

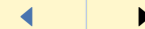
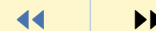
Ostrowski Type Inequality

Zheng Liu

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SOME COMPANIONS OF AN OSTROWSKI TYPE INEQUALITY AND APPLICATIONS

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Abstract: We establish some companions of an Ostrowski type integral inequality for functions whose derivatives are absolutely continuous. Applications for composite quadrature rules are also given.

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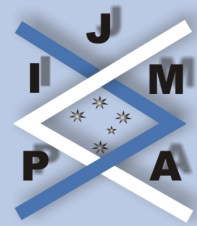
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1. Introduction

Motivated by [1], Dragomir in [2] has proved the following companion of the Ostrowski inequality:

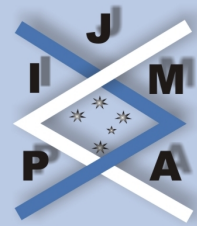
$$(1.1) \quad \left| \frac{1}{2} [f(x) + f(a+b-x)] - \frac{1}{b-a} \int_a^b f(t) dt \right| \leq \begin{cases} \left[\frac{1}{8} + 2 \left(\frac{x - \frac{3a+b}{4}}{b-a} \right)^2 \right] (b-a) \|f'\|_{[a,b],\infty} & \text{if } f' \in L_\infty[a,b]; \\ \frac{2^{\frac{1}{q}}}{(q+1)^{\frac{1}{q}}} \left[\left(\frac{x-a}{b-a} \right)^{q+1} + \left(\frac{a+b-x}{2} \right)^{q+1} \right]^{\frac{1}{q}} (b-a)^{\frac{1}{q}} \|f'\|_{[a,b],p}, & \text{if } p > 1, \frac{1}{p} + \frac{1}{q} = 1, \\ \left[\frac{1}{4} + \left| \frac{x - \frac{3a+b}{4}}{b-a} \right| \right] \|f'\|_{[a,b],1} & \text{and } f' \in L_p[a,b]; \\ & \text{if } f' \in L_1[a,b], \end{cases}$$

for all $x \in [a, \frac{a+b}{2}]$, where $f : [a, b] \rightarrow \mathbb{R}$ is an absolutely continuous function.

In particular, the best result in (1.1) is obtained for $x = \frac{a+3b}{4}$, giving the following trapezoid type inequalities:

$$(1.2) \quad \left| \frac{1}{2} \left[f\left(\frac{3a+b}{4}\right) + f\left(\frac{a+3b}{4}\right) \right] - \frac{1}{b-a} \int_a^b f(t) dt \right| \leq \begin{cases} \frac{1}{8} (b-a) \|f'\|_{[a,b],\infty} & \text{if } f' \in L_\infty[a,b]; \\ \frac{1}{4} \cdot \frac{(b-a)^{\frac{1}{q}}}{(q+1)^{\frac{1}{q}}} \|f'\|_{[a,b],p}, & \text{if } f' \in L_p[a,b], p > 1, \frac{1}{p} + \frac{1}{q} = 1; \\ \frac{1}{4} \|f'\|_{[a,b],1} & \text{if } f' \in L_1[a,b]. \end{cases}$$

Some natural applications of (1.1) and (1.2) are also provided in [2].



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In [3], Dedić et al. have derived the following trapezoid type inequality:

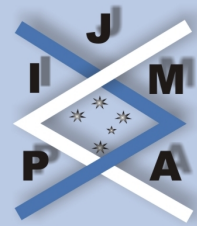
$$(1.3) \quad \left| \frac{1}{2} \left[f \left(\frac{3a+b}{4} \right) + f \left(\frac{a+3b}{4} \right) \right] - \frac{1}{b-a} \int_a^b f(t) dt \right| \leq \frac{(b-a)^2}{32} \|f''\|_\infty,$$

for a function $f : [a, b] \rightarrow \mathbb{R}$ whose derivative f' is absolutely continuous and $f'' \in L_\infty[a, b]$.

In [4], we find that for a function $f : [a, b] \rightarrow \mathbb{R}$ whose derivative f' is absolutely continuous, the following perturbed trapezoid inequalities hold:

$$(1.4) \quad \left| \int_a^b f(t) dt - \frac{b-a}{2} [f(a) + f(b)] + \frac{(b-a)^2}{8} [f'(b) - f'(a)] \right| \leq \begin{cases} \frac{(b-a)^3}{24} \|f''\|_\infty & \text{if } f'' \in L_\infty[a, b]; \\ \frac{(b-a)^{2+\frac{1}{q}}}{8(2q+1)^{\frac{1}{q}}} \|f''\|_p, & \text{if } f'' \in L_p[a, b], p > 1, \frac{1}{p} + \frac{1}{q} = 1; \\ \frac{(b-a)^2}{8} \|f''\|_1 & \text{if } f'' \in L_1[a, b]. \end{cases}$$

In this paper, we provide some companions of Ostrowski type inequalities for functions whose first derivatives are absolutely continuous and whose second derivatives belong to the Lebesgue spaces $L_p[a, b]$, $1 \leq p \leq \infty$. These improve (1.3) and recapture (1.4). Applications for composite quadrature rules are also given.



