Delay-dependent Stability Criteria for Systems with Differentiable Time Delays

ZHU Xun-Lin[1, 2] YANG Guang-Hong[1, 3]

Abstract This paper studies the problem of stability for continuous-time systems with differentiable time-varying delays. By using the information of delay derivative, improved asymptotic stability conditions for time-delay systems are presented. Unlike the previous methods, the upper bound of the delay derivative is taken into consideration even if this upper bound is larger than or equal to 1. It is proved that the obtained results are less conservative than the existing ones. Meanwhile, the computational complexity of the presented stability criteria is reduced greatly since fewer decision variables are involved. Numerical examples are given to illustrate the effectiveness and less conservatism of the obtained stability conditions.

Key words Delay-dependent stability condition, linear matrix inequality (LMI), time-delay systems

During the past decades, considerable attention has been devoted to the problem of stability and stabilization for time-delay systems. When time delays are considered, the dynamics of systems generally become more complicated and the stability problem is more challenging. Two types of stability conditions have been reported in the literature: the so-called delay-dependent conditions (the condition containing delay information) and delay-independent conditions (the condition not containing delay information). Because delay-dependent criteria make use of information on the length of delays, they are less conservative than the delay-independent ones. For delay-dependent criteria[1−9], there are two approaches which are now widely used, the first one is based on model transformation[4], the other is called the free-weighting matrix method[7].

The approaches based on model transformation can be classified into four types[4]. The first is a first-order transformation. The second is a neutral transformation. The third uses the Park′s inequalities and yields a transformed system that is equivalent to the original one. Fridman and Shaked[4] combined a descriptor model transformation with Park and Moon′s inequalities[10] to yield the fourth type of transformation, which produced less conservative stability criteria, and whose main merit is the simpleness of controller design. A simplified descriptor system approach to delay-dependent stability analysis for time-delay systems was provided in [11]. By this approach, the derived results were equivalent to those obtained by the descriptor system approach in [3], but with fewer variables to be determined compared with those in [3] (that is, some variables in [3] are redundant). Hence, they are elegant from a mathematical point of view.

The free-weighting matrix method uses the Newton-Leibniz formula to obtain a delay-dependent condition[7], which is more intuitional than the method afore mentioned. Recently, [8] extended the results in [7] by estimating the upper bound of the derivative of Lyapunov functional without ignoring some useful terms. The result obtained in [8] is one of the latest results in the existing literature.

For systems with time-varying delays, the above mentioned literature usually demand that the upper bound of the derivative of delays must be smaller than 1. If the upper bound of the derivative of delays is larger than 1, the results in [3, 11] are not applicable, whereas [7−8] discard the information of the derivative of the delays, which is obviously unreasonable. In many cases, the upper bound of the derivative of delays may not be less than 1, for example, in the networked control systems[12−13], the derivative of delays is equal to 1 almost everywhere. Thus, how to eliminate the constraint on the upper bound of the delay derivative is very significant, but there is not any answer to this problem in the existing literature.

In this paper, a new method is proposed to eliminate the constraint on the upper bound of the delay derivative, and new stability conditions are also presented for the systems with delay. It is proved that the new results are less conservative than some existing ones. Meanwhile, the obtained stability criteria contain fewer decision variables, hence they are mathematically less complex and computationally more efficient. Without changing the conservatism of the results, the methods for simplifying the delay-dependent stability conditions obtained by the free-weighting matrix approach and the descriptor system approach are also presented.

1 Main results

In this section, the stability of continuous-time systems with differentiable time-varying delay is analyzed, and a sufficient condition is derived by using delay derivative dependent Lyapunov function.

Consider the following linear system

$$\dot{\boldsymbol{x}}(t) = A\boldsymbol{x}(t) + A_d\boldsymbol{x}(t-d(t)), \quad t>0 \tag{1}$$

$$\boldsymbol{x}(t) = \boldsymbol{\phi}(t), \qquad t\in[-\eta,\ 0] \tag{2}$$

where $\boldsymbol{x}(t)\in\mathbf{R}^n$ is the state vector, A and A_d are constant matrices of appropriate dimensions, and the time delay, $d(t)$, is a time-varying continuous function that satisfies

$$\tau\le d(t)\le\eta \tag{3}$$

and

$$\dot{d}(t)\le\mu \tag{4}$$

where $\tau,\ \eta\ (0\le\tau<\eta)$, and μ are constants. The initial condition, $\boldsymbol{\phi}(t)$, is a continuous vector-valued function with $t\in[-\eta,\ 0]$.

Received April 23, 2007; in revised form December 24, 2007
Supported by Program for New Century Excellent Talents in University (NCET-04-0283), the Funds for Creative Research Groups of China (60521003), Program for Changjiang Scholars and Innovative Research Team in University (IRT0421), the State Key Program of National Natural Science Foundation of China (60534010), National Natural Science Foundation of China (60674021) and the Funds of Ph. D. Program of Ministry of Education, China (20060145019)
1. College of Information Science and Engineering, Northeastern University, Shenyang 110004, P. R. China 2. School of Computer and Communication Engineering, Zhengzhou University of Light Industry, Zhengzhou 450002, P. R. China 3. Key Laboratory of Integrated Automation of Process Industry, Ministry of Education, Northeastern University, Shenyang 110004, P. R. China
DOI: 10.3724/SP.J.1004.2008.00765

In the previous work, such as [3] and [6], the upper bound of delay derivative, μ, should be smaller than 1. Though the results in [7−8] can be applied to the case of $\mu \geq 1$, these stability conditions were independent on the upper bound of the delay derivative, μ.

For the time-delay system described by (1) ~ (4), the term

$$\int_{t-d(t)}^{t} \boldsymbol{x}^{\mathrm{T}}(s)Q\boldsymbol{x}(s)\mathrm{d}s \tag{5}$$

with $Q = Q^{\mathrm{T}} \geq 0$ is usually taken as Lyapunov functional (for example, [6−7], [11]). However, if $\mu \geq 1$, then the term $\int_{t-d(t)}^{t} \boldsymbol{x}^{\mathrm{T}}(s)Q\boldsymbol{x}(s)\mathrm{d}s$ is redundant because

$$\begin{aligned}&\left(\int_{t-d(t)}^{t} \boldsymbol{x}^{\mathrm{T}}(s)Q\boldsymbol{x}(s)\mathrm{d}s\right)' = \\ &\boldsymbol{x}^{\mathrm{T}}(t)Q\boldsymbol{x}(t) - (1-\dot{d}(t))\boldsymbol{x}^{\mathrm{T}}(t-d(t))Q\boldsymbol{x}(t-d(t)) \leq \\ &\boldsymbol{x}^{\mathrm{T}}(t)Q\boldsymbol{x}(t) - (1-\mu)\boldsymbol{x}^{\mathrm{T}}(t-d(t))Q\boldsymbol{x}(t-d(t))\end{aligned}$$

and $-(1-\mu) \geq 0$. This implies that the information of the derivative of the time-delay term $d(t)$ is not used, which is obviously unreasonable.

As a matter of fact, the case that the delay derivative is larger than or equal to 1 is universal. For example, in network control systems, the delay $d(t)$ denotes $t - i_k$, where i_k ($k = 1,\ 2,\ \cdots$) are the sampling instants. Thus, this kind of delay satisfies $\dot{d}(t) = 1$ almost for all $t \geq 0$.

