Extended Cubic Uniform B-spline and α-B-spline

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Abstract Spline curve and surface play an important role in CAD and computer graphics. In this paper, we propose several extensions of cubic uniform B-spline. Then, we present the extensions of interpolating α-B-spline based on the new B-splines and the singular blending technique. The advantage of the extensions is that they have global and local shape parameters. Furthermore, we also investigate their applications in data interpolation and polygonal shape deformation.

Key words B-spline, α-B-spline, singular blending, interpolation, curve/surface modeling

Curve and surface modeling is an important subject in CAD and computer graphics[1−3]. Spline curve/surface modeling is the most traditional modeling method based on the theory of computer aided geometric design (CAGD)[4]. Several kinds of splines have been proposed in the field of CAGD, such as B-splines and T-splines[5]. In practice, we often use cubic uniform B-spline for curve/surface modeling. However, once the control points of the cubic uniform B-spline curve are determined, the shape of the curve is determined. In order to overcome this disadvantage, several extensions of cubic uniform B-spline have been proposed[6−8]. On the other hand, cubic B-spline interpolation is the traditional global interpolation method[9−10]. Unfortunately, it also has some disadvantages, making them less desirable for certain applications. First, it cannot provide parameter for curve local modification; second, it may exhibit undesirable oscillations; and finally, solving the linear system is very expensive computationally. Moreover, the interpolation method is global; thus, changes to any data point will require solving again all the linear systems.

In order to avoid the above limitations, Tai et al. proposed a new method to solve the interpolation problem[11] based on the α-B-spline presented in [12]. The generalizations and applications of α-B-spline were investigated in [13−15]. However, the above interpolating splines based on singular blending cannot conveniently modify the curves. If the users want to globally modify the curves, they must set all the local parameters to be equal. In this paper, we first propose two kinds of new extensions of cubic uniform B-spline. Then we propose the extensions of interpolating α-B-spline by the singular blending technique, and name them α-EB-spline. They not only have local shape parameters, but also have global parameters. We also present their applications in data interpolation and shape deformation.

1 Extended cubic uniform B-spline blending functions

The blending function of degree 4 was proposed in [6], which is an extension of cubic uniform B-spline basis function.

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Definition 1. For $t \in [0, 1]$, the blending function of degree 4 is defined as follows.

$$\begin{cases} b_0^4(t) = \dfrac{1}{24}(4-\lambda-3\lambda t)(1-t)^3 \\ b_1^4(t) = \dfrac{1}{24}[16+2\lambda-12(2+\lambda)t^2+12(1+\lambda)t^3-3\lambda t^4] \\ b_2^4(t) = \dfrac{1}{24}[4-\lambda+12t+6(2+\lambda)t^2-12t^3-3\lambda t^4] \\ b_3^4(t) = \dfrac{1}{24}[4(1-\lambda)+3\lambda t]t^3 \end{cases}$$

In particular, in case of $\lambda = 0$, the blending function of degree 4 will degenerate to the cubic uniform B-spline blending function. In the following, we will list two new kinds of extensions of blending functions of degree 4. Hence, they can be considered as further extensions of cubic uniform B-spline blending function.

Definition 2. For $t \in [0, 1]$, the blending function of degree 5 is defined as follows.

$$\begin{cases} b_0^5(t) = \dfrac{1}{40}(5-\lambda-4\lambda t)(1-t)^4 \\ b_1^5(t) = \dfrac{1}{40}[30+2\lambda-20(3+\lambda)t^2+40(1+\lambda)t^3-5(1+ \\ \qquad 7\lambda)t^4+12\lambda t^5] \\ b_2^5(t) = \dfrac{1}{40}[5-\lambda+20t+10(3+\lambda)t^2-20(1+\lambda)t^3- \\ \qquad 5(1-5\lambda)t^4-12\lambda t^5] \\ b_3^5(t) = \dfrac{1}{40}[5(1-\lambda)+4\lambda t]t^4 \end{cases}$$