2. Some Integral Inequalities

Lemma 2.1. Let $f : [a, b] \rightarrow \mathbb{R}$ be such that the derivative f' is absolutely continuous on $[a, b]$. Then we have the equality

$$(2.1) \quad \frac{1}{b-a} \int_a^b f(t) dt - \frac{1}{2} [f(x) + f(a+b-x)] \\ + \frac{1}{2} \left(x - \frac{3a+b}{4} \right) [f'(x) - f'(a+b-x)] \\ = \frac{1}{2(b-a)} \left[\int_a^x (t-a)^2 f''(t) dt + \int_x^{a+b-x} \left(t - \frac{a+b}{2} \right)^2 f''(t) dt \right. \\ \left. + \int_{a+b-x}^b (t-b)^2 f''(t) dt \right]$$

for any $x \in [a, \frac{a+b}{2}]$.

Proof. Using the integration by parts formula for Lebesgue integrals, we have

$$\int_a^x (t-a)^2 f''(t) dt = (x-a)^2 f'(x) - 2(x-a) f(x) + 2 \int_a^x f(t) dt, \\ \int_x^{a+b-x} \left(t - \frac{a+b}{2} \right)^2 f''(t) dt \\ = \left(x - \frac{a+b}{2} \right)^2 [f'(a+b-x) - f'(x)] \\ + 2 \left(x - \frac{a+b}{2} \right) [f(x) + f(a+b-x)] + 2 \int_x^{a+b-x} f(t) dt,$$

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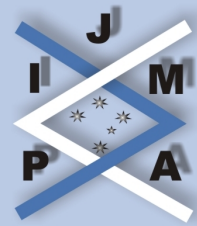
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and

$$\int_{a+b-x}^b (t-b)^2 f''(t) dt \\ = -(x-a)^2 f'(a+b-x) - 2(x-a) f(a+b-x) + 2 \int_{a+b-x}^b f(t) dt.$$

Summing the above equalities, we deduce the desired identity (2.1). \square

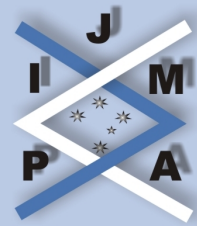
Theorem 2.2. *Let $f : [a, b] \rightarrow \mathbb{R}$ be such that the derivative f' is absolutely continuous on $[a, b]$. Then we have the inequality*

$$(2.2) \quad \left| \frac{1}{b-a} \int_a^b f(t) dt - \frac{1}{2} [f(x) + f(a+b-x)] \right. \\ \left. + \frac{1}{2} \left(x - \frac{3a+b}{4} \right) [f'(x) - f'(a+b-x)] \right| \\ \leq \frac{1}{2(b-a)} \left[\int_a^x (t-a)^2 |f''(t)| dt + \int_x^{a+b-x} \left(t - \frac{a+b}{2} \right)^2 |f''(t)| dt \right. \\ \left. + \int_{a+b-x}^b (t-b)^2 |f''(t)| dt \right] \\ := M(x)$$

for any $x \in [a, \frac{a+b}{2}]$.

If $f'' \in L_\infty[a, b]$, then we have the inequalities

$$(2.3) \quad M(x) \leq \frac{1}{2(b-a)} \left[\frac{(x-a)^3}{3} \|f''\|_{[a,x],\infty} \right]$$



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$$\begin{aligned}
 & + \frac{2}{3} \left(\frac{a+b}{2} - x \right)^3 \|f''\|_{[x, a+b-x], \infty} + \frac{(x-a)^3}{3} \|f''\|_{[a+b-x, b]} \Big] \\
 \leq & \left\{ \begin{array}{l} \left[\frac{1}{96} + \frac{1}{2} \left(\frac{x - \frac{3a+b}{4}}{b-a} \right)^2 \right] (b-a)^2 \|f''\|_{[a, b], \infty}; \\ \left[\frac{1}{2^{\alpha-1}} \left(\frac{x-a}{b-a} \right)^{3\alpha} + \left(\frac{x - \frac{a+b}{2}}{b-a} \right)^{3\alpha} \right]^{\frac{1}{\alpha}} \\ \quad \times \left[\|f''\|_{[a, x], \infty}^\beta + \|f''\|_{[x, a+b-x], \infty}^\beta + \|f''\|_{[a+b-x, b], \infty}^\beta \right]^{\frac{1}{\beta}} \frac{(b-a)^2}{3} \\ \quad \text{if } \alpha > 1, \frac{1}{\alpha} + \frac{1}{\beta} = 1; \\ \max \left\{ \frac{1}{2} \left(\frac{x-a}{b-a} \right)^3, \left(\frac{x - \frac{a+b}{2}}{b-a} \right)^3 \right\} \\ \quad \times \left[\|f''\|_{[a, x], \infty} + \|f''\|_{[x, a+b-x], \infty} + \|f''\|_{[a+b-x, b], \infty} \right] \frac{(b-a)^2}{3}; \end{array} \right.
 \end{aligned}$$

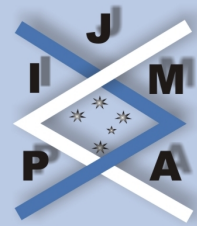
for any $x \in [a, \frac{a+b}{2}]$.