For the case of $\mu \geq 1$, if choosing a positive scalar $0 < \alpha < 1$ satisfying $\alpha\mu < 1$, then it follows that

$$(\alpha d(t))' = \alpha\dot{d}(t) \leq \alpha\mu < 1 \tag{6}$$

and

$$\begin{aligned}&\left(\int_{t-d(t)}^{t} \boldsymbol{x}^{\mathrm{T}}(s)Q\boldsymbol{x}(s)\mathrm{d}s\right)' \leq \\ &\boldsymbol{x}^{\mathrm{T}}(t)Q\boldsymbol{x}(t) - (1-\alpha\mu)\boldsymbol{x}^{\mathrm{T}}(t-\alpha d(t))Q\boldsymbol{x}(t-\alpha d(t))\end{aligned} \tag{7}$$

Thus, the information of the derivative of $d(t)$ can be used.

On the basis of this fact, the following theorem can be obtained.

Theorem 1. For given scalars τ, η ($0 \leq \tau < \eta$), and $0 < \alpha < 1$ satisfying $\alpha\mu < 1$, the system described by (1) ~ (4) is asymptotically stable if there exist matrices $P = P^{\mathrm{T}} > 0$, $Q_i = Q_i^{\mathrm{T}} \geq 0$ ($i = 1,\ 2,\ 3,\ 4$), and $Z_j = Z_j^{\mathrm{T}} > 0$ ($j = 1,\ 2,\ 3$), such that

$$\Pi = \begin{bmatrix} \Pi_1 & \Pi_2 & 0 & 0 & (\alpha\eta)^{-1}Z_3 & A^{\mathrm{T}}U \\ * & \Pi_3 & \theta^{-1}Z_2 & \theta^{-1}Z & (\beta\eta)^{-1}Z_3 & A_d^{\mathrm{T}}U \\ * & * & \Pi_4 & 0 & 0 & 0 \\ * & * & * & \Pi_5 & 0 & 0 \\ * & * & * & * & \Pi_6 & 0 \\ * & * & * & * & * & -U \end{bmatrix} < 0 \tag{8}$$

holds, where

$$\begin{aligned}&\Pi_1 = PA + A^{\mathrm{T}}P + \sum_{i=1}^{4} Q_i - \eta^{-1}Z_1 - (\alpha\eta)^{-1}Z_3 \\ &\Pi_2 = PA_d + \eta^{-1}Z_1 \\ &\Pi_3 = -(1-\mu)Q_3 - \eta^{-1}Z_1 - \theta^{-1}Z - \theta^{-1}Z_2 - (\beta\eta)^{-1}Z_3 \\ &\Pi_4 = -Q_1 - \theta^{-1}Z_2 \\ &\Pi_5 = -Q_2 - \theta^{-1}Z \\ &\Pi_6 = -\gamma Q_4 - (\alpha\eta)^{-1}Z_3 - (\beta\eta)^{-1}Z_3 \\ &Z = \sum_{i=1}^{3} Z_i \\ &U = \eta Z_1 + \theta Z_2 + \eta Z_3 \\ &\theta = \eta - \tau \\ &\beta = 1 - \alpha \\ &\gamma = 1 - \alpha\mu\end{aligned}$$

Proof. Construct a Lyapunov functional as

$$\begin{aligned}V(\boldsymbol{x}_t) = &\boldsymbol{x}^{\mathrm{T}}(t)P\boldsymbol{x}(t) + \int_{t-\tau}^{t} \boldsymbol{x}^{\mathrm{T}}(s)Q_1\boldsymbol{x}(s)\mathrm{d}s + \\ &\int_{t-\eta}^{t} \boldsymbol{x}^{\mathrm{T}}(s)Q_2\boldsymbol{x}(s)\mathrm{d}s + \\ &\int_{t-d(t)}^{t} \boldsymbol{x}^{\mathrm{T}}(s)Q_3\boldsymbol{x}(s)\mathrm{d}s + \\ &\int_{t-\alpha d(t)}^{t} \boldsymbol{x}^{\mathrm{T}}(s)Q_4\boldsymbol{x}(s)\mathrm{d}s + \\ &\int_{-\eta}^{0}\int_{t+\beta}^{t} \dot{\boldsymbol{x}}^{\mathrm{T}}(s)Z_1\dot{\boldsymbol{x}}(s)\mathrm{d}s\mathrm{d}\beta + \\ &\int_{-\eta}^{-\tau}\int_{t+\beta}^{t} \dot{\boldsymbol{x}}^{\mathrm{T}}(s)Z_2\dot{\boldsymbol{x}}(s)\mathrm{d}s\mathrm{d}\beta + \\ &\int_{-\eta}^{0}\int_{t+\beta}^{t} \dot{\boldsymbol{x}}^{\mathrm{T}}(s)Z_3\dot{\boldsymbol{x}}(s)\mathrm{d}s\mathrm{d}\beta\end{aligned} \tag{9}$$

where $P > 0$, $Q_i \geq 0$ ($i = 1,\ 2,\ 3,\ 4$), and $Z_j > 0$ ($j = 1,\ 2,\ 3$) are matrices to be determined.

From the Leibniz-Newton formula, the following equations are true for any matrices N_i, S_i, M_i, Y_i, and T_i ($i = 1,\ 2,\ \cdots,\ 5$) with appropriate dimensions:

$$2\boldsymbol{\zeta}^{\mathrm{T}}(t)N[\boldsymbol{x}(t) - \boldsymbol{x}(t-d(t)) - \int_{t-d(t)}^{t} \dot{\boldsymbol{x}}(s)\mathrm{d}s] = 0 \tag{10}$$

$$2\boldsymbol{\zeta}^{\mathrm{T}}(t)S[\boldsymbol{x}(t-d(t)) - \boldsymbol{x}(t-\eta) - \int_{t-\eta}^{t-d(t)} \dot{\boldsymbol{x}}(s)\mathrm{d}s] = 0 \tag{11}$$

$$2\boldsymbol{\zeta}^{\mathrm{T}}(t)M[\boldsymbol{x}(t-\tau) - \boldsymbol{x}(t-d(t)) - \int_{t-d(t)}^{t-\tau} \dot{\boldsymbol{x}}(s)\mathrm{d}s] = 0 \tag{12}$$

$$2\boldsymbol{\zeta}^{\mathrm{T}}(t)Y[\boldsymbol{x}(t) - \boldsymbol{x}(t-\alpha d(t)) - \int_{t-\alpha d(t)}^{t} \dot{\boldsymbol{x}}(s)\mathrm{d}s] = 0 \tag{13}$$

$$2\boldsymbol{\zeta}^{\mathrm{T}}(t)T[\boldsymbol{x}(t-\alpha d(t)) - \boldsymbol{x}(t-d(t)) - \int_{t-d(t)}^{t-\alpha d(t)} \dot{\boldsymbol{x}}(s)\mathrm{d}s] = 0 \tag{14}$$

where $N = [N_1^{\mathrm{T}}\ N_2^{\mathrm{T}}\ \cdots\ N_5^{\mathrm{T}}]^{\mathrm{T}}$, $S = [S_1^{\mathrm{T}}\ S_2^{\mathrm{T}}\ \cdots\ S_5^{\mathrm{T}}]^{\mathrm{T}}$, $M = [M_1^{\mathrm{T}}\ M_2^{\mathrm{T}}\ \cdots\ M_5^{\mathrm{T}}]^{\mathrm{T}}$, $Y = [Y_1^{\mathrm{T}}\ Y_2^{\mathrm{T}}\ \cdots\ Y_5^{\mathrm{T}}]^{\mathrm{T}}$, $T = [T_1^{\mathrm{T}}\ T_2^{\mathrm{T}}\ \cdots\ T_5^{\mathrm{T}}]^{\mathrm{T}}$ and $\boldsymbol{\zeta}(t) = [\boldsymbol{x}^{\mathrm{T}}(t)\ \ \boldsymbol{x}^{\mathrm{T}}(t-d(t))\ \ \boldsymbol{x}^{\mathrm{T}}(t-\tau)\ \ \boldsymbol{x}^{\mathrm{T}}(t-\eta)\ \ \boldsymbol{x}^{\mathrm{T}}(t-\alpha d(t))]^{\mathrm{T}}$.

