SOME COMPANIONS OF AN OSTROWSKI TYPE INEQUALITY AND APPLICATIONS

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Received: 11 February, 2009
Accepted: 12 May, 2009
Communicated by: S.S. Dragomir
2000 AMS Sub. Class.: 26D15.
Key words: Ostrowski type inequality, Perturbed trapezoid rule, Midpoint rule, Composite quadrature rule.

Abstract: We establish some companions of an Ostrowski type integral inequality for functions whose derivatives are absolutely continuous. Applications for composite quadrature rules are also given.

journal of inequalities in pure and applied mathematics
issn: 1443-5756

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journal of inequalities in pure and applied mathematics
issn: 1443-5756

1. Introduction

Motivated by [1], Dragomir in [2] has proved the following companion of the Ostrowski inequality:

$$\left|\frac{1}{2}\left[f\left(x\right)+f\left(a+b-x\right)\right]-\frac{1}{b-a}\int_a^b f\left(t\right)\,dt\right| \tag{1.1}$$
$$\leq \begin{cases} \left[\frac{1}{8}+2\left(\frac{x-\frac{3a+b}{4}}{b-a}\right)^2\right]\left(b-a\right)\left\|f'\right\|_{[a,b],\infty} & \text{if } f'\in L_\infty\left[a,b\right]; \\ \frac{2^{\frac{1}{q}}}{(q+1)^{\frac{1}{q}}}\left[\left(\frac{x-a}{b-a}\right)^{q+1}+\left(\frac{\frac{a+b}{2}-x}{b-a}\right)^{q+1}\right]^{\frac{1}{q}}\left(b-a\right)^{\frac{1}{q}}\left\|f'\right\|_{[a,b],p}, & \text{if } p>1,\ \frac{1}{p}+\frac{1}{q}=1, \\ & \text{and } f'\in L_p\left[a,b\right]; \\ \left[\frac{1}{4}+\left|\frac{x-\frac{3a+b}{4}}{b-a}\right|\right]\left\|f'\right\|_{[a,b],1} & \text{if } f'\in L_1\left[a,b\right], \end{cases}$$

for all $x\in\left[a,\frac{a+b}{2}\right]$, where $f:[a,b]\to\mathbb{R}$ is an absolutely continuous function.

In particular, the best result in (1.1) is obtained for $x=\frac{a+3b}{4}$, giving the following trapezoid type inequalities:

$$\left|\frac{1}{2}\left[f\left(\frac{3a+b}{4}\right)+f\left(\frac{a+3b}{4}\right)\right]-\frac{1}{b-a}\int_a^b f\left(t\right)dt\right| \tag{1.2}$$
$$\leq \begin{cases} \frac{1}{8}\left(b-a\right)\left\|f'\right\|_{[a,b],\infty} & \text{if } f'\in L_\infty\left[a,b\right]; \\ \frac{1}{4}\cdot\frac{(b-a)^{\frac{1}{q}}}{(q+1)^{\frac{1}{q}}}\left\|f'\right\|_{[a,b],p}, & \text{if } f'\in L_p\left[a,b\right],\ p>1,\ \frac{1}{p}+\frac{1}{q}=1; \\ \frac{1}{4}\left\|f'\right\|_{[a,b],1} & \text{if } f'\in L_1\left[a,b\right]. \end{cases}$$

Some natural applications of (1.1) and (1.2) are also provided in [2].

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In [3], Dedić et al. have derived the following trapezoid type inequality:

$$\left|\frac{1}{2}\left[f\left(\frac{3a+b}{4}\right)+f\left(\frac{a+3b}{4}\right)\right]-\frac{1}{b-a}\int_a^b f(t)\,dt\right| \le \frac{(b-a)^2}{32}\|f''\|_\infty, \tag{1.3}$$

for a function $f : [a,b] \to \mathbb{R}$ whose derivative f' is absolutely continuous and $f'' \in L_\infty[a,b]$.

In [4], we find that for a function $f : [a,b] \to \mathbb{R}$ whose derivative f' is absolutely continuous, the following perturbed trapezoid inequalities hold:

$$\left|\int_a^b f(t)\,dt - \frac{b-a}{2}[f(a)+f(b)] + \frac{(b-a)^2}{8}[f'(b)-f'(a)]\right| \tag{1.4}$$
$$\le \begin{cases} \frac{(b-a)^3}{24}\|f''\|_\infty & \text{if } f'' \in L_\infty[a,b]; \\ \frac{(b-a)^{2+\frac{1}{q}}}{8(2q+1)^{\frac{1}{q}}}\|f''\|_p, & \text{if } f'' \in L_P[a,b],\ p>1,\ \frac{1}{p}+\frac{1}{q}=1; \\ \frac{(b-a)^2}{8}\|f''\|_1 & \text{if } f'' \in L_1[a,b]. \end{cases}$$

In this paper, we provide some companions of Ostrowski type inequalities for functions whose first derivatives are absolutely continuous and whose second derivatives belong to the Lebesgue spaces $L_p[a,b]$, $1 \le p \le \infty$. These improve (1.3) and recapture (1.4). Applications for composite quadrature rules are also given.