The inequality (2.2), the first inequality in (2.3) and the constant $\frac{1}{96}$ are sharp.

Proof. The inequality (2.2) follows by Lemma 2.1 on taking the modulus and using its properties.

If $f'' \in L_\infty[a, b]$, then

$$\begin{aligned}
 \int_a^x (t-a)^2 |f''(t)| dt & \leq \frac{(x-a)^3}{3} \|f''\|_{[a, x], \infty}, \\
 \int_x^{a+b-x} \left(t - \frac{a+b}{2} \right)^2 |f''(t)| dt & \leq \frac{2}{3} \left(\frac{a+b}{2} - x \right)^3 \|f''\|_{[x, a+b-x], \infty}, \\
 \int_{a+b-x}^b (t-b)^2 |f''(t)| dt & \leq \frac{(x-a)^3}{3} \|f''\|_{[a+b-x, b], \infty}
 \end{aligned}$$



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and the first inequality in (2.3) is proved.

Denote

$$\bar{M}(x) := \frac{(x-a)^3}{6} \|f''\|_{[a,x],\infty} + \frac{1}{3} \left(\frac{a+b}{2} - x \right)^3 \|f''\|_{[x,a+b-x],\infty} + \frac{(x-a)^3}{6} \|f''\|_{[a+b-x,b]}$$

for $x \in [a, \frac{a+b}{2}]$.

Firstly, observe that

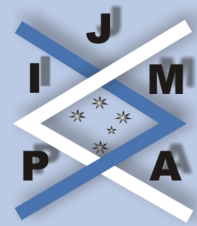
$$\begin{aligned} \bar{M}(x) &\leq \max \left\{ \|f''\|_{[a,x],\infty}, \|f''\|_{[x,a+b-x],\infty}, \|f''\|_{[a+b-x,b],\infty} \right\} \\ &\quad \times \left[\frac{(x-a)^3}{6} + \frac{1}{3} \left(\frac{a+b}{2} - x \right)^3 + \frac{(x-a)^3}{6} \right] \\ &= \|f''\|_{[a,b],\infty} \left[\frac{(b-a)^2}{96} + \frac{1}{2} \left(x - \frac{3a+b}{4} \right)^2 \right] (b-a) \end{aligned}$$

and the first part of the second inequality in (2.3) is proved.

Using the Hölder inequality for $\alpha > 1$, $\frac{1}{\alpha} + \frac{1}{\beta} = 1$, we also have

$$\begin{aligned} \bar{M}(x) &\leq \frac{1}{3} \left\{ \left[\frac{(x-a)^3}{2} \right]^\alpha + \left(x - \frac{a+b}{2} \right)^{3\alpha} + \left[\frac{(x-a)^3}{2} \right]^\alpha \right\}^{\frac{1}{\alpha}} \\ &\quad \times \left[\|f''\|_{[a,x],\infty}^\beta + \|f''\|_{[x,a+b-x],\infty}^\beta + \|f''\|_{[a+b-x,b],\infty}^\beta \right]^{\frac{1}{\beta}} \end{aligned}$$

giving the second part of the second inequality in (2.3)



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Finally, we also observe that

$$\bar{M}(x) \leq \frac{1}{3} \max \left\{ \frac{(x-a)^3}{2}, \left(x - \frac{a+b}{2} \right)^3 \right\} \\ \times \left[\|f''\|_{[a,x],\infty} + \|f''\|_{[x,a+b-x],\infty} + \|f''\|_{[a+b-x,b],\infty} \right].$$

The sharpness of the inequalities mentioned follows from the fact that we can choose a function $f : [a, b] \rightarrow \mathbb{R}$, $f(t) = t^2$ for any $x \in [a, \frac{a+b}{2}]$ to obtain the corresponding equalities. \square

Remark 1. If in Theorem 2.2 we choose $x = a$, then we recapture the first part of the inequality (1.4), i.e.,

$$\left| \frac{1}{b-a} \int_a^b f(t) dt - \frac{1}{2} [f(a) + f(b)] + \frac{b-a}{8} [f'(b) - f'(a)] \right| \leq \frac{1}{24} (b-a)^2 \|f''\|_{\infty}$$

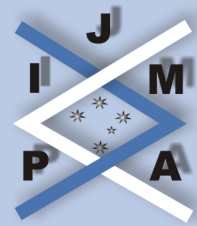
with $\frac{1}{24}$ as a sharp constant. If we choose $x = \frac{a+b}{2}$, then we get

$$\left| \frac{1}{b-a} \int_a^b f(t) dt - f\left(\frac{a+b}{2}\right) \right| \leq \frac{1}{48} \left[\|f''\|_{[a,\frac{a+b}{2}],\infty} + \|f''\|_{[\frac{a+b}{2},b],\infty} \right] \\ \leq \frac{1}{24} (b-a)^2 \|f''\|_{[a,b],\infty}$$

with the constants $\frac{1}{48}$ and $\frac{1}{24}$ being sharp.

