when $\boldsymbol{x}_i^{\rm T}P_iB_i = \boldsymbol{X}_{i,r_i-1}^{\rm T}\boldsymbol{P}_{i2} + p_{i3}x_{i,r_i} = 0$.

By hypothesis H 2, there exists a positive definite matrix F_i such that

$$2\boldsymbol{x}_i^{\rm T}P_iA_i\boldsymbol{x}_i = -\boldsymbol{X}_{i,r_i-1}^{\rm T}F_i\boldsymbol{X}_{i,r_i-1} \tag{14}$$

Thus,

$$\begin{aligned} \boldsymbol{x}^{\rm T}PA\boldsymbol{x} + \boldsymbol{x}^{\rm T}A^{\rm T}P\boldsymbol{x} = &2\sum_{i=1}^{l}\boldsymbol{x}_i^{\rm T}P_iA_i\boldsymbol{x}_i = \\ &-\sum_{i=1}^{l}\boldsymbol{X}_{i,r_i-1}^{\rm T}F_i\boldsymbol{X}_{i,r_i-1} < 0 \end{aligned} \tag{15}$$

when $\boldsymbol{x}^{\rm T}PB = \boldsymbol{0}, \boldsymbol{x} \neq \boldsymbol{0}$.

From (6), (7), and (15), we can deduce that

$$\frac{\partial V}{\partial \boldsymbol{x}}\phi_s(\boldsymbol{x}, \boldsymbol{d}) = \boldsymbol{x}^{\rm T}(A^{\rm T}P + PA)\boldsymbol{x} < 0 \tag{16}$$

for $\dfrac{\partial V}{\partial \boldsymbol{x}}\varphi_s(\boldsymbol{x}, \boldsymbol{d}) = \boldsymbol{0}, \boldsymbol{x} \neq \boldsymbol{0}$.

Thus, $V(\boldsymbol{x}) = \boldsymbol{x}^{\rm T}P\boldsymbol{x}$ is a common CLF for the collection of systems (1). □

The following considers a common feedback to stabilize systems (1) using the common CLF established by Theorem 1. Now, for each $j = 1, 2, \cdots, m$, denote $G_{sj}(\boldsymbol{x}, \boldsymbol{d})$ as the j-th column of $G_s(\boldsymbol{x}, \boldsymbol{d})$ and

$$\begin{aligned} \lambda_j(\boldsymbol{x}) &= \min_s \min_{\boldsymbol{d}\in\Omega} \boldsymbol{x}^{\rm T}PBG_{sj}(\boldsymbol{x}, \boldsymbol{d}) \\ \mu_j(\boldsymbol{x}) &= \max_s \max_{\boldsymbol{d}\in\Omega} \boldsymbol{x}^{\rm T}PBG_{sj}(\boldsymbol{x}, \boldsymbol{d}) \end{aligned} \tag{17}$$

Moreover, define

$$\beta_j(\boldsymbol{x}) = \begin{cases} \lambda_j(\boldsymbol{x}), & \lambda_j(\boldsymbol{x}) > 0 \\ 0, & \lambda_j(\boldsymbol{x}) = 0 \\ \mu_j(\boldsymbol{x}), & \lambda_j(\boldsymbol{x}) < 0 \end{cases} \tag{18}$$

We are ready to verify Theorem 2.

Theorem 2. If there are m positive numbers $r_j > 1$, such that

$$r_j \max_s \max_{\boldsymbol{d}\in\Omega} \boldsymbol{x}^{\rm T}PBG_{sj}(\boldsymbol{x}, \boldsymbol{d}) \le \lambda_j(\boldsymbol{x}) \tag{19}$$

when $\lambda_j(\boldsymbol{x}) < 0, j = 1, 2, \cdots, m$, then there is a common feedback which is independent of $\boldsymbol{d}$ and stabilizes these systems simultaneously.