Alternatively, the following equations are true:

$$-\int_{t-\eta}^{t} \dot{\boldsymbol{x}}^{\mathrm{T}}(s)Z_1\dot{\boldsymbol{x}}(s)\mathrm{d}s = -\int_{t-d(t)}^{t} \dot{\boldsymbol{x}}^{\mathrm{T}}(s)Z_1\dot{\boldsymbol{x}}(s)\mathrm{d}s - \int_{t-\eta}^{t-d(t)} \dot{\boldsymbol{x}}^{\mathrm{T}}(s)Z_1\dot{\boldsymbol{x}}(s)\mathrm{d}s \tag{15}$$

$$-\int_{t-\eta}^{t-\tau} \dot{\boldsymbol{x}}^{\mathrm{T}}(s)Z_2\dot{\boldsymbol{x}}(s)\mathrm{d}s = -\int_{t-d(t)}^{t-\tau} \dot{\boldsymbol{x}}^{\mathrm{T}}(s)Z_2\dot{\boldsymbol{x}}(s)\mathrm{d}s - \int_{t-\eta}^{t-d(t)} \dot{\boldsymbol{x}}^{\mathrm{T}}(s)Z_2\dot{\boldsymbol{x}}(s)\mathrm{d}s \tag{16}$$

$$\begin{aligned}-\int_{t-\eta}^{t} \dot{\boldsymbol{x}}^{\mathrm{T}}(s)Z_3\dot{\boldsymbol{x}}(s)\mathrm{d}s = &-\int_{t-\alpha d(t)}^{t} \dot{\boldsymbol{x}}^{\mathrm{T}}(s)Z_3\dot{\boldsymbol{x}}(s)\mathrm{d}s - \\ &\int_{t-d(t)}^{t-\alpha d(t)} \dot{\boldsymbol{x}}^{\mathrm{T}}(s)Z_3\dot{\boldsymbol{x}}(s)\mathrm{d}s - \int_{t-\eta}^{t-d(t)} \dot{\boldsymbol{x}}^{\mathrm{T}}(s)Z_3\dot{\boldsymbol{x}}(s)\mathrm{d}s\end{aligned} \tag{17}$$

Taking the time derivative of $V(\boldsymbol{x}_t)$ for $t > 0$ along the trajectory of (1) yields that

$$\begin{aligned}
\dot{V}(\boldsymbol{x}_t) &= 2\boldsymbol{x}^{\mathrm{T}}(t)P\dot{\boldsymbol{x}}(t) - \boldsymbol{x}^{\mathrm{T}}(t-\eta)Q_2\boldsymbol{x}(t-\eta) + \\
&\sum_{i=1}^{4}\boldsymbol{x}^{\mathrm{T}}(t)Q_i\boldsymbol{x}(t) - \boldsymbol{x}^{\mathrm{T}}(t-\tau)Q_1\boldsymbol{x}(t-\tau) - \\
&(1-\dot{d}(t))\boldsymbol{x}^{\mathrm{T}}(t-d(t))Q_3\boldsymbol{x}(t-\alpha d(t)) - \\
&(1-\alpha\dot{d}(t))\boldsymbol{x}^{\mathrm{T}}(t-\alpha d(t))Q_4\boldsymbol{x}(t-\alpha d(t)) + \\
&\dot{\boldsymbol{x}}^{\mathrm{T}}(t)\Big(\eta Z_1 + (\eta-\tau)Z_2 + \eta Z_3\Big)\dot{\boldsymbol{x}}(t) - \\
&\int_{t-\eta}^{t}\dot{\boldsymbol{x}}^{\mathrm{T}}(s)Z_1\dot{\boldsymbol{x}}(s)\mathrm{d}s - \int_{t-\eta}^{t-\tau}\dot{\boldsymbol{x}}^{\mathrm{T}}(s)Z_2\dot{\boldsymbol{x}}(s)\mathrm{d}s - \\
&\int_{t-\eta}^{t}\dot{\boldsymbol{x}}^{\mathrm{T}}(s)Z_3\dot{\boldsymbol{x}}(s)\mathrm{d}s \leq \\
&2\boldsymbol{x}^{\mathrm{T}}(t)P\dot{\boldsymbol{x}}(t) - \boldsymbol{x}^{\mathrm{T}}(t-\eta)Q_2\boldsymbol{x}(t-\eta) + \\
&\sum_{i=1}^{4}\boldsymbol{x}^{\mathrm{T}}(t)Q_i\boldsymbol{x}(t) - \boldsymbol{x}^{\mathrm{T}}(t-\tau)Q_1\boldsymbol{x}(t-\tau) - \\
&(1-\mu)\boldsymbol{x}^{\mathrm{T}}(t-d(t))Q_3\boldsymbol{x}(t-d(t)) \\
&-(1-\alpha\mu)\boldsymbol{x}^{\mathrm{T}}(t-\alpha d(t))Q_4\boldsymbol{x}(t-\alpha d(t)) + \\
&\dot{\boldsymbol{x}}^{\mathrm{T}}(t)\Big(\eta Z_1 + (\eta-\tau)Z_2 + \eta Z_3\Big)\dot{\boldsymbol{x}}(t) - \\
&\int_{t-d(t)}^{t}\dot{\boldsymbol{x}}^{\mathrm{T}}(s)Z_1\dot{\boldsymbol{x}}(s)\mathrm{d}s - \\
&\int_{t-\eta}^{t-d(t)}\dot{\boldsymbol{x}}^{\mathrm{T}}(s)(Z_1+Z_2+Z_3)\dot{\boldsymbol{x}}(s)\mathrm{d}s - \\
&\int_{t-d(t)}^{t-\tau}\dot{\boldsymbol{x}}^{\mathrm{T}}(s)Z_2\dot{\boldsymbol{x}}(s)\mathrm{d}s - \int_{t-\alpha d(t)}^{t}\dot{\boldsymbol{x}}^{\mathrm{T}}(s)Z_3\dot{\boldsymbol{x}}(s)\mathrm{d}s - \\
&\int_{t-d(t)}^{t-\alpha d(t)}\dot{\boldsymbol{x}}^{\mathrm{T}}(s)Z_3\dot{\boldsymbol{x}}(s)\mathrm{d}s + \\
&2\boldsymbol{\zeta}^{\mathrm{T}}(t)N[\boldsymbol{x}(t) - \boldsymbol{x}(t-d(t)) - \int_{t-d(t)}^{t}\dot{\boldsymbol{x}}(s)\mathrm{d}s] + \\
&2\boldsymbol{\zeta}^{\mathrm{T}}(t)S[\boldsymbol{x}(t-d(t)) - \boldsymbol{x}(t-\eta) - \int_{t-\eta}^{t-d(t)}\dot{\boldsymbol{x}}(s)\mathrm{d}s] + \\
&2\boldsymbol{\zeta}^{\mathrm{T}}(t)M[\boldsymbol{x}(t-\tau) - \boldsymbol{x}(t-d(t)) - \int_{t-d(t)}^{t-\tau}\dot{\boldsymbol{x}}(s)\mathrm{d}s] + \\
&2\boldsymbol{\zeta}^{\mathrm{T}}(t)Y[\boldsymbol{x}(t) - \boldsymbol{x}(t-\alpha d(t)) - \int_{t-\alpha d(t)}^{t}\dot{\boldsymbol{x}}(s)\mathrm{d}s] + \\
&2\boldsymbol{\zeta}^{\mathrm{T}}(t)T[\boldsymbol{x}(t-\alpha d(t)) - \boldsymbol{x}(t-d(t)) - \\
&\int_{t-d(t)}^{t-\alpha d(t)}\dot{\boldsymbol{x}}(s)\mathrm{d}s] \leq \\
&\boldsymbol{\zeta}^{\mathrm{T}}(t)[\Phi_1 + \Phi_2 + \Phi_2^{\mathrm{T}} + \eta N Z_1^{-1}N^{\mathrm{T}} + \theta S Z^{-1}S^{\mathrm{T}} + \\
&\theta M Z_2^{-1}M^{\mathrm{T}} + \alpha\eta Y Z_3^{-1}Y^{\mathrm{T}} + \beta\eta T Z_3^{-1}T^{\mathrm{T}} + \\
&\bar{A}^{\mathrm{T}}U\bar{A}]\boldsymbol{\zeta}(t)
\end{aligned} \tag{18}$$

