基于终端不变集的 Markov 跳变系统约束预测控制

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摘 要 针对离散 Markov 跳变系统, 研究带输入输出约束的有限时域预测控制问题. 对于给定预测时域内的每条模态轨迹, 设计控制输入序列, 驱动系统状态到达相应的终端不变集内, 在预测时域外, 则寻求一个虚拟的状态反馈控制器以保证系统的随机稳定性, 在此基础上, 分别给出了以线性矩阵不等式 (LMI) 描述的带输入、输出约束预测控制器的设计方法.

关键词 Markov 跳变系统, 约束控制, 滚动时域控制, 终端不变集, 正半定规划

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Constrained Predictive Control of Markov Jump Linear Systems Based on Terminal Invariant Sets

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Abstract In this paper, the receding horizon control (RHC) problem is considered for discrete time Markov jump linear systems, subject to constraints on control inputs and observed outputs. In given receding horizon, for every mode trajectory, control inputs sequence is designed to drive the state trajectory into a terminal invariant set. The invariant set depends on the existence of a fictitious state feedback control law, which guarantees stochastic stability of jump system out of the receding horizon. Based on such stability analysis, a linear matrix inequality approach for designing receding horizon controller subject to input and output constraints is developed.

Key words Markov jump systems, constraint control, receding horizon control (RHC), terminal invariant sets, semidefinite programming (SDP)

Markov 跳变系统代表一类重要的随机系统, 近年来引起广泛关注[1−8]. 而确定性系统的预测控制已获深入研究和应用, 特别是线性矩阵不等式 (Linear matrix inequality, LMI) 的引入, 使得预测控制方法处理输入、输出约束问题更加简便[9−10].

作为一类随机混杂系统, 同时要考虑模态与状态, 使 Markov 跳变系统的预测控制问题具有一定的挑战性, 现有研究结果很少. do Val 等研究了转移模态可测与不可测情况下, 离散 Markov 跳变系统的预测控制问题, 并提出了基于 Riccati 方程的控制器求解方法[3], Vargas 等进而研究了有噪声输入的情形[4], Costa 利用状态反馈和 LMI, 研究了 Markov 跳变系统的约束控制问题[8], 最新的结果是 Park 等人运用 LMI 优化方法, 给出了不确定离散 Markov 跳变系统的单步预测控制器设计[7].

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本文研究了离散 Markov 跳变系统的带约束有限时域预测控制, 主要工作包括: 1) 通过定义一类多步模态轨迹, 将预测控制律的求取, 转化为正半定规划 (Semidefinite programming, SDP) 优化解的搜索, 实现了多步预测控制器设计; 2) 采用终端不变集思想, 解决预测控制器的稳定性分析问题, 具体地, 对于有限预测时域内任意可能的模态轨迹, 采用在线优化的输入变量来控制系统的状态轨迹, 并将其驱动到一个终端不变集合内, 该不变集取决于在预测时域之外, 是否存在一个虚拟的状态反馈控制律, 以此来保证该模态轨迹下系统的随机稳定性; 3) 对输入、输出变量的约束, 先转化为对状态轨迹的约束, 然后将其作为 SDP 搜索的约束条件.

1 问题描述

考虑一类离散 Markov 跳变系统

$$\begin{cases}\boldsymbol{x}_{k+1} = A_{r_k}\boldsymbol{x}_k + B_{r_k}\boldsymbol{u}_k\\ \boldsymbol{y}_k = C_{r_k}\boldsymbol{x}_k\end{cases} \tag{1}$$

其中 $\boldsymbol{x}_k \in \mathbf{R}^n$, $\boldsymbol{u}_k \in \mathbf{R}^m$, $\boldsymbol{y}_k \in \mathbf{R}^p$ 分别是系统的状态、控制输入和观测输出, A_{r_k}, B_{r_k}, C_{r_k} 是依赖模态 r_k 的适当维数的矩阵, r_k 是有限状态集 $\Omega = \{1, \cdots, l\}$ 上的离散时间 Markov 链, 其中一步转移概率阵为 $\Pi = \{p(r_k, r_{k+1})\}$, r_k, $r_{k+1} \in \Omega$, $p(r_k, r_{k+1})$ 为从模态 r_k 跳变到模态 r_{k+1} 的转移率.

定义 1. 考虑 $\boldsymbol{u}_k = 0$, 称跳变系统 (1) 是随机稳定的, 若对于所有初始状态 $\boldsymbol{x}_0$ 和模态 r_0, 满足 $\mathrm{E}\{\sum_{k=0}^{\infty} \| \boldsymbol{x}_k(\boldsymbol{x}_0, r_0) \|^2 | \boldsymbol{x}_0, r_0\} < \infty$.

对于随机跳变系统 (1) 的有限时域控制, 与一般确定性系统不同, 其预测模型不仅仅是状态方程 (1), 还要考虑对模态的预测, 下面给出一类多步模态轨迹预测模型.