Proof. By Theorem 1, $V(\boldsymbol{x}) = \boldsymbol{x}^{\mathrm{T}} P \boldsymbol{x}$ defined as (5) is a common CLF for the collection of systems (1). The derivative of $V(\boldsymbol{x})$ along with the s-th system of (1) is

$$\dot{V}_s(\boldsymbol{x}) = 2\boldsymbol{x}^{\mathrm{T}} PA\boldsymbol{x} + 2\boldsymbol{x}^{\mathrm{T}} PB\boldsymbol{F}_s(\boldsymbol{x}, \boldsymbol{d}) + 2\boldsymbol{x}^{\mathrm{T}} PBG_s(\boldsymbol{x}, \boldsymbol{d})\boldsymbol{u} \tag{20}$$

Define

$$\alpha(\boldsymbol{x}) = \boldsymbol{x}^{\mathrm{T}} PA\boldsymbol{x} + \max_s \max_{\boldsymbol{d}\in\Omega} \boldsymbol{x}^{\mathrm{T}} PB\boldsymbol{F}_s(\boldsymbol{x}, \boldsymbol{d}) \tag{21}$$

and

$$\begin{aligned} \boldsymbol{\beta}(\boldsymbol{x}) &= [\beta_1(\boldsymbol{x}), \beta_2(\boldsymbol{x}), \cdots, \beta_m(\boldsymbol{x})]^{\mathrm{T}} \\ \boldsymbol{\lambda}(\boldsymbol{x}) &= [\lambda_1(\boldsymbol{x}), \lambda_2(\boldsymbol{x}), \cdots, \lambda_m(\boldsymbol{x})]^{\mathrm{T}} \end{aligned} \tag{22}$$

The feedback is

$$\boldsymbol{u} = \boldsymbol{u}(\boldsymbol{x}) = \begin{cases} -\boldsymbol{\beta}(\boldsymbol{x}) \dfrac{\alpha(\boldsymbol{x}) + \sqrt{(\alpha(\boldsymbol{x}))^2 + |\boldsymbol{\beta}(\boldsymbol{x})|^4}}{|\boldsymbol{\beta}(\boldsymbol{x})|^2}, & \boldsymbol{x}^{\mathrm{T}} PB \neq \boldsymbol{0} \\ 0, & \boldsymbol{x}^{\mathrm{T}} PB = \boldsymbol{0} \end{cases} \tag{23}$$

First, we prove that $\boldsymbol{u}(\boldsymbol{x})$ is continuous. For $j = 1, 2, \cdots, m$, if $\lambda_j(\boldsymbol{x})$ goes to zero from the right side of zero, $\beta_j(\boldsymbol{x}) = \lambda_j(\boldsymbol{x})$ goes to zero. On the other hand, by assumption, when $\lambda_j(\boldsymbol{x}) < 0$, $\lambda_j(\boldsymbol{x}) \leq \mu_j(\boldsymbol{x}) \leq (1/r_j)\lambda_j(\boldsymbol{x})$. So, $\beta_j(\boldsymbol{x}) = \mu_j(\boldsymbol{x})$ goes to zero when $\lambda_j(\boldsymbol{x})$ goes to zero from the left side. This implies that $\beta_j(\boldsymbol{x})$ is continuous. In view of (22), $\boldsymbol{\beta}(\boldsymbol{x})$ is continuous. It can be deduced that $\alpha(\boldsymbol{x})$ is also continuous.