Definition 3. For $t \in [0, 1]$, the blending function of degree 6 is defined as follows.

$$\begin{cases} b_0^6(t) = \dfrac{1}{60}(6-\lambda-5\lambda t)(1-t)^5 \\ b_1^6(t) = \dfrac{1}{60}[48+2\lambda-30(4+\lambda)t^2+40(3+2\lambda)t^3-30(2+ \\ \qquad 3\lambda)t^4+6(3+7\lambda)t^5-5\lambda t^6] \\ b_2^6(t) = \dfrac{1}{60}[6-\lambda+30t+15(4+\lambda)t^2-20(3+2\lambda)t^3+ \\ \qquad 15(2+3\lambda)t^4-6(3+2\lambda)t^5-5\lambda t^6] \\ b_3^6(t) = \dfrac{1}{60}[6(1-\lambda)+5\lambda t]t^5 \end{cases}$$

When $\lambda = 0$, $b_i^k(t)$ will be $b_i^{k-1}(t)$ with $\lambda = 1$, $k = 5, 6$. The above three kinds of blending functions have the following theorem.

Theorem 1. The blending functions $b_i^k(t)$, where $k = 4, 5, 6$, and $i = 0, 1, 2, 3$, satisfy

1) $\sum_{i=0}^3 b_i^k(t) = 1$;
2) $b_i^k(t) = b_{3-i}^k(1-t)$;
3) When $-k(k-2) \le \lambda \le 1$, $b_i^k(t) \ge 0$, $t \in [0, 1]$.

2 Extended cubic uniform B-spline curves

The properties mentioned in Theorem 1 enable the blending functions to be used for curve design.

Definition 4. Given control points $\boldsymbol{P}_i \in \mathbf{R}^d$ ($d = 2, 3, i = 0, 1, \cdots, n$), and the knots $u_1 < u_2 < \cdots < u_{n+1}$, for $u \in [u_i, u_{i+1}]$, $i = 3, 4, \cdots, n$, the polynomial curve segments are defined as follows.

$$\boldsymbol{C}_{j,k}(\lambda; t) = \sum_{i=0}^{3} b_i^k(t)\boldsymbol{P}_{j+i-3}, \quad k = 4, 5, 6$$

where $t = \dfrac{u - u_i}{h_i}, h_i = u_{i+1} - u_i$. The polynomial curve is defined as follows.

$$\boldsymbol{C}_k(\lambda; u) = \boldsymbol{C}_{i;k}(\lambda; \frac{u - u_i}{h_i}), \quad u \in [u_i, u_{i+1}]$$

They can be considered as extensions of cubic uniform B-spline curve. They have many nice properties such as convex full property, symmetry, and geometric invariability.

Theorem 2. For the case of uniform knot, $\boldsymbol{C}_k(\lambda; u)$ is C^2 continuous; for the case of nonuniform knot, $\boldsymbol{C}_k(\lambda; u)$ is G^2 continuous.

Proof. After direct computation for $k = 5$ and $k = 6$, we have

$$\begin{aligned}
\boldsymbol{C}_{i,k}(\lambda; 0) =& \frac{1}{2k(k-1)}\{(k - \lambda)\boldsymbol{P}_{i-3} + 2[k(k-2) + \lambda]\boldsymbol{P}_{i-2} + \\
& (k - \lambda)\boldsymbol{P}_{i-1}\} \\
\boldsymbol{C}_{i,k}(\lambda; 1) =& \frac{1}{2k(k-1)}\{(k - \lambda)\boldsymbol{P}_{i-2} + 2[k(k-2) + \lambda]\boldsymbol{P}_{i-1} + \\
& (k - \lambda)\boldsymbol{P}_{i}\} \\
\boldsymbol{C}'_{i,k}(\lambda; 0) =& \frac{1}{2}(\boldsymbol{P}_{i-1} - \boldsymbol{P}_{i-3}), \quad \boldsymbol{C}'_{i,k}(\lambda; 1) = \frac{1}{2}(\boldsymbol{P}_{i} - \boldsymbol{P}_{i-2}) \\
\boldsymbol{C}''_{i,k}(\lambda; 0) =& \frac{1}{2}(k + \lambda - 2)(\boldsymbol{P}_{i-3} - 2\boldsymbol{P}_{i-2} + \boldsymbol{P}_{i-1}) \\
\boldsymbol{C}''_{i,k}(\lambda; 1) =& \frac{1}{2}(k + \lambda - 2)(\boldsymbol{P}_{i-2} - 2\boldsymbol{P}_{i-1} + \boldsymbol{P}_{i})
\end{aligned}$$