Corollary 2.3. *With the assumptions in Theorem 2.2, one has the inequality*

$$(2.4) \quad \left| \frac{1}{b-a} \int_a^b f(t) dt - \frac{f\left(\frac{3a+b}{4}\right) + f\left(\frac{a+3b}{4}\right)}{2} \right| \leq \frac{1}{96} (b-a)^2 \|f''\|_{[a,b],\infty}.$$



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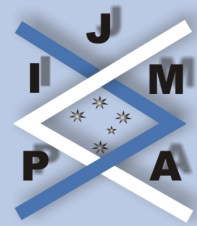
The constant $\frac{1}{96}$ is best possible in the sense that it cannot be replaced by a smaller constant. Clearly (2.4) is an improvement of (1.3).

Theorem 2.4. Let $f : [a, b] \rightarrow \mathbb{R}$ be such that the derivative f' is absolutely continuous on $[a, b]$ and $f'' \in L_p[a, b]$, $p > 1$. If $M(x)$ is as defined in (2.2), then we have the bounds:

$$(2.5) \quad M(x) \leq \frac{1}{2(2q+1)^{\frac{1}{q}}} \left[\left(\frac{x-a}{b-a} \right)^{2+\frac{1}{q}} \|f''\|_{[a,x],p} \right. \\ \left. + 2^{\frac{1}{q}} \left(\frac{\frac{a+b}{2}-x}{b-a} \right)^{2+\frac{1}{q}} \|f''\|_{[x,a+b-x],p} \left(\frac{x-a}{b-a} \right)^{2+\frac{1}{q}} \|f''\|_{[a+b-x,b],p} \right] (b-a)^{1+\frac{1}{q}}$$

$$\leq \frac{1}{2(2q+1)^{\frac{1}{q}}} \times \left\{ \begin{array}{l} \left[2 \left(\frac{x-a}{b-a} \right)^{2+\frac{1}{q}} + 2^{\frac{1}{q}} \left(\frac{\frac{a+b}{2}-x}{b-a} \right)^{2+\frac{1}{q}} \right] \\ \times \max \{ \|f''\|_{[a,x],p}, \|f''\|_{[x,a+b-x],p}, \|f''\|_{[a+b-x,b],p} \} (b-a)^{1+\frac{1}{q}} ; \\ \left[2 \left(\frac{x-a}{b-a} \right)^{2\alpha+\frac{\alpha}{q}} + 2^{\frac{\alpha}{q}} \left(\frac{\frac{a+b}{2}-x}{b-a} \right)^{2\alpha+\frac{\alpha}{q}} \right]^{\frac{1}{\alpha}} \\ \times \left[\|f''\|_{[a,x],p}^\beta + \|f''\|_{[x,a+b-x],p}^\beta + \|f''\|_{[a+b-x,b],p}^\beta \right]^{\frac{1}{\beta}} (b-a)^{1+\frac{1}{q}} \\ \text{if } \alpha > 1, \frac{1}{\alpha} + \frac{1}{\beta} = 1, \\ \max \left\{ \left(\frac{x-a}{b-a} \right)^{2+\frac{1}{q}}, 2^{\frac{1}{q}} \left(\frac{\frac{a+b}{2}-x}{b-a} \right)^{2+\frac{1}{q}} \right\} \\ \times \left[\|f''\|_{[a,x],p} + \|f''\|_{[x,a+b-x],p} + \|f''\|_{[a+b-x,b],p} \right] (b-a)^{1+\frac{1}{q}} ; \end{array} \right.$$

for any $x \in [a, \frac{a+b}{2}]$.



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Proof. Using Hölder's integral inequality for $p > 1$, $\frac{1}{p} + \frac{1}{q} = 1$, we have

$$\begin{aligned} \int_a^x (t-a)^2 |f''(t)| dt &\leq \left(\int_a^x (t-a)^{2q} dt \right)^{\frac{1}{q}} \|f''\|_{[a,x],p} \\ &= \frac{(x-a)^{2+\frac{1}{q}}}{(2q+1)^{\frac{1}{q}}} \|f''\|_{[a,x],p}, \end{aligned}$$

$$\begin{aligned} \int_x^{a+b-x} \left(t - \frac{a+b}{2} \right)^2 |f''(t)| dt &\leq \left(\int_x^{a+b-x} \left| t - \frac{a+b}{2} \right|^{2q} dt \right)^{\frac{1}{q}} \|f''\|_{[x,a+b-x],p} \\ &= \frac{2^{\frac{1}{q}} \left(\frac{a+b}{2} - x \right)^{2+\frac{1}{q}}}{(2q+1)^{\frac{1}{q}}} \|f''\|_{[x,a+b-x],p}, \end{aligned}$$