From the definition of $\boldsymbol{\beta}(\boldsymbol{x})$ and (18), $\boldsymbol{\beta}(\boldsymbol{x}) = \boldsymbol{0}$ is equivalent to $\lambda(\boldsymbol{x}) = \boldsymbol{0}$. Since every $G_s(\boldsymbol{x}, \boldsymbol{d})$ has full row rank, $\lambda(\boldsymbol{x}) = \boldsymbol{0}$ is equivalent to $B^{\mathrm{T}} P\boldsymbol{x} = \boldsymbol{0}$. Thus, $\boldsymbol{\beta}^{\mathrm{T}}(\boldsymbol{x}) = \boldsymbol{0}$ is equivalent to $\boldsymbol{x}^{\mathrm{T}} PB = \boldsymbol{0}$. Furthermore, when $\boldsymbol{x}^{\mathrm{T}} PB = \boldsymbol{0}$ and $\boldsymbol{x} \neq \boldsymbol{0}$, $\alpha(\boldsymbol{x}) = \boldsymbol{x}^{\mathrm{T}} PA\boldsymbol{x} < 0$ by Theorem 1. Thus, $\boldsymbol{u}(\boldsymbol{x})$ is continuous.

Now, let us consider (20) in two cases.

1) $\boldsymbol{\beta}(\boldsymbol{x}) = \boldsymbol{0}, \boldsymbol{x} \neq \boldsymbol{0}$. From the above, $\boldsymbol{\beta}^{\mathrm{T}}(\boldsymbol{x}) = \boldsymbol{0}$ is equivalent to $\boldsymbol{x}^{\mathrm{T}} PB = \boldsymbol{0}$. Then, by (20),

$$\dot{V}_s(\boldsymbol{x}) = 2\boldsymbol{x}^{\mathrm{T}} PA\boldsymbol{x} < 0$$

from Theorem 1.

2) $\boldsymbol{\beta}(\boldsymbol{x}) \neq \boldsymbol{0}$. For $j = 1, 2, \cdots, m$, since $\boldsymbol{x}^{\mathrm{T}} PBG_{sj}(\boldsymbol{x}, \boldsymbol{d}) \geq \lambda_j(\boldsymbol{x})$ and $\beta_j(\boldsymbol{x}) = \lambda_j(\boldsymbol{x}) \geq 0$ when $\lambda_j(\boldsymbol{x}) \geq 0$, we have $\boldsymbol{x}^{\mathrm{T}} PBG_{sj}(\boldsymbol{x}, \boldsymbol{d})\beta_j(\boldsymbol{x}) \geq \beta_j^2(\boldsymbol{x})$ when $\lambda_j(\boldsymbol{x}) \geq 0$; on the other hand, when $\lambda_j(\boldsymbol{x}) < 0$, we have $\beta_j(\boldsymbol{x}) = \mu_j(\boldsymbol{x}) < 0$ by (19), and in view of $\boldsymbol{x}^{\mathrm{T}} PBG_{sj}(\boldsymbol{x}, \boldsymbol{d}) \leq \mu_j(\boldsymbol{x})$, $\boldsymbol{x}^{\mathrm{T}} PBG_{sj}(\boldsymbol{x}, \boldsymbol{d})\beta_j(\boldsymbol{x}) \geq \beta_j^2(\boldsymbol{x})$ when $\lambda_j(\boldsymbol{x}) < 0$. In conclusion, it can be deduced that

$$\sum_{j=1}^{m} \boldsymbol{x}^{\mathrm{T}} PBG_{sj}(\boldsymbol{x}, \boldsymbol{d})\beta_j(\boldsymbol{x}) \geq \sum_{j=1}^{m} \beta_j^2(\boldsymbol{x}) \tag{24}$$

In view of (23),

$$\boldsymbol{u} = -\boldsymbol{\beta}(\boldsymbol{x}) \frac{\alpha(\boldsymbol{x}) + \sqrt{(\alpha(\boldsymbol{x}))^2 + |\boldsymbol{\beta}(\boldsymbol{x})|^4}}{|\boldsymbol{\beta}(\boldsymbol{x})|^2} = \boldsymbol{\beta}(\boldsymbol{x}) \frac{\triangle(\boldsymbol{x})}{\boldsymbol{\beta}^{\mathrm{T}}(\boldsymbol{x})\boldsymbol{\beta}(\boldsymbol{x})}$$

where

$$\triangle(\boldsymbol{x}) = -\alpha(\boldsymbol{x}) - \sqrt{(\alpha(\boldsymbol{x}))^2 + |\boldsymbol{\beta}(\boldsymbol{x})|^4} < 0$$