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2. Some Integral Inequalities

Lemma 2.1. *Let $f : [a,b] \to \mathbb{R}$ be such that the derivative f' is absolutely continuous on $[a,b]$. Then we have the equality*

$$\begin{aligned} (2.1)\quad & \frac{1}{b-a}\int_a^b f(t)\,dt - \frac{1}{2}[f(x)+f(a+b-x)] \\ &\quad + \frac{1}{2}\left(x-\frac{3a+b}{4}\right)[f'(x)-f'(a+b-x)] \\ &= \frac{1}{2(b-a)}\left[\int_a^x (t-a)^2 f''(t)\,dt + \int_x^{a+b-x}\left(t-\frac{a+b}{2}\right)^2 f''(t)\,dt \right. \\ &\qquad \left. + \int_{a+b-x}^b (t-b)^2 f''(t)\,dt\right] \end{aligned}$$

for any $x \in \left[a, \frac{a+b}{2}\right]$.

Proof. Using the integration by parts formula for Lebesgue integrals, we have

$$\int_a^x (t-a)^2 f''(t)\,dt = (x-a)^2 f'(x) - 2(x-a)f(x) + 2\int_a^x f(t)\,dt,$$

$$\begin{aligned} &\int_x^{a+b-x}\left(t-\frac{a+b}{2}\right)^2 f''(t)\,dt \\ &\quad = \left(x-\frac{a+b}{2}\right)^2 [f'(a+b-x)-f'(x)] \\ &\quad\quad + 2\left(x-\frac{a+b}{2}\right)[f(x)+f(a+b-x)] + 2\int_x^{a+b-x} f(t)\,dt, \end{aligned}$$

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and

$$\int_{a+b-x}^{b}(t-b)^2 f''(t)\,dt$$
$$= -(x-a)^2 f'(a+b-x) - 2(x-a)\,f(a+b-x) + 2\int_{a+b-x}^{b} f(t)\,dt.$$

Summing the above equalities, we deduce the desired identity (2.1). □

Theorem 2.2. *Let $f:[a,b]\to\mathbb{R}$ be such that the derivative f' is absolutely continuous on $[a,b]$. Then we have the inequality*

$$\begin{aligned}(2.2)\quad &\left|\frac{1}{b-a}\int_a^b f(t)\;dt - \frac{1}{2}\left[f(x)+f(a+b-x)\right]\right.\\ &\qquad\left.+\frac{1}{2}\left(x-\frac{3a+b}{4}\right)\left[f'(x)-f'(a+b-x)\right]\right|\\ &\le \frac{1}{2(b-a)}\left[\int_a^x (t-a)^2\,|f''(t)|\,dt + \int_x^{a+b-x}\left(t-\frac{a+b}{2}\right)^2 |f''(t)|\,dt\right.\\ &\qquad\left.+\int_{a+b-x}^{b}(t-b)^2\,|f''(t)|\,dt\right]\\ &:= M(x)\end{aligned}$$

for any $x\in\left[a,\frac{a+b}{2}\right]$.

If $f''\in L_\infty[a,b]$, then we have the inequalities

$$(2.3)\quad M(x) \le \frac{1}{2(b-a)}\left[\frac{(x-a)^3}{3}\|f''\|_{[a,x],\infty}\right.$$

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$$+\frac{2}{3}\left(\frac{a+b}{2}-x\right)^3\|f''\|_{[x,a+b-x],\infty}+\frac{(x-a)^3}{3}\|f''\|_{[a+b-x,b]}\Bigg]$$

$$\leq\begin{cases}\left[\frac{1}{96}+\frac{1}{2}\left(\frac{x-\frac{3a+b}{4}}{b-a}\right)^2\right](b-a)^2\|f''\|_{[a,b],\infty};\\ \left[\frac{1}{2^{\alpha-1}}\left(\frac{x-a}{b-a}\right)^{3\alpha}+\left(\frac{x-\frac{a+b}{2}}{b-a}\right)^{3\alpha}\right]^{\frac{1}{\alpha}}\\ \qquad\times\left[\|f''\|^{\beta}_{[a,x],\infty}+\|f''\|^{\beta}_{[x,a+b-x],\infty}+\|f''\|^{\beta}_{[a+b-x,b],\infty}\right]^{\frac{1}{\beta}}\frac{(b-a)^2}{3}\\ \qquad\qquad \text{if } \alpha>1,\ \frac{1}{\alpha}+\frac{1}{\beta}=1;\\ \max\left\{\frac{1}{2}\left(\frac{x-a}{b-a}\right)^3,\left(\frac{x-\frac{a+b}{2}}{b-a}\right)^3\right\}\\ \qquad\times\left[\|f''\|_{[a,x],\infty}+\|f''\|_{[x,a+b-x],\infty}+\|f''\|_{[a+b-x,b],\infty}\right]\frac{(b-a)^2}{3};\end{cases}$$

for any $x\in\left[a,\frac{a+b}{2}\right]$.

The inequality (2.2), the first inequality in (2.3) and the constant $\frac{1}{96}$ *are sharp.*

Proof. The inequality (2.2) follows by Lemma 2.1 on taking the modulus and using its properties.