Hence, for $i = 4, 5, \cdots, n$, we can obtain

$$\boldsymbol{C}^{(l)}_k(\lambda; u_i^-) = (\frac{h_i}{h_{i-1}})^l \boldsymbol{C}^{(l)}_k(\lambda; u_i^+), \quad l = 0, 1, 2, k = 5, 6$$

Thus, the conclusion is proved. □

Fig. 1 presents the curves constructed by various blending functions with different shape parameters. We can find that the approaching degrees of the curves of degree 5 and degree 6 to their control polygon are higher than that of the curves of degree 4 with the same shape parameters.

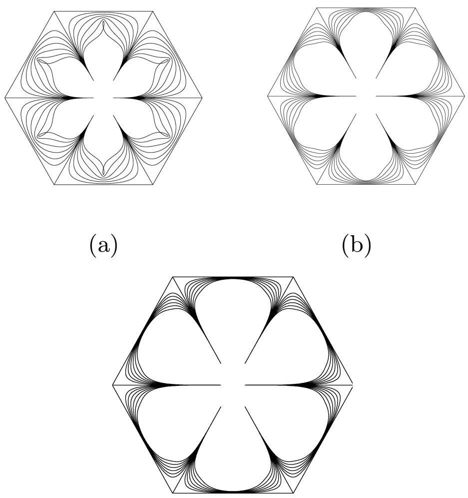

Fig. 1 Curve examples constructed by blending functions with $\lambda = -4, -3, -2, -1, 0, 1$ ((a) Curves constructed by blending function of degree 4; (b) Curves constructed by blending function of degree 5; (c) Curves constructed by blending function of degree 6)

3 Singularly reparameterized line segment

A singularly reparameterized (SR) line segment is a line segment that possesses parametric derivatives equal to zero at each end. It is obtained by blending two endpoints with a singular blending function[11, 15]. An m-level singular blending function $S(t), t \in [0, 1]$, satisfies the following conditions

$$\begin{aligned}
& S(0) = 0, \quad S(1) = 1 \\
& S^{(k)}(0) = S^{(k)}(1) = 0, \quad k = 1, 2, \cdots, m
\end{aligned} \tag{1}$$

Loe used a 2-level singular blending function and suggested $S(t) = 1 - (1 - t^3)^3$ in [12]. Another choice is the quintic Hermite polynomial $S(t) = 10t^3 - 15t^4 + 6t^5$. For the remainder of this paper, we will adopt the singular blending function presented in [14]. It is a piecewise cubic polynomial defined on three subintervals.

$$S(t) = \begin{cases} \dfrac{9}{2}t^3, & 0 \le t \le \dfrac{1}{3} \\ \dfrac{9}{2}[t^3 - 3(t - \dfrac{1}{3})^3], & \dfrac{1}{3} \le t \le \dfrac{2}{3} \\ 1 - \dfrac{9}{2}(1 - t)^3, & \dfrac{2}{3} \le t \le 1 \end{cases} \tag{2}$$

Using the singular blending function $S(t)$, we can blend two adjacent vertices $\boldsymbol{V}_j$ and $\boldsymbol{V}_{j+1}$ to produce a SR line segment $\boldsymbol{L}_j(t)$, that is,

$$\boldsymbol{L}_j(t) = (1 - S(t))\boldsymbol{V}_j + S(t)\boldsymbol{V}_{j+1}, \quad t \in [0,\ 1]$$