(24) is equivalent to

$$\sum_{j=1}^{m} \boldsymbol{x}^{\mathrm{T}} PBG_{sj}(\boldsymbol{x}, \boldsymbol{d})\beta_j(\boldsymbol{x}) \frac{\triangle(\boldsymbol{x})}{\boldsymbol{\beta}^{\mathrm{T}}(\boldsymbol{x})\boldsymbol{\beta}(\boldsymbol{x})} \leq \sum_{j=1}^{m} \beta_j^2(\boldsymbol{x}) \frac{\triangle(\boldsymbol{x})}{\boldsymbol{\beta}^{\mathrm{T}}(\boldsymbol{x})\boldsymbol{\beta}(\boldsymbol{x})} \tag{25}$$

i.e.

$$\boldsymbol{x}^{\mathrm{T}} PBG_s(\boldsymbol{x}, \boldsymbol{d})\boldsymbol{\beta}(\boldsymbol{x}) \frac{\triangle(\boldsymbol{x})}{\boldsymbol{\beta}^{\mathrm{T}}(\boldsymbol{x})\boldsymbol{\beta}(\boldsymbol{x})} - \triangle(\boldsymbol{x}) \leq 0$$

So,

$$\boldsymbol{x}^{\mathrm{T}} PBG_s(\boldsymbol{x}, \boldsymbol{d})\boldsymbol{u} - \triangle(\boldsymbol{x}) \leq 0 \tag{26}$$

By (20) and (26), we have

$$\begin{aligned} \dot{V}_s(\boldsymbol{x}) \leq\ & 2\boldsymbol{x}^{\mathrm{T}} PA\boldsymbol{x} + 2\boldsymbol{x}^{\mathrm{T}} PB\boldsymbol{F}_s(\boldsymbol{x}, \boldsymbol{d}) + 2\triangle(\boldsymbol{x}) = \\ & 2\boldsymbol{x}^{\mathrm{T}} PA\boldsymbol{x} + 2\boldsymbol{x}^{\mathrm{T}} PB\boldsymbol{F}_s(\boldsymbol{x}, \boldsymbol{d}) + \\ & 2(-\alpha(\boldsymbol{x}) - \sqrt{\alpha^2(\boldsymbol{x}) + |\boldsymbol{\beta}(\boldsymbol{x})|^4}) \leq \\ & -2\sqrt{\alpha^2(\boldsymbol{x}) + |\boldsymbol{\beta}(\boldsymbol{x})|^4} < 0 \end{aligned}$$

The conclusion is followed from the Lyapunov theorem. □

3 Example

Consider a collection of multi-input nonlinear systems as follows:

$$Q_s: \begin{cases} \dot{x}_1 = x_2 \\ \dot{x}_2 = f_s(\boldsymbol{x}, \boldsymbol{d}) + g_{s1}(\boldsymbol{x}, \boldsymbol{d})u_1 + g_{s2}(\boldsymbol{x}, \boldsymbol{d})u_2, \ s = 1, 2, 3 \end{cases} \tag{27}$$

with $\boldsymbol{x} = [x_1, x_2]^{\mathrm{T}}$, $\boldsymbol{d} = [d_1, d_2]^{\mathrm{T}}$, $\Omega = \{(d_1, d_2) | d_1 \in [1, 2], d_2 \in [1, 3]\}$, $f_1(\boldsymbol{x}, \boldsymbol{d}) = d_1 x_1 \mathrm{e}^{x_2}$, $f_2(\boldsymbol{x}, \boldsymbol{d}) = d_2 x_1$, $f_3(\boldsymbol{x}, \boldsymbol{d}) = d_1 x_1 \cos x_2$, $g_{11}(\boldsymbol{x}, \boldsymbol{d}) = d_1 \mathrm{e}^{x_1}$, $g_{21}(\boldsymbol{x}, \boldsymbol{d}) = 1 + d_1^2 \sin^2 x_2$, $g_{31}(\boldsymbol{x}, \boldsymbol{d}) = 1 + d_1^2 x_1^2$, $g_{12}(\boldsymbol{x}, \boldsymbol{d}) = -1 - d_2 \cos^2 x_1$, $g_{22}(\boldsymbol{x}, \boldsymbol{d}) = -d_2 \mathrm{e}^{x_1 + x_2}$, and $g_{32}(\boldsymbol{x}, \boldsymbol{d}) = -d_2 x_2^2$.