If $f''\in L_\infty[a,b]$, then

$$\int_a^x(t-a)^2|f''(t)|\,dt\leq\frac{(x-a)^3}{3}\|f''\|_{[a,x],\infty},$$

$$\int_x^{a+b-x}\left(t-\frac{a+b}{2}\right)^2|f''(t)|\,dt\leq\frac{2}{3}\left(\frac{a+b}{2}-x\right)^3\|f''\|_{[x,a+b-x],\infty},$$

$$\int_{a+b-x}^b(t-b)^2|f''(t)|\,dt\leq\frac{(x-a)^3}{3}f''\|_{[a+b-x,b],\infty}$$

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and the first inequality in (2.3) is proved.

Denote

$$\bar{M}(x) := \frac{(x-a)^3}{6}\|f''\|_{[a,x],\infty} + \frac{1}{3}\left(\frac{a+b}{2}-x\right)^3 \|f''\|_{[x,a+b-x],\infty} + \frac{(x-a)^3}{6}\|f''\|_{[a+b-x,b]}$$

for $x \in \left[a, \frac{a+b}{2}\right]$.

Firstly, observe that

$$\begin{aligned}\bar{M}(x) &\le \max\left\{\|f''\|_{[a,x],\infty}, \|f''\|_{[x,a+b-x],\infty}, \|f''\|_{[a+b-x,b],\infty}\right\} \\ &\quad\times\left[\frac{(x-a)^3}{6} + \frac{1}{3}\left(\frac{a+b}{2}-x\right)^3 + \frac{(x-a)^3}{6}\right] \\ &= \|f''\|_{[a,b],\infty}\left[\frac{(b-a)^2}{96} + \frac{1}{2}\left(x-\frac{3a+b}{4}\right)^2\right](b-a)\end{aligned}$$

and the first part of the second inequality in (2.3) is proved.

Using the Hölder inequality for $\alpha > 1$, $\frac{1}{\alpha}+\frac{1}{\beta}=1$, we also have

$$\begin{aligned}\bar{M}(x) \le \frac{1}{3}&\left\{\left[\frac{(x-a)^3}{2}\right]^\alpha + \left(x-\frac{a+b}{2}\right)^{3\alpha} + \left[\frac{(x-a)^3}{2}\right]^\alpha\right\}^{\frac{1}{\alpha}} \\ &\times\left[\|f''\|^\beta_{[a,x],\infty} + \|f''\|^\beta_{[x,a+b-x],\infty} + \|f''\|^\beta_{[a+b-x,b],\infty}\right]^{\frac{1}{\beta}}\end{aligned}$$

giving the second part of the second inequality in (2.3)

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Finally, we also observe that

$$\bar{M}(x) \leq \frac{1}{3}\max\left\{\frac{(x-a)^3}{2}, \left(x-\frac{a+b}{2}\right)^3\right\} \times \left[\|f''\|_{[a,x],\infty} + \|f''\|_{[x,a+b-x],\infty} + \|f''\|_{[a+b-x,b],\infty}\right].$$

The sharpness of the inequalities mentioned follows from the fact that we can choose a function $f:[a,b]\to\mathbb{R}$, $f(t)=t^2$ for any $x\in\left[a,\frac{a+b}{2}\right]$ to obtain the corresponding equalities. □

Remark 1. If in Theorem 2.2 we choose $x=a$, then we recapture the first part of the inequality (1.4), i.e.,

$$\left|\frac{1}{b-a}\int_a^b f(t)\,dt - \frac{1}{2}[f(a)+f(b)] + \frac{b-a}{8}[f'(b)-f'(a)]\right| \leq \frac{1}{24}(b-a)^2\|f''\|_\infty$$

with $\frac{1}{24}$ as a sharp constant. If we choose $x=\frac{a+b}{2}$, then we get

$$\left|\frac{1}{b-a}\int_a^b f(t)\,dt - f\left(\frac{a+b}{2}\right)\right| \leq \frac{1}{48}\left[\|f''\|_{\left[a,\frac{a+b}{2}\right],\infty} + \|f''\|_{\left[\frac{a+b}{2},b\right],\infty}\right] \leq \frac{1}{24}(b-a)^2\|f''\|_{[a,b],\infty}$$

with the constants $\frac{1}{48}$ and $\frac{1}{24}$ being sharp.

Corollary 2.3. *With the assumptions in Theorem 2.2, one has the inequality*

$$\left|\frac{1}{b-a}\int_a^b f(t)\,dt - \frac{f\left(\frac{3a+b}{4}\right)+f\left(\frac{a+3b}{4}\right)}{2}\right| \leq \frac{1}{96}(b-a)^2\|f''\|_{[a,b],\infty}. \tag{2.4}$$

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The constant $\frac{1}{96}$ is best possible in the sense that it cannot be replaced by a smaller constant. Clearly (2.4) is an improvement of (1.3).