定义 2. 定义未来 N 步模态轨迹集合 $\Psi = \{r_k, r_{k+1}, \cdots, r_{k+N+1}\}$, 对于任意一条模态轨迹 $\eta \in \Psi$, 模态 r_k 的 N 步转移概率为 $p_\eta = p(r_k^\eta, r_{k+1}^\eta) \cdots p(r_{k+N-2}^\eta, r_{k+N-1}^\eta)$.

在 k 时刻, 沿模态轨迹 $\eta \in \Psi$, 设未来 N 步的状态和相应的控制输入分别为

$$\boldsymbol{\chi}^\eta(k+1) = [\boldsymbol{x}_{k+1|k}^\eta \cdots \boldsymbol{x}_{k+N|k}^\eta]^{\mathrm{T}},\ \boldsymbol{U}_k^\eta = [\boldsymbol{u}_k^{\eta^{\mathrm{T}}} \cdots \boldsymbol{u}_{k+N-1}^{\eta^{\mathrm{T}}}]^{\mathrm{T}}$$

则 N 步预测时域内跳变系统模型为

$$\boldsymbol{\chi}^\eta(k+1) = A_\eta \boldsymbol{x}_k + B_\eta \boldsymbol{U}_k^\eta \tag{2}$$

其中

$$A_\eta = \begin{bmatrix} A_{r_k} \\ A_{r_{k+1}}A_{r_k} \\ \vdots \\ A_{r_{k+N-1}}\cdots A_{r_k}\end{bmatrix},$$

$$B_\eta = \begin{bmatrix} B_{r_k} & 0 & \cdots & 0 \\ A_{r_{k+1}}B_{r_k} & B_{r_{k+1}} & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ A_{r_{k+N-1}}\cdots A_{r_k}B_{r_k} & A_{r_{k+N-1}}\cdots A_{r_{k+2}}B_{r_{k+1}} & \cdots & B_{r_{k+N-1}}\end{bmatrix}$$

为分析系统的稳定性, 对 N 步预测时域内和预测时域外分别考虑, 不失一般性, 设预测时域 N 以外的一步动态

$$\boldsymbol{x}_{k+N+1}^{\eta}=A_{r_{k+N}}^{\eta}\boldsymbol{x}_{k+N}^{\eta}+B_{r_{k+N}}^{\eta}\boldsymbol{u}_{k+N}^{\eta} \tag{3}$$

其中 $(A_{r_{k+N}}^{\eta},B_{r_{k+N}}^{\eta})$ 为在轨迹 η 基础上预测的第 $k+N$ 步模态所对应的系统参数.

引理 1. 系统 $\boldsymbol{\chi}^{\eta}(k+1)=A_{\eta}\boldsymbol{x}_k$ 沿轨迹 η 演化 N 步后随机稳定, 如果存在一组正定对称矩阵 P_i^{η}, $i\in\Omega$, 满足

$$P_i^{\eta}-(A_{r_{k+N}}^{\eta})^{\mathrm{T}}\sum_{j\in\Omega}p(r_{k+N},j)P_j^{\eta}(A_{r_{k+N}}^{\eta}) \tag{4}$$

证明. 对于 N 步预测时域, 相应的模态轨迹 $\eta=\{r_k,r_{k+1},\cdots,r_{k+N+1}\}\in\Psi$, 有

$$\boldsymbol{x}_{k+N}^{\eta}=A_{r_{k+N-1}}\cdots A_{r_k}\boldsymbol{x}_k$$

若已知当前 k 时刻 $\boldsymbol{x}_k$ 及模态轨迹 η, 则可唯一确定 $\boldsymbol{x}_{k+N}^{\eta}$, 而此后 $k+N+1$ 时刻的动态为

$$\boldsymbol{x}_{k+N+1}^{\eta}=A_{r_{k+N}}\boldsymbol{x}_{k+N}^{\eta} \tag{5}$$

由此, 根据文献 [2], 式 (4) 为跳变系统随机稳定的充分条件. □

注 1. 引理 1 实际上给出的是 N 步预测时域外的第 $k+N$ 步以后的稳定条件.

注 2. 若在引理 1 中增加约束条件

$$\boldsymbol{x}_{k+N}^{\mathrm{T}}P_{r_{k+N}}^{\eta}\boldsymbol{x}_{k+N}\leq 1 \tag{6}$$

则 $k+N$ 步后, 系统的状态将被驱动到集合 $\{\boldsymbol{x}\mid\boldsymbol{x}^{\mathrm{T}}P_{r_{k+N}}^{\eta}\boldsymbol{x}\leq 1\}$ 内, 即对于轨迹 η, 一组正定对称矩阵 $P_{r_{k+N}}^{\eta}$ 定义了系统运动的终端椭圆不变集.