By Theorem 1, there exists a common CLF for systems (27). Let $p_{13} = 1$ and $P_{12} = 1$. Then,

$$V(\boldsymbol{x}) = \boldsymbol{x}^{\mathrm{T}} \begin{bmatrix} 1.5 & 1 \\ 1 & 1 \end{bmatrix} \boldsymbol{x}$$

is a common CLF for systems (27).

It is easy to check that the condition (19) in Theorem 2 holds. Denote

$$\begin{aligned} \alpha(\boldsymbol{x}) =\ & (1.5x_1 + x_2)x_2 + \max_{\boldsymbol{d}\in\Omega}\{(x_1 + x_2)d_1 x_1 \mathrm{e}^{x_2}, \\ & (x_1 + x_2)d_2 x_1, (x_1 + x_2)d_1 x_1 \cos x_2\} \\ \lambda_1(\boldsymbol{x}) =\ & \min_{d_1\in[1,2]}\{(x_1 + x_2)d_1 \mathrm{e}^{x_1}, (x_1 + x_2)(1 + d_1^2 \sin^2 x_2), \\ & (x_1 + x_2)(1 + d_1^2 x_1^2)\} \\ \lambda_2(\boldsymbol{x}) =\ & \min_{d_2\in[1,3]}\{-(x_1 + x_2)(1 + d_2 \cos^2 x_1), \\ & -(x_1 + x_2)d_2 \mathrm{e}^{x_1 + x_2}, -(x_1 + x_2)d_2 x_2^2\} \\ \mu_1(\boldsymbol{x}) =\ & \max_{d_1\in[1,2]}\{(x_1 + x_2)d_1 \mathrm{e}^{x_1}, (x_1 + x_2)(1 + d_1^2 \sin^2 x_2), \\ & (x_1 + x_2)(1 + d_1^2 x_1^2)\} \\ \mu_2(\boldsymbol{x}) =\ & \max_{d_2\in[1,3]}\{-(x_1 + x_2)(1 + d_2 \cos^2 x_1), \\ & -(x_1 + x_2)d_2 \mathrm{e}^{x_1 + x_2}, -(x_1 + x_2)d_2 x_2^2\} \end{aligned}$$

$$\beta_1(\boldsymbol{x}) = \begin{cases} \lambda_1(\boldsymbol{x}), & \lambda_1(\boldsymbol{x}) > 0 \\ 0, & \lambda_1(\boldsymbol{x}) = 0 \\ \mu_1(\boldsymbol{x}), & \lambda_1(\boldsymbol{x}) < 0 \end{cases}$$

$$\beta_2(\boldsymbol{x}) = \begin{cases} \lambda_2(\boldsymbol{x}), & \lambda_2(\boldsymbol{x}) > 0 \\ 0, & \lambda_2(\boldsymbol{x}) = 0 \\ \mu_2(\boldsymbol{x}), & \lambda_2(\boldsymbol{x}) < 0 \end{cases}$$

$$\boldsymbol{\beta}(\boldsymbol{x}) = \begin{bmatrix} \beta_1(\boldsymbol{x}) \\ \beta_2(\boldsymbol{x}) \end{bmatrix}$$