Theorem 2.4. *Let $f : [a,b] \to \mathbb{R}$ be such that the derivative f' is absolutely continuous on $[a,b]$ and $f'' \in L_p[a,b]$, $p > 1$. If $M(x)$ is as defined in (2.2), then we have the bounds:*

$$M(x) \leq \frac{1}{2(2q+1)^{\frac{1}{q}}} \left[\left(\frac{x-a}{b-a} \right)^{2+\frac{1}{q}} \|f''\|_{[a,x],p} + 2^{\frac{1}{q}} \left(\frac{\frac{a+b}{2}-x}{b-a} \right)^{2+\frac{1}{q}} \|f''\|_{[x,a+b-x],p} \left(\frac{x-a}{b-a} \right)^{2+\frac{1}{q}} \|f''\|_{[a+b-x,b],p} \right] (b-a)^{1+\frac{1}{q}} \tag{2.5}$$

$$\leq \frac{1}{2(2q+1)^{\frac{1}{q}}} \times \begin{cases} \left[2\left(\frac{x-a}{b-a}\right)^{2+\frac{1}{q}} + 2^{\frac{1}{q}} \left(\frac{\frac{a+b}{2}-x}{b-a}\right)^{2+\frac{1}{q}} \right] \\ \quad \times \max\left\{ \|f''\|_{[a,x],p}, \|f''\|_{[x,a+b-x],p}, \|f''\|_{[a+b-x,b],p} \right\} (b-a)^{1+\frac{1}{q}}; \\ \left[2\left(\frac{x-a}{b-a}\right)^{2\alpha+\frac{\alpha}{q}} + 2^{\frac{\alpha}{q}} \left(\frac{\frac{a+b}{2}-x}{b-a}\right)^{2\alpha+\frac{\alpha}{q}} \right]^{\frac{1}{\alpha}} \\ \quad \times \left[\|f''\|_{[a,x],p}^{\beta} + \|f''\|_{[x,a+b-x],p}^{\beta} + \|f''\|_{[a+b-x,b],p}^{\beta} \right]^{\frac{1}{\beta}} (b-a)^{1+\frac{1}{q}} \\ \qquad \text{if } \alpha > 1, \frac{1}{\alpha} + \frac{1}{\beta} = 1, \\ \max\left\{ \left(\frac{x-a}{b-a}\right)^{2+\frac{1}{q}}, 2^{\frac{1}{q}} \left(\frac{\frac{a+b}{2}-x}{b-a}\right)^{2+\frac{1}{q}} \right\} \\ \quad \times \left[\|f''\|_{[a,x],p} + \|f''\|_{[x,a+b-x],p} + \|f''\|_{[a+b-x,b],p} \right] (b-a)^{1+\frac{1}{q}}; \end{cases}$$

for any $x \in \left[a, \frac{a+b}{2}\right]$.

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Proof. Using Hölder's integral inequality for $p > 1$, $\frac{1}{p} + \frac{1}{q} = 1$, we have

$$\int_a^x (t-a)^2 |f''(t)|\, dt \le \left(\int_a^x (t-a)^{2q}\, dt\right)^{\frac{1}{q}} \|f''\|_{[a,x],p}$$
$$= \frac{(x-a)^{2+\frac{1}{q}}}{(2q+1)^{\frac{1}{q}}} \|f''\|_{[a,x],p},$$

$$\int_x^{a+b-x} \left(t - \frac{a+b}{2}\right)^2 |f''(t)|\, dt \le \left(\int_x^{a+b-x} |t - \frac{a+b}{2}|^{2q}\, dt\right)^{\frac{1}{q}} \|f''\|_{[x,a+b-x],p}$$
$$= \frac{2^{\frac{1}{q}}\left(\frac{a+b}{2} - x\right)^{2+\frac{1}{q}}}{(2q+1)^{\frac{1}{q}}} \|f''\|_{[x,a+b-x],p},$$

and

$$\int_{a+b-x}^b (t-b)^2 |f''(t)|\, dt \le \left(\int_{a+b-x}^b (b-t)^{2q}\, dt\right)^{\frac{1}{q}} \|f''\|_{[a+b-x,b],p}$$
$$= \frac{(x-a)^{2+\frac{1}{q}}}{(2q+1)^{\frac{1}{q}}} \|f''\|_{[a+b-x,b],p}.$$

Summing the above inequalities, we deduce the first bound in (2.5).

The last part may be proved in a similar fashion to the one in Theorem 2.2, and we omit the details. □

Remark 2. If in (2.5) we choose $\alpha = q$, $\beta = p$, $\frac{1}{p} + \frac{1}{q} = 1$, $p > 1$, then we get the

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inequality

$$M(x) \le \frac{2^{\frac{1}{q}}}{2(2q+1)^{\frac{1}{q}}} \left[\left(\frac{x-a}{b-a} \right)^{2q+1} + \left(\frac{\frac{a+b}{2}-x}{b-a} \right)^{2q+1} \right]^{\frac{1}{q}} \times (b-a)^{1+\frac{1}{q}} \|f''\|_{[a,b],p}, \tag{2.6}$$

for any $x \in \left[a, \frac{a+b}{2}\right]$.