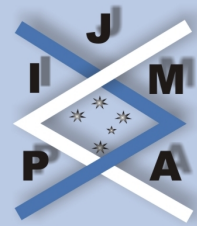
and

$$\begin{aligned} \int_{a+b-x}^b (t-b)^2 |f''(t)| dt &\leq \left(\int_{a+b-x}^b (b-t)^{2q} dt \right)^{\frac{1}{q}} \|f''\|_{[a+b-x,b],p} \\ &= \frac{(x-a)^{2+\frac{1}{q}}}{(2q+1)^{\frac{1}{q}}} \|f''\|_{[a+b-x,b],p}. \end{aligned}$$

Summing the above inequalities, we deduce the first bound in (2.5).

The last part may be proved in a similar fashion to the one in Theorem 2.2, and we omit the details. \square

Remark 2. If in (2.5) we choose $\alpha = q$, $\beta = p$, $\frac{1}{p} + \frac{1}{q} = 1$, $p > 1$, then we get the



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inequality

$$(2.6) \quad M(x) \leq \frac{2^{\frac{1}{q}}}{2(2q+1)^{\frac{1}{q}}} \left[\left(\frac{x-a}{b-a} \right)^{2q+1} + \left(\frac{\frac{a+b}{2}-x}{b-a} \right)^{2q+1} \right]^{\frac{1}{q}} \\ \times (b-a)^{1+\frac{1}{q}} \|f''\|_{[a,b],p},$$

for any $x \in [a, \frac{a+b}{2}]$.

Remark 3. If in Theorem 2.4 we choose $x = a$, then we recapture the second part of the inequality (1.4), i.e.,

$$(2.7) \quad \left| \frac{1}{b-a} \int_a^b f(t) dt - \frac{1}{2} [f(a) + f(b)] + \frac{b-a}{8} [f'(b) - f'(a)] \right| \\ \leq \frac{1}{8} \cdot \frac{(b-a)^{1+\frac{1}{q}} \|f''\|_{[a,b],p}}{(2q+1)^{\frac{1}{q}}}.$$

The constant $\frac{1}{8}$ is best possible in the sense that it cannot be replaced by a smaller constant.

Proof. Indeed, if we assume that (2.7) holds with a constant $C > 0$, instead of $\frac{1}{8}$, i.e.,

$$(2.8) \quad \left| \frac{1}{b-a} \int_a^b f(t) dt - \frac{1}{2} [f(a) + f(b)] + \frac{b-a}{8} [f'(b) - f'(a)] \right| \\ \leq C \cdot \frac{(b-a)^{1+\frac{1}{q}} \|f''\|_{[a,b],p}}{(2q+1)^{\frac{1}{q}}},$$



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then for the function $f : [a, b] \rightarrow \mathbb{R}$, $f(x) = k \left(x - \frac{a+b}{2}\right)^2$, $k > 0$, we have

$$\begin{aligned}\frac{f(a) + f(b)}{2} &= k \cdot \frac{(b-a)^2}{4}, \\ f'(b) - f'(a) &= 2k(b-a), \\ \frac{1}{b-a} \int_a^b f(t) dt &= k \cdot \frac{(b-a)^2}{12}, \\ \|f''\|_{[a,b],p} &= 2k(b-a)^{\frac{1}{p}};\end{aligned}$$

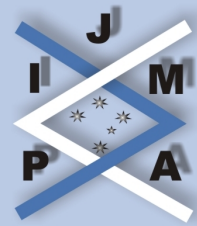
and by (2.8) we deduce

$$\left| \frac{k(b-a)^2}{12} - \frac{k(b-a)^2}{4} + \frac{k(b-a)^2}{4} \right| \leq \frac{2C \cdot k(b-a)^2}{(2q+1)^{\frac{1}{q}}},$$

giving $C \geq \frac{(2q+1)^{\frac{1}{q}}}{24}$. Letting $q \rightarrow 1+$, we deduce $C \geq \frac{1}{8}$, and the sharpness of the constant is proved. \square

Remark 4. If in Theorem 2.4 we choose $x = \frac{a+b}{2}$, then we get the midpoint inequality

$$\begin{aligned}(2.9) \quad & \left| \frac{1}{b-a} \int_a^b f(t) dt - f\left(\frac{a+b}{2}\right) \right| \\ & \leq \frac{1}{8} \cdot \frac{(b-a)^{1+\frac{1}{q}}}{2^{\frac{1}{q}}(2q+1)^{\frac{1}{q}}} \left[\|f''\|_{[a, \frac{a+b}{2}],p} + \|f''\|_{[\frac{a+b}{2}, b],p} \right] \\ & \leq \frac{1}{8} \cdot \frac{(b-a)^{1+\frac{1}{q}}}{(2q+1)^{\frac{1}{q}}} \|f''\|_{[a,b],p}, \quad p > 1, \frac{1}{p} + \frac{1}{q} = 1.\end{aligned}$$



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In both inequalities the constant $\frac{1}{8}$ is sharp in the sense that it cannot be replaced by a smaller constant.