From (1), we have

$$\begin{aligned}
& \boldsymbol{L}_j(0) = \boldsymbol{V}_j, \quad \boldsymbol{L}_j(1) = \boldsymbol{V}_{j+1} \\
& \boldsymbol{L}'_j(0) = \boldsymbol{L}'_j(1) = \boldsymbol{L}''_j(0) = \boldsymbol{L}''_j(1) = 0
\end{aligned}$$

4 α-EB-Spline curves

For the interpolating α-B-spline presented in [11] and [13], a local tension parameter α_j will be assigned to each vertex $\boldsymbol{P}_j$. And we interpolate these tension parameters using the singular blending function.

$$\alpha_j(t) = (1 - S(t))\alpha_j + S(t)\alpha_{j+1}$$

By using the singular blending technique, we construct the interpolating α-EB-spline as follows.

$$\begin{aligned}
& \boldsymbol{Q}^k_j(\lambda; t) = (1 - \alpha_j(t))\boldsymbol{C}_{j,k}(\lambda; t) + \alpha_j(t)\boldsymbol{L}_j(t) = \\
& (1 - \alpha_j(t))\boldsymbol{C}_{j,k}(\lambda; t) + \alpha_j(t)[(1 - S(t))\boldsymbol{V}_j + S(t)\boldsymbol{V}_{j+1}]
\end{aligned}$$

where $0 \le \alpha_j \le 1, S(t)$ is defined as (2), and

$$\boldsymbol{V}_j = \boldsymbol{P}_j + \frac{1 - \alpha_j}{\alpha_j}(\boldsymbol{P}_j - \boldsymbol{C}_{j,k}(\lambda; 0))$$

We easily obtain that

$$\boldsymbol{Q}_j(0) = \boldsymbol{P}_j, \quad \boldsymbol{Q}_j(1) = \boldsymbol{P}_{j+1}$$

For open control polygon, we introduce two additional points $\boldsymbol{P}_{-1}$ and $\boldsymbol{P}_{n+2}$, where

$$\boldsymbol{P}_{-1} = 2\boldsymbol{P}_0 - \boldsymbol{P}_1, \boldsymbol{P}_{n+2} = 2\boldsymbol{P}_{n+1} - \boldsymbol{P}_n$$

The plantom points are chosen so that the extended B-spline curve interpolates the endpoints $\boldsymbol{P}_0$ and $\boldsymbol{P}_{n+1}$ (See Fig. 2).

To produce the closed curve for closed polygon, three additional points $\boldsymbol{P}_{-1}$, $\boldsymbol{P}_{n+2}$, and $\boldsymbol{P}_{n+3}$ are introduced as follows (See Fig. 3).

$$\boldsymbol{P}_{-1} = \boldsymbol{P}_n, \boldsymbol{P}_{n+2} = \boldsymbol{P}_0, \boldsymbol{P}_{n+3} = \boldsymbol{P}_1$$

The interpolating α-EB-spline interpolates the polygon vertices as shown in Figs. 2 and 3. As $\boldsymbol{L}_j(t)$ and $\boldsymbol{C}_{j,k}(\lambda;t)$ are both C^2 continuous, the α-EB-spline is also C^2 continuous. We can also change the shape of the curves by adjusting the shape parameters α_j and λ (See Figs. 2 and 3).

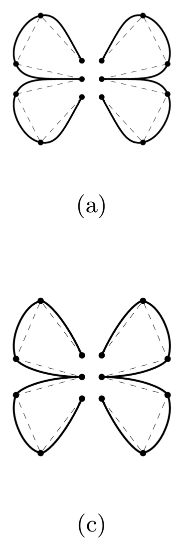

Fig. 2 The butterfly-like curves constructed by the blending function of degree 5