2 多步预测控制问题

对于滚动时域预测控制问题, 可采用最坏情况下的无穷范数指标, 定义模态轨迹 $\eta\in\Psi$ 下的 min-max 性能指标

$$\min_{\eta,\boldsymbol{U}_k^{\eta},\boldsymbol{u}_{k+N}^{\eta},K_{\eta},P_i^{\eta}}\ \max_{A_{\eta},B_{\eta},p(i,j),p_{\eta}} J \tag{7}$$

式中 $J=J_1+J_2$, 且有

$$J_1=\mathrm{E}\{\sum_{n=0}^{N}[\boldsymbol{x}_{k+n|k}^{\eta}{}^{\mathrm{T}}Q_n\boldsymbol{x}_{k+n|k}^{\eta}+\boldsymbol{u}_{k+n|k}^{\eta}{}^{\mathrm{T}}R_n\boldsymbol{u}_{k+n|k}^{\eta}]\}$$

$$J_2=\mathrm{E}\{\boldsymbol{x}_{k+N+1|k}^{\eta}{}^{\mathrm{T}}P_{r_{k+N+1}}^{\eta}\boldsymbol{x}_{k+N+1|k}^{\eta}\}$$

其中 $Q_n,R_n\geq 0$ 为权矩阵, 且不同时为 0.

上述 min-max 预测控制问题, 可从两方面分析: 一是在 N 步滚动预测时域内, 通过指标优化获得控制序列 $\boldsymbol{U}_k^{\eta}$; 二是在 $k+N$ 时刻, 通过引入虚拟的状态反馈控制器

$$\boldsymbol{u}_{k+N}^{\eta}=K_{\eta}\boldsymbol{x}_{k+N|k}^{\eta} \tag{8}$$

保证系统在 N 步时域后的稳定性. 具体对于闭环系统 (2)、(3) 和 (8), 令

$$\hat{A}_{r_{k+N}}^{\eta}=A_{r_{k+N}}^{\eta}+B_{r_{k+N}}^{\eta}K_{\eta} \tag{9}$$

考虑到引理 1, 有如下定义

定义 3. 称 $\boldsymbol{U}_k^{\eta}$ 为多步 min-max 预测控制器或预测控制序列集合, 如果在满足

$$P_{r_{k+N}}^{\eta}-(\hat{A}_{r_{k+N}}^{\eta})^{\mathrm{T}}\sum_{i\in\Omega}p(r_{k+N},i)P_i^{\eta}\hat{A}_{r_{k+N}}^{\eta}>0 \tag{10}$$

的条件下, min-max 性能指标 (7) 有最优解.

显然这是一个二次约束二次规划 (Quadratic constrained quadratic programming, QCQP) 问题, 引理 2 将其进一步等价为 SDP 问题.

引理 2. 多步 min-max 预测控制器存在, 当且仅当下列 SDP 问题存在最优解 $\boldsymbol{U}^{\eta}$, $\boldsymbol{u}_{k+N}^{\eta}$, K_{η} 和 $P_{r_{k+N}}^{\eta}$.

$$\min_{\gamma_1,\gamma_2}\ \gamma_1+\gamma_2 \tag{11}$$

$$\begin{aligned}\text{s.t.}\quad &\max_{\eta,\boldsymbol{U}_k^{\eta},\boldsymbol{u}_{k+N}^{\eta},K_{\eta},P_i^{\eta}} J_1\leq\gamma_1\\ &\max_{\eta,\boldsymbol{U}_k^{\eta},\boldsymbol{u}_{k+N}^{\eta},K_{\eta},P_i^{\eta}} J_2\leq\gamma_2\end{aligned}$$

证明. 略. □

注 3. 引入虚拟控制器 (8) 进行稳定性分析, 只是在求解 min-max 优化问题时, 增加了一组保证稳定性的约束条件 (10), 实际上该控制器并不用于实施.

定理 1. 如果在 k 时刻存在多步 min-max 预测控制 $\boldsymbol{U}_k^{\eta}$, 使得上述 SDP 有解, 则预测控制器

$$\boldsymbol{U}_k=\sum_{\eta\in\Psi}p_{\eta}\boldsymbol{U}_k^{\eta} \tag{12}$$

使 Markov 跳变系统 (1) 随机稳定.

证明. 在时刻 k 任给预测轨迹 η, SDP 问题有解, 则可以找到一组解 $\boldsymbol{U}_k^{\eta^*}$, $\boldsymbol{u}_{k+N}^{\eta^*}$, $P_i^{\eta^*}$, 使性能指标达到最优 $J_k^{\eta^*}$, 因 Q_n, R_n 的选择并不影响系统的稳定性, 为讨论方便, 不失一般性, 设 $Q_n=I$, $R_n=0$.