To show this fact, assume that (2.9) holds with $C, D > 0$, i.e.,

$$\begin{aligned} (2.10) \quad & \left| \frac{1}{b-a} \int_a^b f(t) dt - f\left(\frac{a+b}{2}\right) \right| \\ & \leq C \cdot \frac{(b-a)^{1+\frac{1}{q}}}{2^{\frac{1}{q}}(2q+1)^{\frac{1}{q}}} \left[\|f''\|_{[a, \frac{a+b}{2}], p} + \|f''\|_{[\frac{a+b}{2}, b], p} \right] \\ & \leq D \cdot \frac{(b-a)^{1+\frac{1}{q}}}{(2q+1)^{\frac{1}{q}}} \|f''\|_{[a, b], p}. \end{aligned}$$

For the function $f : [a, b] \rightarrow \mathbb{R}$, $f(x) = k(x - \frac{a+b}{2})^2$, $k > 0$, we have

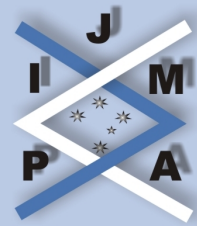
$$f\left(\frac{a+b}{2}\right) = 0, \quad \frac{1}{b-a} \int_a^b f(t) dt = \frac{k(b-a)^2}{12},$$

$$\begin{aligned} \|f''\|_{[a, \frac{a+b}{2}], p} + \|f''\|_{[\frac{a+b}{2}, b], p} &= 4k \left(\frac{b-a}{2}\right)^{\frac{1}{p}} \\ &= 2^{1+\frac{1}{q}} (b-a)^{\frac{1}{p}} k, \end{aligned}$$

$$\|f''\|_{[a, b], p} = 2(b-a)^{\frac{1}{p}} k;$$

and then by (2.10) we deduce

$$\frac{k(b-a)^2}{12} \leq C \cdot \frac{2k(b-a)^2}{(2q+1)^{\frac{1}{q}}} \leq D \cdot \frac{2k(b-a)^2}{(2q+1)^{\frac{1}{q}}},$$

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giving $C, D \geq \frac{(2q+1)^{\frac{1}{q}}}{24}$ for any $q > 1$. Letting $q \rightarrow 1+$, we deduce $C, D \geq \frac{1}{8}$ and the sharpness of the constants in (2.9) is proved.

The following result is useful in providing the best quadrature rule in the class for approximating the integral of a function $f : [a, b] \rightarrow \mathbb{R}$ whose first derivative is absolutely continuous on $[a, b]$ and whose second derivative is in $L_p[a, b]$.

Corollary 2.5. *With the assumptions in Theorem 2.4, one has the inequality*

$$(2.11) \quad \left| \frac{1}{2} \left[f \left(\frac{3a+b}{4} \right) + f \left(\frac{a+3b}{4} \right) \right] - \frac{1}{b-a} \int_a^b f(t) dt \right| \\ \leq \frac{1}{32} \cdot \frac{(b-a)^{1+\frac{1}{q}}}{(2q+1)^{\frac{1}{q}}} \|f''\|_{[a,b],p},$$

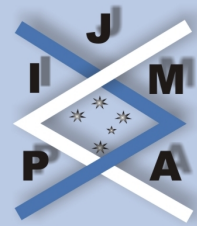
where $\frac{1}{p} + \frac{1}{q} = 1$.

The constant $\frac{1}{32}$ is the best possible in the sense that it cannot be replaced by a smaller constant.

Proof. The inequality follows by Theorem 2.4 and (2.6) on choosing $x = \frac{3a+b}{4}$.

To prove the sharpness of the constant, assume that (2.11) holds with a constant $E > 0$, i.e.,

$$(2.12) \quad \left| \frac{1}{2} \left[f \left(\frac{3a+b}{4} \right) + f \left(\frac{a+3b}{4} \right) \right] - \frac{1}{b-a} \int_a^b f(t) dt \right| \\ \leq E \cdot \frac{(b-a)^{1+\frac{1}{q}}}{(2q+1)^{\frac{1}{q}}} \|f''\|_{[a,b],p}.$$