where

$$\Phi_1 = \begin{bmatrix} \Phi_{11} & PA_d & 0 & 0 & 0 \\ * & -(1-\mu)Q_3 & 0 & 0 & 0 \\ * & * & -Q_1 & 0 & 0 \\ * & * & * & -Q_2 & 0 \\ * & * & * & * & -\gamma Q_4 \end{bmatrix}$$

$$\Phi_{11} = \sum_{i=1}^{4} Q_i + PA + (PA)^{\mathrm{T}}$$

$$\Phi_2 = \begin{bmatrix} N+Y & S-N-M-T & M & -S & T-Y \end{bmatrix}$$

$$\bar{A} = \begin{bmatrix} A & A_d & 0 & 0 & 0 \end{bmatrix}$$

By the Schur complement, inequality $\Phi_1 + \Phi_2 + \Phi_2^{\mathrm{T}} + \eta N Z_1^{-1}N^{\mathrm{T}} + \theta S Z^{-1}S^{\mathrm{T}} + \theta M Z_2^{-1}M^{\mathrm{T}} + \alpha\eta Y Z_3^{-1}Y^{\mathrm{T}} + \beta\eta T Z_3^{-1}T^{\mathrm{T}} + \bar{A}^{\mathrm{T}}U\bar{A} < 0$ is equivalent to

$$\Phi = \begin{bmatrix} \Phi_1 + \Phi_2 + \Phi_2^{\mathrm{T}} & \Phi_3 \\ * & \Phi_4 \end{bmatrix} < 0 \tag{19}$$

where

$$\Phi_3 = \begin{bmatrix} \eta N_1 & \theta S_1 & \theta M_1 & \alpha\eta Y_1 & \beta\eta T_1 & A^{\mathrm{T}}U \\ \eta N_2 & \theta S_2 & \theta M_2 & \alpha\eta Y_2 & \beta\eta T_2 & A_d^{\mathrm{T}}U \\ \eta N_3 & \theta S_3 & \theta M_3 & \alpha\eta Y_3 & \beta\eta T_3 & 0 \\ \eta N_4 & \theta S_4 & \theta M_4 & \alpha\eta Y_4 & \beta\eta T_4 & 0 \\ \eta N_5 & \theta S_5 & \theta M_5 & \alpha\eta Y_5 & \beta\eta T_5 & 0 \end{bmatrix}$$

$$\Phi_4 = \mathrm{diag}\{-\eta Z_1, -\theta Z, -\theta Z_2, -\alpha\eta Z_3, -\beta\eta Z_3, -U\}$$

Thus, if $\Phi < 0$ holds, then $\dot{V}(\boldsymbol{x}_t) < 0$ for all $t > 0$.

Note that

$$\Gamma_1 \Phi \Gamma_1^{\mathrm{T}} = \begin{bmatrix} \Pi & \Pi_7 \\ * & \Pi_8 \end{bmatrix} \tag{20}$$

where

$$\Gamma_1 = \begin{bmatrix} I & \Gamma_2 \\ 0 & \Gamma_3 \end{bmatrix}$$

$$\Gamma_2 = \begin{bmatrix} -\frac{1}{\eta}I & 0 & 0 & -\frac{1}{\alpha\eta}I & 0 & 0 \\ \frac{1}{\eta}I & -\frac{1}{\theta}I & \frac{1}{\theta}I & 0 & \frac{1}{\beta\eta}I & 0 \\ 0 & 0 & -\frac{1}{\theta}I & 0 & 0 & 0 \\ 0 & \frac{1}{\theta}I & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{\alpha\eta}I & -\frac{1}{\beta\eta}I & 0 \end{bmatrix}$$

$$\Gamma_3 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & I \\ I & 0 & 0 & 0 & 0 & 0 \\ 0 & I & 0 & 0 & 0 & 0 \\ 0 & 0 & I & 0 & 0 & 0 \\ 0 & 0 & 0 & I & 0 & 0 \\ 0 & 0 & 0 & 0 & I & 0 \end{bmatrix}$$

$$\Pi_7 = \begin{bmatrix} \Pi_{71} & \theta S_1 & \theta M_1 & \alpha\eta Y_1 + Z_3 & \beta\eta T_1 \\ \Pi_{72} & \Pi_{73} & \theta M_2 - Z_2 & \alpha\eta Y_2 & \beta\eta T_2 - Z_3 \\ \eta N_3 & \theta S_3 & \theta M_3 + Z_2 & \alpha\eta Y_3 & \beta\eta T_3 \\ \eta N_4 & \Pi_{74} & \theta M_4 & \alpha\eta Y_4 & \beta\eta T_4 \\ \eta N_5 & \theta S_5 & \theta M_5 & \alpha\eta Y_5 - Z_3 & \beta\eta T_5 + Z_3 \end{bmatrix}$$

$$\Pi_{71} = \eta N_1 + Z_1$$

$$\Pi_{72} = \eta N_2 - Z_1$$

$$\Pi_{73} = \theta S_2 + Z$$

$$\Pi_{74} = \theta S_4 - Z$$

$$\Pi_8 = \mathrm{diag}\{-\eta Z_1, -\theta Z, -\theta Z_2, -\alpha\eta Z_3, -\beta\eta Z_3\}$$

and Γ_1 is nonsingular, so it follows that $\Pi < 0$ if $\Phi < 0$.

Contrarily, if $\Pi < 0$ holds, then $\Phi < 0$ is also true from (20) by letting

$$N = \begin{bmatrix} -\eta^{-1}Z_1 \\ \eta^{-1}Z_1 \\ 0 \\ 0 \\ 0 \end{bmatrix},\ S = \begin{bmatrix} 0 \\ -\theta^{-1}Z \\ 0 \\ \theta^{-1}Z \\ 0 \end{bmatrix},\ M = \begin{bmatrix} 0 \\ \theta^{-1}Z_2 \\ -\theta^{-1}Z_2 \\ 0 \\ 0 \end{bmatrix}$$

$$Y = \begin{bmatrix} -(\alpha\eta)^{-1}Z_3 \\ 0 \\ 0 \\ 0 \\ (\alpha\eta)^{-1}Z_3 \end{bmatrix},\ T = \begin{bmatrix} 0 \\ (\beta\eta)^{-1}Z_3 \\ 0 \\ 0 \\ -(\beta\eta)^{-1}Z_3 \end{bmatrix}$$

Thus, $\Phi < 0$ holds if and only if $\Pi < 0$ holds. □

Remark 1. Theorem 1 gives a new stability criterion, which is based on linear matrix inequalities (LMIs), and this criterion is different from existing ones for ordinary time-delay systems. Its novelty is shown in two aspects. The first one is that the information of the delay derivative can be used even if the upper bound of the delay derivative is not smaller than 1. The second one is that the free-weighting matrices N_i, S_i, M_i, Y_i, and $T_i (i = 1, 2, \cdots, 5)$

introduced by using the Newton-Leibniz formula are all eliminated in the final stability criterion by (20). Compared with the existing results, the stability criterion in Theorem 1 contains fewer decision variables, hence it is mathematically less complex and computationally more efficient.