Remark 3. If in Theorem 2.4 we choose $x = a$, then we recapture the second part of the inequality (1.4), i.e.,

$$\left| \frac{1}{b-a} \int_a^b f(t)\, dt - \frac{1}{2}[f(a) + f(b)] + \frac{b-a}{8}[f'(b) - f'(a)] \right| \le \frac{1}{8} \cdot \frac{(b-a)^{1+\frac{1}{q}} \|f''\|_{[a,b],p}}{(2q+1)^{\frac{1}{q}}}. \tag{2.7}$$

The constant $\frac{1}{8}$ is best possible in the sense that it cannot be replaced by a smaller constant.

Proof. Indeed, if we assume that (2.7) holds with a constant $C > 0$, instead of $\frac{1}{8}$, i.e.,

$$\left| \frac{1}{b-a} \int_a^b f(t)\, dt - \frac{1}{2}[f(a) + f(b)] + \frac{b-a}{8}[f'(b) - f'(a)] \right| \le C \cdot \frac{(b-a)^{1+\frac{1}{q}} \|f''\|_{[a,b],p}}{(2q+1)^{\frac{1}{q}}}, \tag{2.8}$$

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then for the function $f:[a,b]\to\mathbb{R}$, $f(x)=k\left(x-\frac{a+b}{2}\right)^2$, $k>0$, we have

$$\frac{f(a)+f(b)}{2}=k\cdot\frac{(b-a)^2}{4},$$
$$f'(b)-f'(a)=2k(b-a),$$
$$\frac{1}{b-a}\int_a^b f(t)\,dt=k\cdot\frac{(b-a)^2}{12},$$
$$\|f''\|_{[a,b],p}=2k(b-a)^{\frac{1}{p}};$$

and by (2.8) we deduce

$$\left|\frac{k(b-a)^2}{12}-\frac{k(b-a)^2}{4}+\frac{k(b-a)^2}{4}\right|\le\frac{2C\cdot k(b-a)^2}{(2q+1)^{\frac{1}{q}}},$$

giving $C\ge\frac{(2q+1)^{\frac{1}{q}}}{24}$. Letting $q\to 1+$, we deduce $C\ge\frac{1}{8}$, and the sharpness of the constant is proved. □

Remark 4. If in Theorem 2.4 we choose $x=\frac{a+b}{2}$, then we get the midpoint inequality

$$\left|\frac{1}{b-a}\int_a^b f(t)\,dt-f\left(\frac{a+b}{2}\right)\right| \tag{2.9}$$
$$\le\frac{1}{8}\cdot\frac{(b-a)^{1+\frac{1}{q}}}{2^{\frac{1}{q}}(2q+1)^{\frac{1}{q}}}\left[\|f''\|_{[a,\frac{a+b}{2}],p}+\|f''\|_{[\frac{a+b}{2},b],p}\right]$$
$$\le\frac{1}{8}\cdot\frac{(b-a)^{1+\frac{1}{q}}}{(2q+1)^{\frac{1}{q}}}\|f''\|_{[a,b],p},\quad p>1,\ \frac{1}{p}+\frac{1}{q}=1.$$

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In both inequalities the constant $\frac{1}{8}$ is sharp in the sense that it cannot be replaced by a smaller constant.

To show this fact, assume that (2.9) holds with $C, D > 0$, i.e.,

$$\left|\frac{1}{b-a}\int_a^b f(t)\,dt - f\left(\frac{a+b}{2}\right)\right| \tag{2.10}$$
$$\leq C \cdot \frac{(b-a)^{1+\frac{1}{q}}}{2^{\frac{1}{q}}(2q+1)^{\frac{1}{q}}}\left[\|f''\|_{[a,\frac{a+b}{2}],p} + \|f''\|_{[\frac{a+b}{2},b],p}\right]$$
$$\leq D \cdot \frac{(b-a)^{1+\frac{1}{q}}}{(2q+1)^{\frac{1}{q}}}\|f''\|_{[a,b],p}.$$

For the function $f : [a, b] \to \mathbb{R}$, $f(x) = k\left(x - \frac{a+b}{2}\right)^2$, $k > 0$, we have

$$f\left(\frac{a+b}{2}\right) = 0, \qquad \frac{1}{b-a}\int_a^b f(t)\,dt = \frac{k(b-a)^2}{12},$$

$$\|f''\|_{[a,\frac{a+b}{2}],p} + \|f''\|_{[\frac{a+b}{2},b],p} = 4k\left(\frac{b-a}{2}\right)^{\frac{1}{p}}$$
$$= 2^{1+\frac{1}{q}}(b-a)^{\frac{1}{p}}k,$$

$$\|f''\|_{[a,b],p} = 2(b-a)^{\frac{1}{p}}k;$$

and then by (2.10) we deduce

$$\frac{k(b-a)^2}{12} \leq C \cdot \frac{2k(b-a)^2}{(2q+1)^{\frac{1}{q}}} \leq D \cdot \frac{2k(b-a)^2}{(2q+1)^{\frac{1}{q}}},$$

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giving $C, D \geq \frac{(2q+1)^{\frac{1}{q}}}{24}$ for any $q > 1$. Letting $q \to 1+$, we deduce $C, D \geq \frac{1}{8}$ and the sharpness of the constants in (2.9) is proved.