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Consider the function $f : [a, b] \rightarrow \mathbb{R}$,

$$f(x) = \begin{cases} -\frac{1}{2} \left(x - \frac{3a+b}{4}\right)^2 & \text{if } x \in \left[a, \frac{3a+b}{4}\right], \\ \frac{1}{2} \left(x - \frac{3a+b}{4}\right)^2 & \text{if } x \in \left(\frac{3a+b}{4}, \frac{a+b}{2}\right], \\ -\frac{1}{2} \left(x - \frac{a+3b}{4}\right)^2 & \text{if } x \in \left(\frac{a+b}{2}, \frac{a+3b}{4}\right], \\ \frac{1}{2} \left(x - \frac{a+3b}{4}\right)^2 & \text{if } x \in \left(\frac{a+3b}{4}, b\right]. \end{cases}$$

We have

$$f'(x) = \begin{cases} \left|x - \frac{3a+b}{4}\right| & \text{if } x \in \left[a, \frac{a+b}{2}\right], \\ \left|x - \frac{a+3b}{4}\right| & \text{if } x \in \left(\frac{a+b}{2}, b\right]. \end{cases}$$

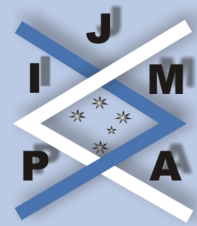
Then f' is absolutely continuous and $f'' \in L_p[a, b]$, $p > 1$. We also have

$$\begin{aligned} \frac{1}{2} \left[f\left(\frac{3a+b}{4}\right) + f\left(\frac{a+3b}{4}\right) \right] &= 0, \\ \frac{1}{b-a} \int_a^b f(t) dt &= \frac{(b-a)^2}{96}, \\ \|f''\|_{[a,b],p} &= (b-a)^{\frac{1}{p}}, \end{aligned}$$

and then, by (2.12), we obtain

$$\frac{(b-a)^2}{96} \leq E \cdot \frac{(b-a)^2}{(2q+1)^{\frac{1}{q}}},$$

giving $E \geq \frac{(2q+1)^{\frac{1}{q}}}{96}$ for any $q > 1$, i.e., $E \geq \frac{1}{32}$, and the corollary is proved. \square



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Theorem 2.6. Let $f : [a, b] \rightarrow \mathbb{R}$ be such that the derivative f' is absolutely continuous on $[a, b]$ and $f'' \in L_1[a, b]$. If $M(x)$ is as defined in (2.2), then we have the bounds:

$$(2.13) \quad M(x) \leq \frac{b-a}{2} \left[\left(\frac{x-a}{b-a} \right)^2 \|f''\|_{[a,x],1} + \left(\frac{\frac{a+b}{2}-x}{b-a} \right)^2 \|f''\|_{[x,a+b-x],1} + \left(\frac{x-a}{b-a} \right)^2 \|f''\|_{[a+b-x,b],1} \right]$$

$$\leq \begin{cases} \frac{b-a}{2} \left[2 \left(\frac{x-a}{b-a} \right)^2 + \left(\frac{\frac{a+b}{2}-x}{b-a} \right)^2 \right] \\ \times \max \left[\|f''\|_{[a,x],1}, \|f''\|_{[x,a+b-x],1}, \|f''\|_{[a+b-x,b],1} \right]; \\ \frac{b-a}{2} \left[2 \left(\frac{x-a}{b-a} \right)^{2\alpha} + \left(\frac{\frac{a+b}{2}-x}{b-a} \right)^{2\alpha} \right]^{\frac{1}{\alpha}} \\ \times \left[\|f''\|_{[a,x],1}^\beta + \|f''\|_{[x,a+b-x],1}^\beta + \|f''\|_{[a+b-x,b],1}^\beta \right]^{\frac{1}{\beta}} \\ \text{if } \alpha > 1, \frac{1}{\alpha} + \frac{1}{\beta} = 1; \\ \frac{b-a}{2} \left[\left| \frac{x-\frac{3a+b}{4}}{b-a} \right| + \frac{1}{4} \right]^2 \|f''\|_{[a,b],1}; \end{cases}$$

for any $x \in [a, \frac{a+b}{2}]$.

The proof is as in Theorem 2.2 and we need only to prove the third inequality of the last part as

$$M(x) \leq \frac{b-a}{2} \max \left\{ \left(\frac{x-a}{b-a} \right)^2, \left(\frac{\frac{a+b}{2}-x}{b-a} \right)^2 \right\} \\ \times \left[\|f''\|_{[a,x],1} + \|f''\|_{[x,a+b-x],1} + \|f''\|_{[a+b-x,b],1} \right]$$

$$= \frac{b-a}{2} \left[\left| \frac{x - \frac{3a+b}{4}}{b-a} \right| + \frac{1}{4} \right]^2 \|f''\|_{[a,b],1}.$$

Remark 5. By the use of Theorem 2.6, for $x = a$, we recapture the third part of the inequality (1.4), i.e.,

$$\left| \frac{1}{b-a} \int_a^b f(t) dt - \frac{1}{2} [f(a) + f(b)] + \frac{b-a}{8} [f'(b) - f'(a)] \right| \leq \frac{1}{8} (b-a) \|f''\|_{[a,b],1}.$$