Remark 2. Theorem 1 presents a new stability condition for the system with a single delay. Using the similar method proposed in Theorem 1, we can also get new stability conditions for the system with multiple delays, here they are omitted for space limit.

For the case of $\tau = 0$, we can directly derive the following corollary from Theorem 1.

Corollary 1. For given scalars $\eta > 0$, $\tau = 0$, and $0 < \alpha < 1$ satisfying $\alpha\mu < 1$, the system described by (1) $\sim$ (4) is asymptotically stable if there exist matrices $P = P^{\rm T} > 0$, $Q_i = Q_i^{\rm T} \geq 0$ $(i = 2,\ 3,\ 4)$ and $Z_j = Z_j^{\rm T} > 0$ $(j = 1,\ 3)$, such that

$$\Theta = \begin{bmatrix} \Theta_{11} & \Theta_{12} & 0 & (\alpha\eta)^{-1}Z_3 & \eta A^{\rm T}(Z_1+Z_3) \\ * & \Theta_{22} & \Theta_{23} & (\beta\eta)^{-1}Z_3 & \eta A_d^{\rm T}(Z_1+Z_3) \\ * & * & \Theta_{33} & 0 & 0 \\ * & * & * & \Theta_{44} & 0 \\ * & * & * & * & -\eta(Z_1+Z_3) \end{bmatrix} < 0 \tag{21}$$

holds, where

$$\Theta_{11} = PA + A^{\rm T}P + \sum_{i=2}^{4} Q_i - \eta^{-1}Z_1 - (\alpha\eta)^{-1}Z_3$$
$$\Theta_{12} = PA_d + \eta^{-1}Z_1$$
$$\Theta_{22} = -(1-\mu)Q_3 - \eta^{-1}(2Z_1+Z_3) - (\beta\eta)^{-1}Z_3$$
$$\Theta_{23} = \eta^{-1}(Z_1+Z_3)$$
$$\Theta_{33} = -Q_2 - \eta^{-1}(Z_1+Z_3)$$
$$\Theta_{44} = -\gamma Q_4 - (\alpha\eta)^{-1}Z_3 - (\beta\eta)^{-1}Z_3$$
$$\beta = 1-\alpha$$
$$\gamma = 1-\alpha\mu$$

2 Relation to the existing results

In the above section, new LMI-based delay-dependent conditions ensuring the stability of system (1) $\sim$ (4) have been presented. In the following, we will prove that the stability conditions obtained in [3], [7], and [8] are more conservative than Theorem 1. In addition, compared with [11], the result in [3] can be simplified further.

For convenience of comparison, the result in [8] is listed as the following lemma.

Lemma 1[8]. For given scalars τ, and η $(0 \leq \tau < \eta)$, the linear system (1) $\sim$ (4) is asymptotically stable if there exist matrices $P = P^{\rm T} > 0$, $Q_i = Q_i^{\rm T} > 0$ $(i = 1,\ 2,\ 3)$, $Z_j = Z_j^{\rm T} > 0$ $(j = 1,\ 2,\ 3)$, and N_i, M_i, S_i $(i = 1,\ 2)$, such that the following LMI holds:

$$\Lambda = \begin{bmatrix} \Lambda_1 & \Lambda_2 \\ * & \Lambda_3 \end{bmatrix} < 0 \tag{22}$$

where

$$\Lambda_1 = \begin{bmatrix} \Lambda_{11} & \Lambda_{12} & M_1 & -S_1 \\ * & \Lambda_{22} & M_2 & -S_2 \\ * & * & -Q_1 & 0 \\ * & * & * & -Q_2 \end{bmatrix}$$

$$\Lambda_{11} = PA + A^{\rm T}P + \sum_{i=1}^{3} Q_i + N_1 + N_1^{\rm T}$$
$$\Lambda_{12} = PA_d + N_2^{\rm T} - N_1 + S_1 - M_1$$
$$\Lambda_{22} = -(1-\mu)Q_3 + S_2 + S_2^{\rm T} - N_2 - N_2^{\rm T} - M_2 - M_2^{\rm T}$$

$$\Lambda_2 = \begin{bmatrix} \eta N_1 & (\eta-\tau)S_1 & (\eta-\tau)M_1 & A^{\rm T}U_1 \\ \eta N_2 & (\eta-\tau)N_2 & (\eta-\tau)M_2 & A_d^{\rm T}U_1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\Lambda_3 = {\rm diag}\{-\eta Z_1, -(\eta-\tau)\sum_{i=1}^{2} Z_i, -(\eta-\tau)Z_2, -U_1\}$$
$$U_1 = \eta Z_1 + (\eta-\tau)Z_2$$

In the following, we will prove that Theorem 1 is less conservative than Lemma 1.

Theorem 2. If inequality (22) is feasible, then inequality (8) is also feasible.

Proof. If inequality (22) is feasible, then there exists a sufficient small positive scalar $\varepsilon > 0$, such that

$$\Delta = \Lambda + \varepsilon\eta \begin{bmatrix} 0 & 0 & 0 & \cdots & 0 & A^{\rm T} \\ * & 0 & 0 & \cdots & 0 & A_d^{\rm T} \\ * & * & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ * & * & * & \cdots & 0 & 0 \\ * & * & * & \cdots & * & -I \end{bmatrix} + {\rm diag}\{\varepsilon I,\ 0,\ \cdots,\ 0,\ 0\} < 0 \tag{23}$$

where Λ is defined in (22). By taking $Y_i = 0$, $T_i = 0$ $(i = 1,\ 2,\ \cdots,\ 5)$, $N_j = 0$, $S_j = 0$, $M_j = 0$ $(j = 3,\ 4,\ 5)$ and $Q_4 = \varepsilon I$, $Z_3 = \varepsilon I$ in (19), (19) is equivalent to $\Delta < 0$ by the Schur complement. Thus, (19) is feasible too. This implies that (8) is also feasible because (8) is equivalent to (19). □

Next, to compare the stability result reported in [7] with Corollary 1 in this paper, the following lemma is needed.

Lemma 2. If Z is a positive definite matrix and $\mathcal{A}$ is a symmetrical matrix, and if Z, $\mathcal{A}$, Y are of appropriate dimensions and h is a positive constant, then there exists a symmetrical matrix $\mathcal{N} \geq 0$, such that

$$\mathcal{A} + h\mathcal{N} < 0 \tag{24}$$

and

$$\begin{bmatrix} \mathcal{N} & Y \\ Y^{\rm T} & Z \end{bmatrix} \geq 0 \tag{25}$$

hold, if and only if

$$\begin{bmatrix} \mathcal{A} & hY \\ hY^{\rm T} & -hZ \end{bmatrix} < 0 \tag{26}$$

Proof. (Necessity) From (25), we have

$$0 \leq YZ^{-1}Y^{\rm T} \leq \mathcal{N} \tag{27}$$

so, from (24), it is seen that $\mathcal{A}$ is a negative definite matrix and that

$$\mathcal{A} + hYZ^{-1}Y^{\rm T} \leq \mathcal{A} + h\mathcal{N} < 0 \tag{28}$$

Thus, it implies that (26) holds by the Schur complement.

(Sufficiency) If (26) holds, by letting $\mathcal{N} = YZ^{-1}Y^{\rm T}$ and using the Schur complement, (24) is equivalent to (26), and

$$\begin{bmatrix} \mathcal{N} & Y \\ Y^{\rm T} & Z \end{bmatrix} = \begin{bmatrix} I & YZ^{-1} \\ 0 & I \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 0 & Z \end{bmatrix} \begin{bmatrix} I & YZ^{-1} \\ 0 & I \end{bmatrix}^{\rm T} \tag{29}$$

so, (25) holds. □

Now, we re-write Theorem 2 in [7] as follows.