The following result is useful in providing the best quadrature rule in the class for approximating the integral of a function $f : [a, b] \to \mathbb{R}$ whose first derivative is absolutely continuous on $[a, b]$ and whose second derivative is in $L_p [a, b]$.

Corollary 2.5. *With the assumptions in Theorem 2.4, one has the inequality*

$$\left| \frac{1}{2} \left[f\left(\frac{3a+b}{4}\right) + f\left(\frac{a+3b}{4}\right) \right] - \frac{1}{b-a} \int_a^b f(t)\, dt \right| \leq \frac{1}{32} \cdot \frac{(b-a)^{1+\frac{1}{q}}}{(2q+1)^{\frac{1}{q}}} \|f''\|_{[a,b],p}, \tag{2.11}$$

where $\frac{1}{p} + \frac{1}{q} = 1$.

The constant $\frac{1}{32}$ *is the best possible in the sense that it cannot be replaced by a smaller constant.*

Proof. The inequality follows by Theorem 2.4 and (2.6) on choosing $x = \frac{3a+b}{4}$.

To prove the sharpness of the constant, assume that (2.11) holds with a constant $E > 0$, i.e.,

$$\left| \frac{1}{2} \left[f\left(\frac{3a+b}{4}\right) + f\left(\frac{a+3b}{4}\right) \right] - \frac{1}{b-a} \int_a^b f(t)\, dt \right| \leq E \cdot \frac{(b-a)^{1+\frac{1}{q}}}{(2q+1)^{\frac{1}{q}}} \|f''\|_{[a,b],p}. \tag{2.12}$$

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Consider the function $f:[a,b]\to\mathbb{R}$,

$$f(x)=\begin{cases} -\frac{1}{2}\left(x-\frac{3a+b}{4}\right)^2 & \text{if} \quad x\in\left[a,\frac{3a+b}{4}\right], \\ \frac{1}{2}\left(x-\frac{3a+b}{4}\right)^2 & \text{if} \quad x\in\left(\frac{3a+b}{4},\frac{a+b}{2}\right], \\ -\frac{1}{2}\left(x-\frac{a+3b}{4}\right)^2 & \text{if} \quad x\in\left(\frac{a+b}{2},\frac{a+3b}{4}\right], \\ \frac{1}{2}\left(x-\frac{a+3b}{4}\right)^2 & \text{if} \quad x\in\left(\frac{a+3b}{4},b\right]. \end{cases}$$

We have

$$f'(x)=\begin{cases} \left|x-\frac{3a+b}{4}\right| & \text{if} \quad x\in\left[a,\frac{a+b}{2}\right], \\ \left|x-\frac{a+3b}{4}\right| & \text{if} \quad x\in\left(\frac{a+b}{2},b\right]. \end{cases}$$

Then f' is absolutely continuous and $f''\in L_p[a,b]$, $p>1$. We also have

$$\frac{1}{2}\left[f\left(\frac{3a+b}{4}\right)+f\left(\frac{a+3b}{4}\right)\right]=0,$$

$$\frac{1}{b-a}\int_a^b f(t)\,dt=\frac{(b-a)^2}{96},$$

$$\|f''\|_{[a,b],p}=(b-a)^{\frac{1}{p}},$$

and then, by (2.12), we obtain

$$\frac{(b-a)^2}{96}\le E\cdot\frac{(b-a)^2}{(2q+1)^{\frac{1}{q}}},$$

giving $E\ge\frac{(2q+1)^{\frac{1}{q}}}{96}$ for any $q>1$, i.e., $E\ge\frac{1}{32}$, and the corollary is proved. □

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Theorem 2.6. *Let $f : [a, b] \to \mathbb{R}$ be such that the derivative f' is absolutely continuous on $[a, b]$ and $f'' \in L_1[a, b]$. If $M(x)$ is as defined in (2.2), then we have the bounds:*

$$M(x) \leq \frac{b-a}{2}\left[\left(\frac{x-a}{b-a}\right)^2 \|f''\|_{[a,x],1} + \left(\frac{\frac{a+b}{2}-x}{b-a}\right)^2 \|f''\|_{[x,a+b-x],1} + \left(\frac{x-a}{b-a}\right)^2 \|f''\|_{[a+b-x,b],1}\right] \tag{2.13}$$

$$\leq \begin{cases} \frac{b-a}{2}\left[2\left(\frac{x-a}{b-a}\right)^2 + \left(\frac{\frac{a+b}{2}-x}{b-a}\right)^2\right] \\ \quad \times \max\left[\|f''\|_{[a,x],1}, \|f''\|_{[x,a+b-x],1}, \|f''\|_{[a+b-x,b],1}\right]; \\ \frac{b-a}{2}\left[2\left(\frac{x-a}{b-a}\right)^{2\alpha} + \left(\frac{\frac{a+b}{2}-x}{b-a}\right)^{2\alpha}\right]^{\frac{1}{\alpha}} \\ \quad \times \left[\|f''\|^{\beta}_{[a,x],1} + \|f''\|^{\beta}_{[x,a+b-x],1} + \|f''\|^{\beta}_{[a+b-x,b],1}\right]^{\frac{1}{\beta}} \\ \qquad \text{if} \quad \alpha > 1, \frac{1}{\alpha}+\frac{1}{\beta} = 1; \\ \frac{b-a}{2}\left[\left|\frac{x-\frac{3a+b}{4}}{b-a}\right| + \frac{1}{4}\right]^2 \|f''\|_{[a,b],1}; \end{cases}$$

for any $x \in \left[a, \frac{a+b}{2}\right]$.