设对应于轨迹 η, 在 $\boldsymbol{U}_k^{\eta}$, $\boldsymbol{u}_{k+N}^{\eta}$ 输入下, $k+1$ 时刻的指标值为 J_{k+1}^{η}, 而 $J_{k+1}^{\eta^*}$ 为 $\boldsymbol{u}_{k+N}^{\eta^*}$ 输入下的最优值, 显然有 $J_{k+1}^{\eta}\geq J_{k+1}^{\eta^*}$, 则

$$\begin{aligned}&\mathrm{E}\{J_k^{\eta^*}-J_{k+1}^{\eta^*}\}\geq J_k^{\eta^*}-\mathrm{E}\{J_{k+1}^{\eta}\}=\\ &\sum_{n=0}^{N}[\boldsymbol{x}_{k+n|k+1}^{\eta^*}{}^{\mathrm{T}}Q_n\boldsymbol{x}_{k+n|k+1}^{\eta^*}]+\\ &\boldsymbol{x}_{k+N+1|k+1}^{\eta^*}{}^{\mathrm{T}}P_{r_{k+N+1}}^{*}\boldsymbol{x}_{k+N+1|k+1}^{\eta^*}-\\ &\sum_{n=0}^{N}[\boldsymbol{x}_{k+n+1|k}^{\eta^*}{}^{\mathrm{T}}Q_n\boldsymbol{x}_{k+n+1|k}^{\eta^*}]-\\ &\boldsymbol{x}_{k+N+2|k}^{\eta^*}{}^{\mathrm{T}}\bar{P}_{r_{k+N+2}}\boldsymbol{x}_{k+N+2|k}^{\eta^*}\end{aligned} \tag{13}$$

其中 $(\bar{P})_i=\sum_{j=1}^{l}p(i,j)P_j$. 由于在 $\boldsymbol{U}_k^{\eta^*}$, $\boldsymbol{u}_{k+N}^{\eta^*}$ 控制输入下

$$\begin{aligned}&\boldsymbol{x}_{k+i|k+1}^{\eta}{}^{\mathrm{T}}\boldsymbol{x}_{k+i|k+1}^{\eta}=\boldsymbol{x}_{k+i|k}^{\eta^*}{}^{\mathrm{T}}\boldsymbol{x}_{k+i|k}^{\eta^*},\quad i=1,\cdots,N\\ &\boldsymbol{x}_{k+N+1|k+1}^{\eta}{}^{\mathrm{T}}P_{r_{k+N+1}}^{*}\boldsymbol{x}_{k+N+1|k+1}^{\eta}=\\ &\boldsymbol{x}_{k+N+1|k}^{\eta^*}{}^{\mathrm{T}}P_{r_{k+N+1}}^{*}\boldsymbol{x}_{k+N+1|k}^{\eta^*}\end{aligned}$$

则式 (13) 可表示为

$$\begin{aligned}&\mathrm{E}\{J_k^{\eta^*}-J_{k+1}^{\eta^*}\}\geq J_k^{\eta^*}-\mathrm{E}\{J_{k+1}^{\eta}\}=\\ &\boldsymbol{x}_{k|k}^{\eta^*}{}^{\mathrm{T}}\boldsymbol{x}_{k|k}^{\eta^*}-\boldsymbol{x}_{k+N+2|k+1}^{\eta^*}{}^{\mathrm{T}}\boldsymbol{x}_{k+N+2|k+1}^{\eta^*}+\\ &\boldsymbol{x}_{k+N+1|k+1}^{\eta^*}{}^{\mathrm{T}}P_{r_{k+N+1}}^{*}\boldsymbol{x}_{k+N+1|k+1}^{\eta^*}-\\ &\boldsymbol{x}_{k+N+2|k}^{\eta^*}{}^{\mathrm{T}}\bar{P}_{r_{k+N+2}}\boldsymbol{x}_{k+N+2|k}^{\eta^*}\end{aligned} \tag{14}$$

下面首先证明: $\boldsymbol{x}_{k|k}^{\eta^*\,\mathrm{T}}\boldsymbol{x}_{k|k}^{\eta^*} \geq \boldsymbol{x}_{k+N+1|k+1}^{\eta^*\,\mathrm{T}}\boldsymbol{x}_{k+N+1|k+1}^{\eta^*}$ 成立.

假设 $\boldsymbol{x}_{k|k}^{\eta^*} = \boldsymbol{x}_{k|k-N}^{\eta''}$, 并且 $k-N$ 时刻得到第 k 时刻的预测反馈控制器为 $\boldsymbol{u}_{k|k-N}^{\eta''} = K_\eta''\boldsymbol{x}_{k|k-N}^{\eta''}$, 设 $\boldsymbol{u}_{k+i|k-N}^{\eta''} = K_\eta''\boldsymbol{x}_{k+i|k-N}^{\eta''}$, $i=1,\cdots,N$, 则该反馈控制器加入后的指标函数为 $J_k^{\eta''}$, 显然有 $J_k^{\eta^*}\leq J_k^{\eta''}$, 因 $P_{r_k}^\eta - (\hat{A}_{r_{k+1}}^{\eta''})^\mathrm{T}\sum_{i\in\Omega}p(r_{k+1},i)P_i^\eta\hat{A}_{r_{k+1}}^{\eta''}>0$, 其中 $\hat{A}_{r_{k+1}}^{\eta''} = A_{r_{k+1}}^\eta + B_{r_{k+1}}^\eta K_\eta''$, 则

$$\|\boldsymbol{x}_{k|k-N}^\eta\|_{P_{r_k}} \geq \mathrm{E}\{\|\boldsymbol{x}_{k+1|k-N}^\eta\|_{P_{r_{k+1}}}\}$$