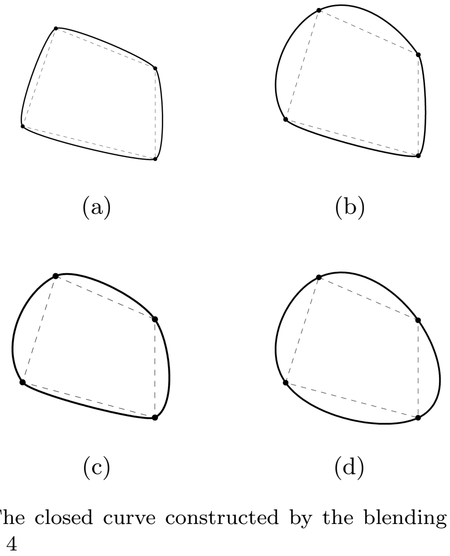

Fig. 3 The closed curve constructed by the blending function of degree 4

5 α-EB-spline surface

For the blending of extended B-spline curve with SR line segments, we can blend extended B-spline surface with network of singularly reparametrized bilinear patches. An SR bilinear patch is determined by four vertices $\{\boldsymbol{V}_{i,j}, \boldsymbol{V}_{i,j+1}, \boldsymbol{V}_{i+1,j}, \boldsymbol{V}_{i+1,j+1}\}$:

$$\boldsymbol{L}_{i,j}(u,v) = (1-S(u))(1-S(v))\boldsymbol{V}_{i,j} + (1-S(u))S(v)\times \boldsymbol{V}_{i,j+1} + S(u)(1-S(v))\boldsymbol{V}_{i+1,j} + S(u)S(v)\boldsymbol{V}_{i+1,j+1}$$

where $S(u)$ and $S(v)$ are defined as (2).

Obviously, its first and second order partial derivatives vanish at the boundaries.

$$\partial_u \boldsymbol{L}_{i,j}(0,v) = \partial_u \boldsymbol{L}_{i,j}(1,v) = \partial_v \boldsymbol{L}_{i,j}(u,0) = \partial_v \boldsymbol{L}_{i,j}(u,1) = 0$$
$$\partial_u^2 \boldsymbol{L}_{i,j}(0,v) = \partial_u^2 \boldsymbol{L}_{i,j}(1,v) = \partial_v^2 \boldsymbol{L}_{i,j}(u,0) = \partial_v^2 \boldsymbol{L}_{i,j}(u,1) = 0$$
$$\partial_{uv}^2 \boldsymbol{L}_{i,j}(0,v) = \partial_{uv}^2 \boldsymbol{L}_{i,j}(1,v) = \partial_{uv}^2 \boldsymbol{L}_{i,j}(u,0) = \partial_{uv}^2 \boldsymbol{L}_{i,j}(u,1) = 0$$

The extended B-spline surface is defined by

$$\boldsymbol{C}_{i,j}^k(u,v) = \sum_{n=0}^{3}\sum_{m=0}^{3} b_{n,k}(\lambda;u) b_{m,k}(\lambda;v) \boldsymbol{P}_{i+n-1,j+m-1}$$

where $b_{n,k}(\lambda;u)$ and $b_{m,k}(\lambda;v)$ are the extended blending functions, and $\boldsymbol{P}_{i+n-1,j+m-1}$ is the control point. The interpolating α-EB-spline surface is given by

$$\boldsymbol{Q}_{i,j}^k(u,v) = (1-\alpha_{ij}(u,v))\boldsymbol{C}_{i,j}^k(u,v) + \alpha_{ij}(u,v)\boldsymbol{L}_{ij}(u,v)$$

where

$$\alpha_{i,j}(u,v) = (1-S(u))(1-S(v))\alpha_{i,j} + (1-S(u))S(v)\times \alpha_{i,j+1} + S(u)(1-S(v))\alpha_{i+1,j} + S(u)S(v)\alpha_{i+1,j+1}$$

$\boldsymbol{V}_{i,j}$ in $\boldsymbol{L}_{ij}(u,v)$ are defined as follows.

$$\boldsymbol{V}_{i,j} = \boldsymbol{P}_{i,j} + \frac{1-\alpha_{ij}}{\alpha_{ij}}(\boldsymbol{P}_{ij} - \boldsymbol{C}_{i,j}^k(0,0))$$

Obviously, $\boldsymbol{Q}_{i,j}^k(0,0) = \boldsymbol{P}_{i,j}$. That is, interpolating α-EB-spline surface interpolates the vertices of the network. It is also C^2 continuous. We can change the shape of the surface by adjusting the shape parameters λ and α_{ij}.