The proof is as in Theorem 2.2 and we need only to prove the third inequality of the last part as

$$M(x) \leq \frac{b-a}{2}\max\left\{\left(\frac{x-a}{b-a}\right)^2, \left(\frac{\frac{a+b}{2}-x}{b-a}\right)^2\right\} \times \left[\|f''\|_{[a,x],1} + \|f''\|_{[x,a+b-x],1} + \|f''\|_{[a+b-x,b],1}\right]$$

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$$= \frac{b-a}{2}\left[\left|\frac{x-\frac{3a+b}{4}}{b-a}\right|+\frac{1}{4}\right]^2 \|f''\|_{[a,b],1}.$$

Remark 5. By the use of Theorem 2.6, for $x = a$, we recapture the third part of the inequality (1.4), i.e.,

$$\left|\frac{1}{b-a}\int_a^b f(t)\ dt - \frac{1}{2}\left[f(a)+f(b)\right] + \frac{b-a}{8}\left[f'(b)-f'(a)\right]\right| \le \frac{1}{8}(b-a)\|f''\|_{[a,b],1}.$$

If in (2.13) we choose $x = \frac{a+b}{2}$, then we get the mid-point inequality

$$\left|\frac{1}{b-a}\int_a^b f(t)\ dt - f\left(\frac{a+b}{2}\right)\right| \le \frac{1}{8}(b-a)\|f''\|_{[a,b],1}.$$

Corollary 2.7. *With the assumptions in Theorem 2.6, one has the inequality*

$$\left|\frac{1}{b-a}\int_a^b f(t)\ dt - \frac{f\left(\frac{3a+b}{4}\right)+f\left(\frac{a+3b}{4}\right)}{2}\right| \le \frac{1}{32}(b-a)\|f''\|_{[a,b],1}.$$

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3. A Composite Quadrature Formula

We use the following inequalities obtained in the previous section:

$$\left|\frac{1}{2}\left[f\left(\frac{3a+b}{4}\right)+f\left(\frac{a+3b}{4}\right)\right]-\frac{1}{b-a}\int_a^b f(t)\,dt\right| \tag{3.1}$$
$$\leq \begin{cases} \frac{1}{96}(b-a)^2\|f''\|_{[a,b],\infty} & \text{if } f''\in L_\infty[a,b]\,; \\ \frac{1}{32}\cdot\frac{(b-a)^{1+\frac{1}{q}}}{(2q+1)^{\frac{1}{q}}}\|f''\|_{[a,b],p} & \text{if } f''\in L_p[a,b]\,,\ p>1,\ \frac{1}{p}+\frac{1}{q}=1; \\ \frac{1}{32}(b-a)\|f''\|_{[a,b],1} & \text{if } f''\in L_1[a,b]\,. \end{cases}$$

Let $I_n : a = x_0 < x_1 < \cdots < x_{n-1} < x_n = b$ be a division of the interval $[a,b]$ and $h_i := x_{i+1} - x_i$ $(i = 0,\ldots,n-1)$ and $\nu(I_n) := \max\{h_i | i = 0,\ldots,n-1\}$.

Consider the composite quadrature rule

$$Q_n(I_n,f) := \frac{1}{2}\sum_{i=0}^{n-1}\left[f\left(\frac{3x_i+x_{i+1}}{4}\right)+f\left(\frac{x_i+3x_{i+1}}{4}\right)\right]h_i. \tag{3.2}$$

The following result holds.

Theorem 3.1. *Let $f : [a,b] \to \mathbb{R}$ be such that the derivative f' is absolutely continuous on $[a,b]$. Then we have*

$$\int_a^b f(t)\,dt = Q_n(I_n,f) + R_n(I_n,f)\,,$$

where $Q_n(I_n,f)$ is defined by the formula (3.2), and the remainder satisfies the

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estimates

$$(3.3)\quad |R_n(I_n,f)| \le \begin{cases} \frac{1}{96}\|f''\|_{[a,b],\infty}\sum_{i=0}^{n-1} h_i^3 & \text{if } f''\in L_\infty[a,b]; \\ \frac{1}{32(2q+1)^{\frac{1}{q}}}\|f''\|_{[a,b],p}\left(\sum_{i=0}^{n-1} h_i^{2q+1}\right)^{\frac{1}{q}} & \text{if } f''\in L_p[a,b], \\ & p>1,\ \frac{1}{p}+\frac{1}{q}=1; \\ \frac{1}{32}\|f''\|_{[a,b],1}\left[\nu(I_n)\right]^2 & \text{if } f''\in L_1[a,b]. \end{cases}$$