By Theorem 2, the feedback

$$\boldsymbol{u} = \boldsymbol{u}(\boldsymbol{x}) = \begin{cases} -\boldsymbol{\beta}(\boldsymbol{x})\dfrac{\alpha(\boldsymbol{x}) + \sqrt{(\alpha(\boldsymbol{x}))^2 + |\boldsymbol{\beta}(\boldsymbol{x})|^4}}{|\boldsymbol{\beta}(\boldsymbol{x})|^2}, \\ \qquad\qquad\qquad\qquad x_1 + x_2 \neq 0 \\ 0, \qquad\qquad\qquad\quad x_1 + x_2 = 0 \end{cases} \tag{28}$$

globally asymptotically stabilizes all systems of (27) simultaneously.

Figs. 1 and 2 show the state trajectories and control inputs of these three systems (The initial states are $[-0.7, 0.4]^{\mathrm{T}}$, $[0.8, -0.6]^{\mathrm{T}}$, and $[0.5, 0.7]^{\mathrm{T}}$, and random choices of $d_1 \in [1, 2]$ and $d_2 \in [1, 3]$ for example $d_1 = 1.5$ and $d_2 = 2$) with the same feedback (28), respectively.

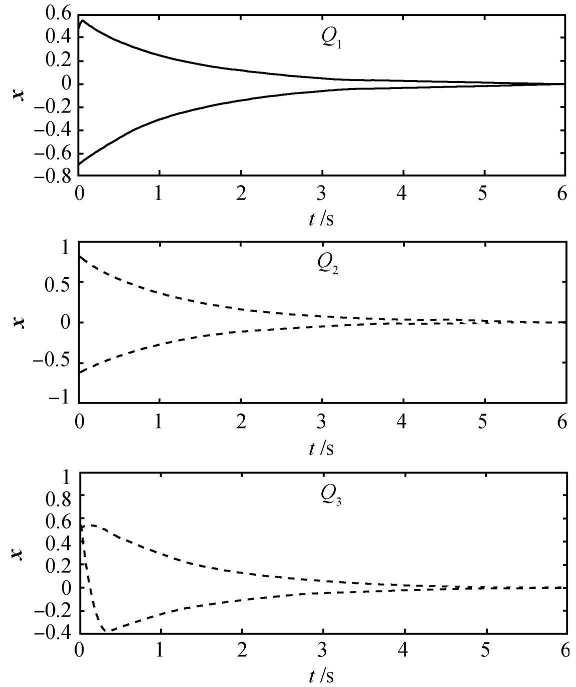

Fig. 1 State trajectories of the three systems in (27) with the feedback (28)

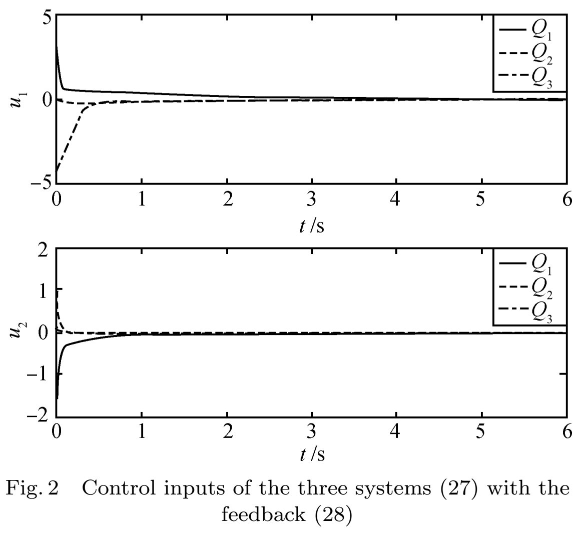

Fig. 2 Control inputs of the three systems (27) with the feedback (28)

4 Conclusion

In this paper, a systematic method is given for obtaining a common CLF of the collection of multi-input nonlinear systems with uncertain parameters. Furthermore, a continuous state feedback is designed which simultaneously stabilizes these systems. Finally, the effectiveness of the proposed scheme is illustrated by a simulation example.

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