Lemma 3[7]**.** System (1) $\sim$ (4) with $\eta > 0$, $\tau = 0$, and $\mu < 1$ is asymptotically stable if there exist positive definite matrices $P > 0$, $Q > 0$, and $Z > 0$, semipositive definite matrix $N = \begin{bmatrix} N_{11} & N_{12} \\ * & N_{22} \end{bmatrix} \geq 0$, and any appropriately dimensioned matrices Y and T, such that

$$\begin{bmatrix} \Omega_1 & PA_d - Y + T^{\mathrm{T}} + \eta N_{12} & \eta A^{\mathrm{T}} Z \\ * & -(1-\mu)Q - T - T^{\mathrm{T}} + \eta N_{22} & \eta A_d^{\mathrm{T}} Z \\ * & * & -\eta Z \end{bmatrix} < 0 \quad (30)$$

and

$$\begin{bmatrix} N_{11} & N_{12} & Y \\ * & N_{22} & T \\ * & * & Z \end{bmatrix} \geq 0 \quad (31)$$

where

$$\Omega_1 = PA + A^{\mathrm{T}}P + Y + Y^{\mathrm{T}} + Q + \eta N_{11}$$

The following theorem shows that the stability condition in Corollary 1 is less conservative than the one in Lemma 3.

Theorem 3. If inequalities (30) and (31) are feasible, then inequality (21) is also feasible.

Proof. By Lemma 2 and under (31), inequality (30) is equivalent to

$$\begin{bmatrix} \Omega_2 & PA_d - Y + T^{\mathrm{T}} & \eta A^{\mathrm{T}} Z & \eta Y \\ * & -(1-\mu)Q - T - T^{\mathrm{T}} & \eta A_d^{\mathrm{T}} Z & \eta T \\ * & * & -\eta Z & 0 \\ * & * & * & -\eta Z \end{bmatrix} < 0 \quad (32)$$

where $\Omega_2 = PA + A^{\mathrm{T}}P + Y + Y^{\mathrm{T}} + Q$.

Similar to (20), inequality (32) is equivalent to

$$\Omega = \begin{bmatrix} \Omega_3 & PA_d + \frac{1}{\eta}Z & \eta A^{\mathrm{T}} Z \\ * & -(1-\mu)Q - \frac{1}{\eta}Z & \eta A_d^{\mathrm{T}} Z \\ * & * & -\eta Z \end{bmatrix} < 0 \quad (33)$$

where $\Omega_3 = PA + A^{\mathrm{T}}P + Q - \frac{1}{\eta}Z$.

Thus, from (33), there exists a small enough positive scalar $\varepsilon > 0$ such that

$$\Omega + \varepsilon\eta \begin{bmatrix} 0 & 0 & A^{\mathrm{T}} \\ * & 0 & A_d^{\mathrm{T}} \\ * & * & -I \end{bmatrix} + \varepsilon\alpha(1-\alpha)\eta\Omega_4^{\mathrm{T}}\Omega_4 < 0 \quad (34)$$

where $\Omega_4 = \begin{bmatrix} \frac{1}{\alpha\eta}I & \frac{1}{(1-\alpha)\eta}I & 0 \end{bmatrix}$.

Taking $Q_3 = Q$, $Z_1 = Z$, $Q_2 = Q_4 = 0$, $Z_3 = \varepsilon I$ and using the Schur complement, we can see that inequality (21) is true from (34). □

Finally, to show the relationship between the delay-dependent stability condition in [3] and Corollary 1 in this paper, we rewrite Lemma 1 in [3] as follows.

Lemma 4[3]**.** System (1) $\sim$ (4) with $\tau = 0$ and $\mu < 1$ is asymptotically stable if there exist matrices $P_1 > 0$, $S > 0$, $Z > 0$, $N_1 \geq 0$, $N_3 \geq 0$, and P_2, P_3, Y_1, Y_2, N_2 satisfying the following linear matrix inequalities (LMIs):

$$\begin{bmatrix} \Psi & P^{\mathrm{T}}\begin{bmatrix} 0 \\ A_d \end{bmatrix} - Y^{\mathrm{T}} \\ * & -S(1-\mu) \end{bmatrix} < 0 \quad (35)$$

and

$$\begin{bmatrix} Z & Y \\ * & N \end{bmatrix} \geq 0 \quad (36)$$

where

$$Y = \begin{bmatrix} Y_1 & Y_2 \end{bmatrix}, \ N = \begin{bmatrix} N_1 & N_2 \\ * & N_3 \end{bmatrix}, \ P = \begin{bmatrix} P_1 & 0 \\ P_2 & P_3 \end{bmatrix} \quad (37)$$

and

$$\begin{aligned} \Psi = P^{\mathrm{T}}\begin{bmatrix} 0 & I \\ A & -I \end{bmatrix} + \begin{bmatrix} 0 & I \\ A & -I \end{bmatrix}^{\mathrm{T}} P + \\ \eta N + \begin{bmatrix} S & 0 \\ 0 & \eta Z \end{bmatrix} + \begin{bmatrix} Y \\ 0 \end{bmatrix} + \begin{bmatrix} Y \\ 0 \end{bmatrix}^{\mathrm{T}} \end{aligned} \quad (38)$$

To compare Lemma 4 with Corollary 1, we compare Lemma 4 with Lemma 3.

Theorem 4. If inequalities (35) and (36) are feasible, then inequalities (30) and (31) are also feasible.

Proof. Substituting Ψ, Y, N, and P, in (35) and (36) by (37) and (38), and permutating some rows and columns in (35) and (36), respectively, we can get

$$\begin{bmatrix} \Psi_1 & P_2^{\mathrm{T}}A_d - Y_1^{\mathrm{T}} & \Psi_2 \\ * & -(1-\mu)S & A_d^{\mathrm{T}}P_3 - Y_2 \\ * & * & \Psi_3 \end{bmatrix} < 0 \quad (39)$$

and

$$\begin{bmatrix} N_1 & N_2 & Y_1^{\mathrm{T}} \\ * & N_3 & Y_2^{\mathrm{T}} \\ * & * & Z \end{bmatrix} \geq 0 \quad (40)$$

where

$$\begin{aligned} \Psi_1 &= P_2^{\mathrm{T}}A + A^{\mathrm{T}}P_2 + S + Y_1 + Y_1^{\mathrm{T}} + \eta N_1 \\ \Psi_2 &= P_1 - P_2^{\mathrm{T}} + A^{\mathrm{T}}P_3 + Y_2 + \eta N_2 \\ \Psi_3 &= -P_3 - P_3^{\mathrm{T}} + \eta Z + \eta N_3 \end{aligned}$$

Using Lemma 2, we have

$$\begin{bmatrix} \Psi_4 & P_2^{\mathrm{T}}A_d - Y_1^{\mathrm{T}} & \Psi_5 & \eta Y_1^{\mathrm{T}} \\ * & -(1-\mu)S & A_d^{\mathrm{T}}P_3 - Y_2 & 0 \\ * & * & \Psi_6 & \eta Y_2^{\mathrm{T}} \\ * & * & * & -\eta Z \end{bmatrix} < 0 \quad (41)$$

where

$$\begin{aligned} \Psi_4 &= P_2^{\mathrm{T}}A + A^{\mathrm{T}}P_2 + S + Y_1 + Y_1^{\mathrm{T}} \\ \Psi_5 &= P_1 - P_2^{\mathrm{T}} + A^{\mathrm{T}}P_3 + Y_2 \\ \Psi_6 &= -P_3 - P_3^{\mathrm{T}} + \eta Z \end{aligned}$$