Proof. Applying inequality (3.1) on the interval $[x_i, x_{i+1}]$, we may state that

$$(3.4)\quad \left|\int_{x_i}^{x_{i+1}} f(t)\,dt - \frac{1}{2}\left[f\left(\frac{3x_i+x_{i+1}}{4}\right)+f\left(\frac{x_i+3x_{i+1}}{4}\right)\right]h_i\right| \le \begin{cases} \frac{1}{96}h_i^3\|f''\|_{[x_i,x_{i+1}],\infty}; \\ \frac{1}{32(2q+1)^{\frac{1}{q}}}h_i^{2+\frac{1}{q}}\|f''\|_{[x_i,x_{i+1}],p}, \quad p>1,\ \frac{1}{p}+\frac{1}{q}=1; \\ \frac{1}{32}h_i^2\|f''\|_{[x_i,x_{i+1}],1}; \end{cases}$$

for each $i\in\{0,\dots,n-1\}$.

Summing the inequality (3.4) over i from 0 to $n-1$ and using the generalized triangle inequality, we get

$$(3.5)\quad |R_n(I_n,f)| \le \begin{cases} \frac{1}{96}\sum_{i=0}^{n-1} h_i^3\|f''\|_{[x_i,x_{i+1}],\infty}; \\ \frac{1}{32(2q+1)^{\frac{1}{q}}}\sum_{i=0}^{n-1} h_i^{2+\frac{1}{q}}\|f''\|_{[x_i,x_{i+1}],p}, \quad p>1,\ \frac{1}{p}+\frac{1}{q}=1; \\ \frac{1}{32}\sum_{i=0}^{n-1} h_i^2\|f''\|_{[x_i,x_{i+1}],1}. \end{cases}$$

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Now, we observe that

$$\sum_{i=0}^{n-1} h_i^3 \|f''\|_{[x_i,x_{i+1}],\infty} \leq \|f''\|_{[a,b],\infty} \sum_{i=0}^{n-1} h_i^3.$$

Using Hölder's discrete inequality, we may write that

$$\begin{aligned}
\sum_{i=0}^{n-1} h_i^{2+\frac{1}{q}} \|f''\|_{[x_i,x_{i+1}],p} &\leq \left(\sum_{i=0}^{n-1} h_i^{\left(2+\frac{1}{q}\right)q}\right)^{\frac{1}{q}} \left(\sum_{i=0}^{n-1} \|f''\|_{[x_i,x_{i+1}],p}^p\right)^{\frac{1}{p}} \\
&= \left(\sum_{i=0}^{n-1} h_i^{2q+1}\right)^{\frac{1}{q}} \left(\sum_{i=0}^{n-1} \int_{x_i}^{x_{i+1}} |f''(t)|^p \, dt\right)^{\frac{1}{p}} \\
&= \left(\sum_{i=0}^{n-1} h_i^{2q+1}\right)^{\frac{1}{q}} \|f''\|_{[a,b],p}.
\end{aligned}$$

Also, we note that

$$\begin{aligned}
\sum_{i=0}^{n-1} h_i^2 \|f''\|_{[x_i,x_{i+1}],1} &\leq \max_{0\leq i\leq n-1} \{h_i^2\} \sum_{i=0}^{n-1} \|f''\|_{[x_i,x_{i+1}],1} \\
&= [\nu(I_n)]^2 \|f''\|_{[a,b],1}.
\end{aligned}$$

Consequently, by the use of (3.5), we deduce the desired result (3.3). □

For the particular case where the division I_n is equidistant, i.e.,

$$I_n := x_i = a + i \cdot \frac{b-a}{n}, \qquad i = 0, \dots, n,$$

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we may consider the quadrature rule:

$$Q_n(f) := \frac{b-a}{2n}\sum_{i=0}^{n-1}\left\{f\left[a+\left(\frac{4i+1}{4n}\right)(b-a)\right] + f\left[a+\left(\frac{4i+3}{4n}\right)(b-a)\right]\right\}. \tag{3.6}$$

The following corollary will be more useful in practice.

Corollary 3.2. *With the assumption of Theorem 3.1, we have*

$$\int_a^b f(t)\,dt = Q_n(f) + R_n(f),$$

where $Q_n(f)$ is defined by (3.6) and the remainder $R_n(f)$ satisfies the estimate:

$$|R_n(I_n,f)| \le \begin{cases} \frac{1}{96}\|f''\|_{[a,b],\infty}\frac{(b-a)^3}{n^2}; \\ \frac{1}{32(2q+1)^{\frac{1}{q}}}\|f''\|_{[a,b],p}\frac{(b-a)^{2+\frac{1}{q}}}{n^2}, \quad p>1,\ \frac{1}{p}+\frac{1}{q}=1; \\ \frac{1}{32}\|f''\|_{[a,b],1}\frac{(b-a)^2}{n^2}. \end{cases}$$

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journal of inequalities in pure and applied mathematics
issn: 1443-5756

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journal of inequalities in pure and applied mathematics

issn: 1443-5756