同理

$$\|\boldsymbol{x}_{k+i|k-N}^\eta\|_{P_{r_{k+1}}}\geq \mathrm{E}\{\|\boldsymbol{x}_{k+i+1|k-N}^\eta\|_{P_{r_{k+2}}}\}$$

即在该反馈控制器作用下, 系统 k 时刻以后的状态的期望值在不变集 $\{\boldsymbol{x}\mid\boldsymbol{x}^\mathrm{T}P_{r_k}^\eta\boldsymbol{x}\leq 1\}$ 内, 也即

$$\boldsymbol{x}_{k|k-N}^{\eta''\,\mathrm{T}}\boldsymbol{x}_{k|k-N}^{\eta''}\geq \boldsymbol{x}_{k+N|k-N}^{\eta''\,\mathrm{T}}\boldsymbol{x}_{k+N|k-N}^{\eta''}$$

考虑到 $J_k^{\eta^*}\leq J_k^{\eta''}$, 有

$$\begin{aligned}&\boldsymbol{x}_{k|k}^{\eta^*\,\mathrm{T}}\boldsymbol{x}_{k|k}^{\eta^*} = \boldsymbol{x}_{k|k-N}^{\eta''\,\mathrm{T}}\boldsymbol{x}_{k|k-N}^{\eta''} \geq\\ &\boldsymbol{x}_{k+N|k-N}^{\eta''\,\mathrm{T}}\boldsymbol{x}_{k+N|k-N}^{\eta''} \geq \boldsymbol{x}_{k+N|k}^{\eta^*\,\mathrm{T}}\boldsymbol{x}_{k+N|k}^{\eta^*}\end{aligned}$$

得到

$$\boldsymbol{x}_{k|k}^{\eta^*\,\mathrm{T}}\boldsymbol{x}_{k|k}^{\eta^*}\geq \boldsymbol{x}_{k+N|k}^{\eta^*\,\mathrm{T}}\boldsymbol{x}_{k+N|k}^{\eta^*} = \boldsymbol{x}_{k+N|k+1}^{\eta\,\mathrm{T}}\boldsymbol{x}_{k+N|k+1}^{\eta} \tag{15}$$

又由 $P_{r_{k+N}}^\eta - (\hat{A}_{r_{k+N}}^{\eta''})^\mathrm{T}\sum_{i\in\Omega}p(r_{k+N},i)P_i^\eta\hat{A}_{r_{k+N}}^{\eta''}>0$, 其中 $\hat{A}_{r_{k+N}}^{\eta''} = A_{r_{k+N}}^\eta + B_{r_{k+N}}^\eta K_\eta''$, 显然 $\boldsymbol{x}_{k+N|k+1}^\eta$ 在不变集 $\{\boldsymbol{x}\mid\boldsymbol{x}^\mathrm{T}P_{r_{k+N}}^\eta\boldsymbol{x}\leq 1\}$ 内, 由此

$$\boldsymbol{x}_{k+N|k+1}^{\eta^*\,\mathrm{T}}\boldsymbol{x}_{k+N|k+1}^{\eta^*}\geq \boldsymbol{x}_{k+N+1|k+1}^{\eta^*\,\mathrm{T}}\boldsymbol{x}_{k+N+1|k+1}^{\eta^*} \tag{16}$$

综合式 (15), (16), 得 $\boldsymbol{x}_{k|k}^{\eta^*\,\mathrm{T}}\boldsymbol{x}_{k|k}^{\eta^*}\geq \boldsymbol{x}_{k+N+1|k+1}^{\eta^*\,\mathrm{T}}\boldsymbol{x}_{k+N+1|k+1}^{\eta^*}$, 则式 (14) 进一步表示为

$$\begin{aligned}&\mathrm{E}\{J_k^{\eta^*}-J_{k+1}^{\eta^*}\}\geq J_k^{\eta^*}-\mathrm{E}\{J_{k+1}^{\eta}\}\geq\\ &\boldsymbol{x}_{k+N+1|k+1}^{\eta^*\,\mathrm{T}}P_{r_{k+N+1}}^{*}\boldsymbol{x}_{k+N+1|k+1}^{\eta^*}-\\ &\boldsymbol{x}_{k+N+2|k+1}^{\eta^*\,\mathrm{T}}\bar{P}_{r_{k+N+2}}\boldsymbol{x}_{k+N+2|k+1}^{\eta^*}=\\ &\boldsymbol{x}_{k+N+1|k+1}^{\eta^*\,\mathrm{T}}(P_{r_{k+N+1}}^{*}-(\hat{A}_{r_{k+N}}^{\eta''})^\mathrm{T}\times\\ &\sum_{i\in\Omega}p(r_{k+N},i)P_i^\eta\hat{A}_{r_{k+N}}^{\eta''})\boldsymbol{x}_{k+N+1|k+1}^{\eta^*}\end{aligned}\tag{17}$$