Obviously, the above inequality can be rewritten as

$$\Xi + \mathcal{C}^{\mathrm{T}}\begin{bmatrix} P_2^{\mathrm{T}} \\ P_3^{\mathrm{T}} \end{bmatrix}\mathcal{B} + \mathcal{B}^{\mathrm{T}}\begin{bmatrix} P_2 & P_3 \end{bmatrix}\mathcal{C} < 0 \quad (42)$$

where

$$\Xi = \begin{bmatrix} S + Y_1 + Y_1^{\mathrm{T}} & -Y_1^{\mathrm{T}} & P_1 + Y_2 & \eta Y_1^{\mathrm{T}} \\ * & -(1-\mu)S & -Y_2 & 0 \\ * & * & \eta Z & \eta Y_2^{\mathrm{T}} \\ * & * & * & -\eta Z \end{bmatrix}$$

$$\mathcal{C} = \begin{bmatrix} I & 0 & 0 & 0 \\ 0 & 0 & I & 0 \end{bmatrix}, \ \mathcal{B} = \begin{bmatrix} A & A_d & -I & 0 \end{bmatrix}$$

By the elimination Lemma[14], it is readily seen that there exist matrices P_i ($i = 2, 3$) that solve inequality (42), if and only if

$$\mathcal{N}_{\mathcal{B}}^{\mathrm{T}}\Xi\mathcal{N}_{\mathcal{B}} < 0, \qquad \mathcal{N}_{\mathcal{C}}^{\mathrm{T}}\Xi\mathcal{N}_{\mathcal{C}} < 0 \quad (43)$$

hold, where $\mathcal{N}_{\mathcal{B}}$ and $\mathcal{N}_{\mathcal{C}}$ denote the full-rank matrix representations of the right annihilators of $\mathcal{B}$ and $\mathcal{C}$, respectively,

$$\mathcal{N}_{\mathcal{B}} = \begin{bmatrix} I & 0 & 0 \\ 0 & I & 0 \\ A & A_d & 0 \\ 0 & 0 & I \end{bmatrix}, \ \mathcal{N}_{\mathcal{C}} = \begin{bmatrix} 0 & 0 \\ I & 0 \\ 0 & 0 \\ 0 & I \end{bmatrix}$$

Because

$$\mathcal{N}_{\mathcal{B}}{}^{\mathrm{T}} \Xi \mathcal{N}_{\mathcal{B}} = \begin{bmatrix} \Xi_1 & \Xi_2 & \eta \bar{Y}_1^{\mathrm{T}} \\ * & \Xi_3 & \eta \bar{Y}_2^{\mathrm{T}} \\ * & * & -\eta Z \end{bmatrix} \tag{44}$$

where

$$\begin{aligned} &\Xi_1 = S + \bar{Y}_1 + \bar{Y}_1^{\mathrm{T}} + P_1 A + A^{\mathrm{T}} P_1 + \eta A^{\mathrm{T}} Z A \\ &\Xi_2 = -\bar{Y}_1^{\mathrm{T}} + P_1 A_d + \bar{Y}_2 + \eta A^{\mathrm{T}} Z A_d \\ &\Xi_3 = -(1-\mu) S - \bar{Y}_2 - \bar{Y}_2^{\mathrm{T}} + \eta A_d^{\mathrm{T}} Z A_d \\ &\bar{Y}_1 = Y_1 + Y_2 A \\ &\bar{Y}_2 = Y_2 A_d \end{aligned}$$

and

$$\mathcal{N}_{\mathcal{C}}{}^{\mathrm{T}} \Xi \mathcal{N}_{\mathcal{C}} = \begin{bmatrix} -(1-\mu) S & 0 \\ 0 & -\eta Z \end{bmatrix} \tag{45}$$

(45) is obviously true because S and Z are positive definite matrices, so by the Schur complement and (44), $\mathcal{N}_{\mathcal{B}}{}^{\mathrm{T}} \Xi \mathcal{N}_{\mathcal{B}} < 0$ is equivalent to

$$\begin{bmatrix} \Psi_7 & P_1 A_d - \bar{Y}_1^{\mathrm{T}} + \bar{Y}_2 & \eta \bar{Y}_1^{\mathrm{T}} & \eta A^{\mathrm{T}} Z \\ * & -(1-\mu) S - \bar{Y}_2 - \bar{Y}_2^{\mathrm{T}} & \eta \bar{Y}_2^{\mathrm{T}} & \eta A_d^{\mathrm{T}} Z \\ * & * & -\eta Z & 0 \\ * & * & * & -\eta Z \end{bmatrix} < 0 \tag{46}$$

where $\Psi_7 = S + \bar{Y}_1 + \bar{Y}_1^{\mathrm{T}} + P_1 A + A^{\mathrm{T}} P_1, \ \bar{Y}_2 = Y_2 A_d$ and $\bar{Y}_1, \ Y_2$ are slack variables. This means that (35) with (36) is equivalent to (46).

Therefore, if $Y = \bar{Y}_1^{\mathrm{T}}$, $T = \bar{Y}_2^{\mathrm{T}}$, $Q = S$, and $P = P_1$ in (32), then (32) is feasible when (35) and (36) are feasible. This implies that Lemma 3 is less conservative than Lemma 4. □

Remark 3. Theorem 2 shows that the stability condition in Theorem 1 of this paper is less conservative than the one in [8], which stated that its result is less conservative than the one in [7]. As shown in Theorem 4, the stability condition in [7] is less conservative than the one in [3]. From a mathematical point of view, the condition in Theorem 1 of this paper is more efficient than those in [3], [7−8], and [11] because it involves the least number of variables and provides the least conservatism. Table 1 (see next page) provides a comparison of the numbers of the variables involved in [3], [7−8], and [11].

Remark 4. As shown in [11], the slack variables $N_1, \ N_2$, and N_3 in Lemma 4 are all redundant. Compared with [11], we further prove that the slack variables P_2 and P_3 are redundant too in the proof of Theorem 4, which can be seen from the equivalence between (35) under (36) and (46). Therefore, the delay-dependent stability conditions obtained by the descriptor system approach in [3] can be further simplified. Similarly, the results in [11, 15 − 16] can also be simplified in this way. In addition, the variables $Y_1, \ Y_2$ in (41), and $\bar{Y}_1$ and $\bar{Y}_2$ in (46) can be eliminated by the method given in (20), thus more simplified results can be easily derived.

Theorem 2 shows that the obtained stability condition in Theorem 1 is less conservative than the one in [8] for any given scalar α satisfying $0 < \alpha < 1$. To seek an appropriate α satisfying $0 < \alpha < 1$ such that the upper bound η of delay $d(t)$ satisfying (3) and (4) is maximal, we give an algorithm as follows.

Algorithm 1. (Maximizing $\eta > 0$):

Step 1. Set appropriate step lengths, η_{step} and α_{step}, for η and α, respectively. Set $\alpha = \alpha_{step}$. For given μ and τ, choose an upper bound on η satisfying (8), and then select this upper bound as the initial value η_0 of η.

Step 2. Set k as a counter, and choose $k = 1$. Meanwhile, let $\eta = \eta_0 + \eta_{step}$ and the initial value α_0 of α equal α_{step}.

Step 3. Let $\alpha = k\alpha_{step}$, and if inequality (8) is feasible, go to Step 4; otherwise, go to Step 5.

Step 4. Let $\eta_0 = \eta$, $\alpha_0 = \alpha$, $k = 1$, and $\eta = \eta_0 + \eta_{step}$, and go to Step 3.

Step 5. Let $k = k + 1$. If $k\alpha_{step} < 1$ and $k\alpha_{step}\mu < 1$, then go to Step 3; otherwise, stop.