6 Applications

As the α-B-spline[11−14], the interpolating α-EB-spline can also be used for curve/surface interpolation. It avoids the necessity of solving linear systems for interpolation of B-spline curve/surface (See Fig. 4). As the interpolating α-B-spline[15], we can also use α-EB-spline for deforming polygonal shapes into smooth surfaces. The user only needs to input a polygonal shape, then the modeling system will generate an original smooth surface interpolating all the polygonal vertices. The user can modify the smooth surface, both globally and locally, by changing the global shape parameter λ and the local tension parameters α_{ij}, respectively. Fig. 5 shows the effects of the global shape parameter in a interpolating surfaces constructed by the blending functions of degree 6.

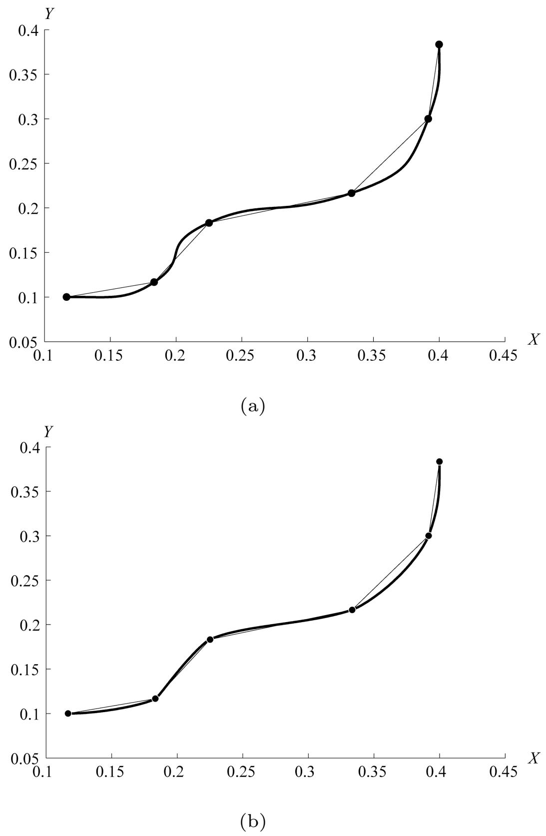

Fig. 4 Data points interpolation with the same λ but different α_j

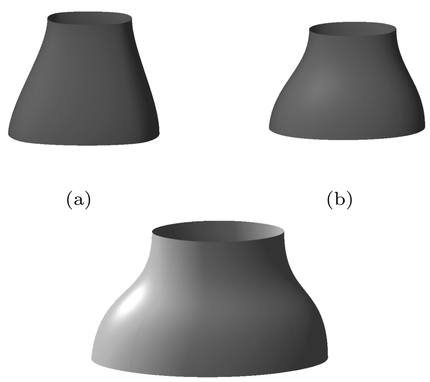

Fig. 5 The effects of global shape parameter λ ((a) $\lambda = -2$; (b) $\lambda = 0$; (c) $\lambda = 1$)

7 Conclusion

In this paper, we first proposed two kinds of new extensions of cubic uniform B-spline. The extensions have global shape parameters, so the users can modify the curve globally by adjusting the shape parameters. Then, we proposed the extensions of interpolating α-B-spline by the singular blending technique. We also studied their application in curve interpolation and deformation. The advantage of the extensions is that they have global and local parameters. The modeling examples illustrate that these new interpolating splines are valuable for curve/surface design in CAD systems. Convexity-preserving interpolation and monotone-preserving interpolation are very important in practice. In the future, we will study the convexity-preserving conditions and the monotone-preserving conditions of these interpolation splines.

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