其中 $\hat{A}_{r_{k+N+1}}^\eta = A_{r_{k+N+1}}^\eta + B_{r_{k+N+1}}^\eta K_\eta$. 又因为系统需满足条件 (10), 取 $k=k+1$ 代入可知

$$\mathrm{E}\{J_k^{\eta^*}-J_{k+1}^{\eta^*}\}\geq J_k^{\eta^*}-\mathrm{E}\{J_{k+1}^{\eta}\}\geq 0$$

当且仅当 $\boldsymbol{x}_{k+N+2|k+1}^\eta = \boldsymbol{0}$ 时, $\mathrm{E}\{J_k^{\eta^*}-J_{k+1}^{\eta^*}\}=0$. 对任意一条无穷模态轨迹 η,

$$\sum_{k=0}^{\infty}\mathrm{E}\{J_k^{\eta^*}-J_{k+1}^{\eta^*}\} = J_0^{\eta^*}-\mathrm{E}\{J_\infty^{\eta^*}\}\leq J_0^{\eta^*} \tag{18}$$

由于 $J_0^{\eta^*}$ 是有限值, 故必有 $\lim_{k\to\infty}\mathrm{E}\{\boldsymbol{x}^*(k\mid k)\}=0$, 由文献 [1] 可知系统是随机稳定的. □

注 4. 在稳定性证明的时候, 为了简化证明, 默认 $\boldsymbol{x}_{k+N+1}^\eta = (A_{r_{k+N}}^\eta + B_{r_{k+N}}^\eta K_\eta)\boldsymbol{x}_{k+N|k}^\eta$, 即预测时域外采用状态反馈控制器.

3 输入输出约束控制器设计

定理 2. 考虑控制输入有约束 $|\boldsymbol{u}_k|\leq u_{\max}$, 任意时刻 k, 给定系统状态 $\boldsymbol{x}_k$, 如果存在正定对称矩阵 $X_i^\eta\in\mathbf{R}^{n\times n}$, 矩阵 $Y_i^\eta\in\mathbf{R}^{m\times n}$, $\boldsymbol{U}_k^\eta\in\mathbf{R}^{Nm\times 1}$, $\boldsymbol{u}_{k+N}^\eta\in\mathbf{R}^{m\times 1}$, 以及 $\gamma_{1\eta}\in\mathbf{R}$, $\gamma_{2\eta}\in\mathbf{R}$, $i\in\Omega$, $\eta\in\Psi$, 使下面 SDP 优化问题有最优解, 则带输入约束的预测控制器为 $\boldsymbol{U}_k=\sum_\eta p_\eta\boldsymbol{U}_k^\eta$.

$$\min\{\sum_{\eta\in\Psi}p_\eta(\gamma_{1\eta}+\gamma_{2\eta})\}$$

$$\text{s.t.}\quad\begin{bmatrix}\gamma_{1\eta}I-\Theta & \boldsymbol{U}_k^{\eta\,\mathrm{T}}\\ \boldsymbol{U}_k^\eta & (B_\eta^\mathrm{T}QB_\eta+R)^{-1}\end{bmatrix}\geq 0 \tag{19}$$

$$\begin{bmatrix}\gamma_{2\eta} & * & \cdots & *\\ \sqrt{p(r_{k+N},1)}\boldsymbol{x}_{k+N+1|k}^\eta & X_1^\eta & \cdots & *\\ \vdots & \vdots & \vdots & \vdots\\ \sqrt{p(r_{k+N},l)}\boldsymbol{x}_{k+N+1|k}^\eta & 0 & \cdots & X_l^\eta\end{bmatrix}\geq 0 \tag{20}$$

$$\begin{bmatrix}X_{r_{k+N}}^\eta & * & \cdots & *\\ \sqrt{p(r_{k+N},1)}Z_{r_{k+N}}^\eta & X_1^\eta & \cdots & *\\ \vdots & \vdots & \vdots & \vdots\\ \sqrt{p(r_{k+N},l)}Z_{r_{k+N}}^\eta & 0 & \cdots & X_l^\eta\end{bmatrix}\geq 0 \tag{21}$$

$$\begin{bmatrix}u_{\max}^2 & M(\alpha)\boldsymbol{U}_k^{\eta\,\mathrm{T}}\\ M(\alpha)\boldsymbol{U}_k^\eta & I\end{bmatrix}\geq 0 \tag{22}$$

$$\begin{bmatrix}1 & \boldsymbol{x}_{k+N|k}^{\eta\,\mathrm{T}}\\ \boldsymbol{x}_{k+N|k}^\eta & X_{r_{k+N}}^\eta\end{bmatrix}\geq 0 \tag{23}$$

$$\begin{bmatrix}u_{\max}^2 & Y_{r_{k+N}}^\eta\\ Y_{r_{k+N}}^{\eta\,\mathrm{T}} & X_{r_{k+N}}^\eta\end{bmatrix}\geq 0 \tag{24}$$