Remark 5. In the above algorithm, the final η_0 is the desired maximum of the upper bound of delay $d(t)$ satisfying (8), and α_0 is the corresponding value of α.

3 Numerical examples

In this section, we use two numerical examples to show the benefits of our results. Here, we set step lengths, $\eta_{step} = 0.01$ and $\alpha_{step} = 0.01$, for η and α, respectively.

Example 1. Consider the following system[8]:

$$\dot{\boldsymbol{x}}(t) = \begin{bmatrix} -2 & 0 \\ 0 & -0.9 \end{bmatrix} \boldsymbol{x}(t) + \begin{bmatrix} -1 & 0 \\ -1 & -1 \end{bmatrix} \boldsymbol{x}(t - d(t)) \tag{47}$$

and

$$0 \le d(t) \le \eta, \quad \dot{d}(t) \le \mu \tag{48}$$

where η and μ are constants.

According to Algorithm 1, for various μ, the computed upper bounds, η, which guarantee the stability of system (47) for $\tau = 0$ are listed in Table 2 (see next page). It is clear that our results are superior to those in [8], and the results of [8] are better than the ones obtained by [3].

Example 2. Consider the following system described by

$$\dot{\boldsymbol{x}}(t) = \begin{bmatrix} -2 & 0 \\ 1 & -3 \end{bmatrix} \boldsymbol{x}(t) + \begin{bmatrix} -1.4 & 0 \\ -0.8 & -1.5 \end{bmatrix} \boldsymbol{x}(t - d(t)) \tag{49}$$

and

$$\tau \le d(t) \le \eta, \qquad \dot{d}(t) \le \mu \tag{50}$$

where $\tau, \ \eta$, and $\mu \ge 1$ are constants.

For various τ, the computed upper bounds, η, which guarantee the stability of system (49), are listed in Table 3 (see next page), which also illustrates the merits of the method proposed in this paper.

From (48) and (50), we observe that the lower bound of $d(t)$ used in Example 1 is 0, whereas in this example, it may be larger than 0, which is the main difference between Example 1 and Example 2.

4 Conclusion

In this paper, the problem of stability analysis for continuous-time systems with time-varying delays has been investigated. The information about the upper bound of delay derivative is taken into consideration even if this upper bound is not smaller than 1. The obtained stability conditions are less conservative and have fewer decision variables than the corresponding ones in the existing literature. In addition, the methods for simplifying the delay-dependent stability conditions obtained by the descriptor

Table 1 Comparison of the numbers of the variables involved

Methods	Number of the variables involved
Lemma 1 in [8]	$9.5n^2 + 3.5n$
Lemma 3 in [7]	$5.5n^2 + 2.5n$
Lemma 4 in [3]	$7.5n^2 + 2.5n$
Theorem 1 in [11]	$5.5n^2 + 1.5n$
Theorem 1 in this paper	$4n^2 + 4n$

Table 2 Allowable upper bounds of η and the corresponding α for various μ

Methods	$\mu = 0.9$	$\mu = 1.0$	$\mu = 1.2$	$\mu = 1.5$
Lemma 4 in [3]	1.18	0.99	0.99	0.99
Lemma 1 in [8]	1.37	1.34	1.34	1.34
Corollary 1	1.43 ($\alpha = 0.80$)	1.39 ($\alpha = 0.62$)	1.36 ($\alpha = 0.6$)	1.35 ($\alpha = 0.5$)

Table 3 Allowable upper bounds of η and the corresponding α for given τ

μ	Methods	$\tau = 0$	$\tau = 0.1$	$\tau = 0.2$	$\tau = 0.3$	$\tau = 0.4$
−	Lemma 1 in [8]	0.92	0.95	1.02	1.09	1.16
1.0	Theorem 1	1.09 ($\alpha = 0.73$)	1.10 ($\alpha = 0.70$)	1.12 ($\alpha = 0.72$)	1.14 ($\alpha = 0.73$)	1.17 ($\alpha = 0.60$)
1.2	Theorem 1	1.03 ($\alpha = 0.62$)	1.04 ($\alpha = 0.58$)	1.06 ($\alpha = 0.58$)	1.10 ($\alpha = 0.52$)	1.16 ($\alpha = 0.01$)

system approach and by the free-weighting matrix approach are presented, and the proposed methods can reduce the computational complexity without changing the conservatism of those conditions. Two numerical examples are given to illustrate the effectiveness and less conservatism of the presented stability conditions.

References

1 Fridman E, Shaked U. Delay-dependent stability and H_∞ control: constant and time-varying delays. *International Journal of Control*, 2003, **76**(1): 48−60

2 Fridman E. New Lyapunov-Krasovskii functionals for stability of linear retarded and neutral type systems. *Systems and Control Letters*, 2001, **43**(4): 309−319

3 Fridman E, Shaked U. An improved stabilization method for linear time-delay systems. *IEEE Transactions on Automatic Control*, 2002, **47**(11): 1931−1937

4 Fridman E, Shaked U. A descriptor system approach to H_∞ control of linear time-delay systems. *IEEE Transactions on Automatic Control*, 2002, **47**(2): 253−270

5 Suplin V, Fridman E, Shaked U. H_∞ control of linear uncertain time-delay systems – a projection approach. *IEEE Transactions on Automatic Control*, 2006, **51**(4): 680−685

6 Fridman E, Shaked U. Parameter dependent stability and stabilization of uncertain time-delay systems. *IEEE Transactions on Automatic Control*, 2003, **48**(5): 861−866

7 Wu M, He Y, She J H, Liu G P. Delay-dependent criteria for robust stability of time-varying delay systems. *Automatica*, 2004, **40**(8): 1435−1439

8 He Y, Wang Q G, Lin C, Wu M. Delay-range-dependent stability for systems with time-varying delay. *Automatica*, 2007, **43**(2): 371−376

9 Moon Y S, Park P G, Kwon W H, Lee Y S. Delay-dependent robust stabilization of uncertain state-delayed systems. *International Journal of Control*, 2001, **74**(14): 1447−1455

10 Lee Y S, Moon Y S, Kwon W H, Park P G. Delay-dependent robust H_∞ control for uncertain systems with a state-delay. *Automatica*, 2004, **40**(1): 65−72

11 Xu S Y, Lam J, Zuo Y. Simplified descriptor system approach to delay-dependent stability and performance analyses for time-delay systems. *IEE Proceedings Control Theorey and Applications*, 2005, **152**(2): 147−151

12 Naghshtabrizi P, Hespanha J P. Designing an observer-based controller for a network control system. In: Proceedings of the 44th IEEE Conference on Decision and Control. Piscataway, USA: IEEE, 2005. 848−853

13 Yue D, Han Q L, Lam J. Network-based robust H_∞ control of systems with uncertainty. *Automatica*, 2005, **41**(6): 999−1007

14 Boyd S, Ghaoui L E, Feron E, Balakrishnan V. *Linear Matrix Inequality in Systems and Control Theory*. Philadelphia: Society for Industrial and Applied Mathematics, 1994. 22

15 Su J H. Futher results on the robust stability of linear systems with a single time delay. *Systems and Control Letters*, 1994, **23**(5): 375−379

16 Xu S Y, Lam J. On equivalence and efficiency of certain stability criteria for time-delay systems. *IEEE Transactions on Automatic Control*, 2007, **52**(1): 95−101

ZHU Xun-Lin Ph. D. candidate in the College of Information Science and Engineering at Northeastern University. He is also an associate professor at Zhengzhou University of Light Industry. His research interest covers networked control systems and quantized control. Corresponding author of this paper.
E-mail: hntjxx@163.com

YANG Guang-Hong Professor at Northeastern University. His research interest covers fault-tolerant control, fault detection and isolation, and robust control.
E-mail: yangguanghong@ise.neu.edu.cn