其中

$$\begin{aligned}&\Theta = \boldsymbol{x}_k^\mathrm{T}A_\eta^\mathrm{T}QA_\eta\boldsymbol{x}_k+\boldsymbol{x}_k^\mathrm{T}A_\eta^\mathrm{T}QB_\eta\boldsymbol{U}_k^\eta+\boldsymbol{U}_k^{\eta\,\mathrm{T}}B_\eta{}^\mathrm{T}QA_\eta\boldsymbol{x}_k\\ &Q=\mathrm{diag}(Q_1,\cdots,Q_N),\ R=\mathrm{diag}(R_1,\cdots,R_N)\\ &M(\alpha)=\left[\overbrace{\underbrace{0\ \ \cdots\ \ I}_{\alpha}\ \ \cdots\ \ 0}^{N}\right],(\alpha=1,\cdots,N-1)\\ &\boldsymbol{x}_{k+N+1|k}^\eta = A_{r_{k+N}}^\eta\boldsymbol{x}_{k+N|k}^\eta+B_{r_{k+N}}^\eta\boldsymbol{u}_{k+N}^\eta\\ &\boldsymbol{x}_{k+N|k}^\eta = M(N)(A_\eta\boldsymbol{x}_k+B_\eta\boldsymbol{U}_k^\eta)\\ &Z_{r_{k+N}}^\eta = A_{r_{k+N}}^\eta X_{r_{k+N}}^\eta+B_{r_{k+N}}^\eta Y_{r_{k+N}}^\eta\end{aligned}$$

证明. 首先考虑指标

$$\begin{aligned}J_1=\sum_{\eta\in\Psi}p_\eta\sum_{n=0}^{N-1}[&\boldsymbol{x}_{k+n+1|k}^{\eta\,\mathrm{T}}Q_{n+1}\boldsymbol{x}_{k+n+1|k}^\eta+\\ &\boldsymbol{u}_{k+n|k}^{\eta\,\mathrm{T}}R_n\boldsymbol{u}_{k+n|k}^\eta]+\boldsymbol{x}_{k|k}^{\eta\,\mathrm{T}}Q_1\boldsymbol{x}_{k|k}^\eta\leq\gamma_1'\end{aligned}\tag{25}$$

由于 ${\boldsymbol{x}^\eta_{k|k}}^{\rm T} Q_1 \boldsymbol{x}^\eta_{k|k}$ 是常数, 不参与优化, 设 $\gamma_1 = \gamma_1' - {\boldsymbol{x}^\eta_{k|k}}^{\rm T} Q_1 \boldsymbol{x}^\eta_{k|k}$, 则

$$\gamma_1 - \sum_{\eta\in\Psi} p_\eta \sum_{n=0}^{N-1} [{\boldsymbol{x}^\eta_{k+n+1|k}}^{\rm T} Q_{n+1} \boldsymbol{x}^\eta_{k+n+1|k} + {\boldsymbol{u}^\eta_{k+n|k}}^{\rm T} R_n \boldsymbol{u}^\eta_{k+n|k}] \geq 0 \tag{26}$$

若设 $\gamma_1 = \sum_{\eta\in\Psi} p_\eta \gamma_{1\eta}$, 式 (26) 可以表示为

$$\gamma_{1\eta} - \sum_{\eta\in\Psi} p_\eta \sum_{n=0}^{N-1} [{\boldsymbol{x}^\eta_{k+n+1|k}}^{\rm T} Q_{n+1} \boldsymbol{x}^\eta_{k+n+1|k} + {\boldsymbol{u}^\eta_{k+n|k}}^{\rm T} R_n \boldsymbol{u}^\eta_{k+n|k}] \geq 0 \tag{27}$$

即

$$\gamma_{1\eta} I - \Theta + {\boldsymbol{U}^\eta_k}^{\rm T} (B^{\rm T}_\eta Q B_\eta + R) \boldsymbol{U}^\eta_k \geq 0 \tag{28}$$

对式 (28) 运用 Schur 补可得式 (19). 再考虑指标

$$J_2 = {\rm E}\{{\boldsymbol{x}^\eta_{k+N+1|k}}^{\rm T} P^\eta_{r_{k+N+1}} \boldsymbol{x}^\eta_{k+N+1|k}\} \leq \gamma_2 \tag{29}$$

即

$$\gamma_2 - \sum_{\eta\in\Psi} p_\eta \left[{\boldsymbol{x}^\eta_{k+N+1|k}}^{\rm T} \bar{P}^\eta_{r_{k+N}} \boldsymbol{x}^\eta_{k+N+1|k}\right] \geq 0 \tag{30}$$

设 $\gamma_2 = \sum_{\eta\in\Psi} p_\eta \gamma_{2\eta}$, 式 (30) 可以写成

$$\gamma_{2\eta} - {\boldsymbol{x}^\eta_{k+N+1|k}}^{\rm T} \sum_{i\in\Omega} p(r_{k+N}, i) P_i \boldsymbol{x}^\eta_{k+N+1|k} \geq 0 \tag{31}$$

将 $\boldsymbol{x}^\eta_{k+N+1|k} = A^\eta_{r_{k+N}} \boldsymbol{x}^\eta_{k+N|k} + B^\eta_{r_{k+N}} \boldsymbol{u}^\eta_{k+N|k}$ 代入, 令 $X^\eta_{r_{k+N}} = {P^\eta_{r_{k+N}}}^{-1}$, 运用 Schur 补可得式 (20), 其中

$$\boldsymbol{x}^\eta_{k+N|k} = M(N) \boldsymbol{\chi}^\eta(k+1) = M(N)(A_\eta \boldsymbol{x}_k + B_\eta \boldsymbol{U}^\eta_k)$$

$M(N)$ 的作用是从 $\boldsymbol{\chi}^\eta(k+1)$ 中取出 $\boldsymbol{x}^\eta_{k+N|k}$. 再考虑 SDP 问题的约束条件 (10), 经矩阵运算, 令 $Y^\eta_{r_{k+N}} = K_\eta {P^\eta_{r_{k+N}}}^{-1}$, 可得式 (21). 对于输入约束 $|\boldsymbol{u}_k| \leq u_{\max}$, $\alpha = 1, \cdots, N-1$, 即 ${\boldsymbol{u}^\eta_\alpha}^{\rm T} \boldsymbol{u}^\eta_\alpha \leq u^2_{\max}$, 将 $\boldsymbol{u}^\eta_\alpha = M(\alpha) \boldsymbol{U}^\eta_k$ 代入, 运用 Schur 补得式 (22).

对于预测终端即第 N 步控制 $|\boldsymbol{u}^\eta_{k+N}| \leq u_{\max}, \alpha = 1, \cdots, N-1$, 即 $|K_\eta \boldsymbol{x}^\eta_{k+N+1|k}| \leq u_{\max}$, 结合终端不变集 ${\boldsymbol{x}^\eta_{k+N|k}}^{\rm T} P^\eta_{r_{k+N}} \boldsymbol{x}^\eta_{k+N|k} \leq 1$ (即式 (23)), 可得 $K^{\rm T}_\eta {P^\eta_{r_{k+N}}}^{-1} K_\eta \leq u^2_{\max}$, 进一步经矩阵变换得式 (24). □

注 5. 定理 2 实际是在预测时域外采用带约束的状态反馈控制器, 其基本出发点是: 既然当前预测时域外存在带约束的状态反馈控制器使系统稳定, 那么一定存在预测控制序列使系统稳定.

定理 3. 考虑系统输出约束 $|\boldsymbol{y}_k| \leq y_{\max}$, 任意时刻 k, 给定系统状态 $\boldsymbol{x}_k$, 如果存在 X^η_i, Y^η_i, $\boldsymbol{U}^\eta_k$, $\boldsymbol{u}^\eta_{k+N}$, $\gamma_{1\eta}$ 和 $\gamma_{2\eta}$, 使下面 SDP 问题有最优解, 则带输出约束的预测控制器为 $\boldsymbol{U}_k = \sum_\eta p_\eta \boldsymbol{U}^\eta_k$.

$$\min \sum_{\eta\in\Psi} p_\eta (\gamma_{1\eta} + \gamma_{2\eta})$$

$$\text{s.t.} \quad (19)(20)(21)(23)$$

$$\begin{bmatrix} y^2_{\max} & M(\alpha) C_\eta (A_\eta \boldsymbol{x}_k + B_\eta \boldsymbol{U}^\eta_k) \\ * & I \end{bmatrix} \geq 0 \tag{32}$$

$$\begin{bmatrix} y^2_{\max} & C^\eta_{r_{k+N}} Z^\eta_{r_{k+N}} \\ * & X^\eta_{r_{k+N}} \end{bmatrix} \geq 0 \tag{33}$$

其中 $C_\eta = \begin{bmatrix} C^\eta_{r_{k+1}} & \cdots & C^\eta_{r_{k+N-1}} \end{bmatrix}$.

证明. 略. □

注 6. 上述讨论中, 无论是输入控制约束还是输出约束, 均转化为对状态空间的约束来处理, 而对状态空间约束利用凸不变集来实现, 由此带来一定的保守性.

4 结论

随机 Markov 跳变系统的多步预测控制问题, 对于有限预测时域内任意可能的模态轨迹, 本文通过在线优化的控制变量驱动系统状态到一个终端不变集内, 在此基础上研究带输入、输出约束的预测控制器. 作为一种多步预测控制器设计方法, 其计算的复杂程度随预测时域的线性增加而呈指数增加.

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