If in (2.13) we choose $x = \frac{a+b}{2}$, then we get the mid-point inequality

$$\left| \frac{1}{b-a} \int_a^b f(t) dt - f\left(\frac{a+b}{2}\right) \right| \leq \frac{1}{8} (b-a) \|f''\|_{[a,b],1}.$$

Corollary 2.7. *With the assumptions in Theorem 2.6, one has the inequality*

$$\left| \frac{1}{b-a} \int_a^b f(t) dt - \frac{f\left(\frac{3a+b}{4}\right) + f\left(\frac{a+3b}{4}\right)}{2} \right| \leq \frac{1}{32} (b-a) \|f''\|_{[a,b],1}.$$



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3. A Composite Quadrature Formula

We use the following inequalities obtained in the previous section:

$$(3.1) \quad \left| \frac{1}{2} \left[f \left(\frac{3a+b}{4} \right) + f \left(\frac{a+3b}{4} \right) \right] - \frac{1}{b-a} \int_a^b f(t) dt \right| \leq \begin{cases} \frac{1}{96} (b-a)^2 \|f''\|_{[a,b],\infty} & \text{if } f'' \in L_\infty [a,b]; \\ \frac{1}{32} \cdot \frac{(b-a)^{1+\frac{1}{q}}}{(2q+1)^{\frac{1}{q}}} \|f''\|_{[a,b],p} & \text{if } f'' \in L_p [a,b], p > 1, \frac{1}{p} + \frac{1}{q} = 1; \\ \frac{1}{32} (b-a) \|f''\|_{[a,b],1} & \text{if } f'' \in L_1 [a,b]. \end{cases}$$

Let $I_n : a = x_0 < x_1 < \dots < x_{n-1} < x_n = b$ be a division of the interval $[a, b]$ and $h_i := x_{i+1} - x_i$ ($i = 0, \dots, n-1$) and $\nu(I_n) := \max \{h_i | i = 0, \dots, n-1\}$.

Consider the composite quadrature rule

$$(3.2) \quad Q_n(I_n, f) := \frac{1}{2} \sum_{i=0}^{n-1} \left[f \left(\frac{3x_i + x_{i+1}}{4} \right) + f \left(\frac{x_i + 3x_{i+1}}{4} \right) \right] h_i.$$

The following result holds.

Theorem 3.1. *Let $f : [a, b] \rightarrow \mathbb{R}$ be such that the derivative f' is absolutely continuous on $[a, b]$. Then we have*

$$\int_a^b f(t) dt = Q_n(I_n, f) + R_n(I_n, f),$$

where $Q_n(I_n, f)$ is defined by the formula (3.2), and the remainder satisfies the



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$$(3.3) \quad |R_n(I_n, f)| \leq \begin{cases} \frac{1}{96} \|f''\|_{[a,b],\infty} \sum_{i=0}^{n-1} h_i^3 & \text{if } f'' \in L_\infty[a, b]; \\ \frac{1}{32(2q+1)^{\frac{1}{q}}} \|f''\|_{[a,b],p} \left(\sum_{i=0}^{n-1} h_i^{2q+1} \right)^{\frac{1}{q}} & \text{if } f'' \in L_p[a, b], \\ & p > 1, \frac{1}{p} + \frac{1}{q} = 1; \\ \frac{1}{32} \|f''\|_{[a,b],1} [\nu(I_n)]^2 & \text{if } f'' \in L_1[a, b]. \end{cases}$$

Proof. Applying inequality (3.1) on the interval $[x_i, x_{i+1}]$, we may state that

$$(3.4) \quad \left| \int_{x_i}^{x_{i+1}} f(t) dt - \frac{1}{2} \left[f\left(\frac{3x_i + x_{i+1}}{4}\right) + f\left(\frac{x_i + 3x_{i+1}}{4}\right) \right] h_i \right| \leq \begin{cases} \frac{1}{96} h_i^3 \|f''\|_{[x_i, x_{i+1}], \infty}; \\ \frac{1}{32(2q+1)^{\frac{1}{q}}} h_i^{2+\frac{1}{q}} \|f''\|_{[x_i, x_{i+1}], p}, & p > 1, \frac{1}{p} + \frac{1}{q} = 1; \\ \frac{1}{32} h_i^2 \|f''\|_{[x_i, x_{i+1}], 1}; \end{cases}$$

for each $i \in \{0, \dots, n-1\}$.

Summing the inequality (3.4) over i from 0 to $n-1$ and using the generalized triangle inequality, we get

$$(3.5) \quad |R_n(I_n, f)| \leq \begin{cases} \frac{1}{96} \sum_{i=0}^{n-1} h_i^3 \|f''\|_{[x_i, x_{i+1}], \infty}; \\ \frac{1}{32(2q+1)^{\frac{1}{q}}} \sum_{i=0}^{n-1} h_i^{2+\frac{1}{q}} \|f''\|_{[x_i, x_{i+1}], p}, & p > 1, \frac{1}{p} + \frac{1}{q} = 1; \\ \frac{1}{32} \sum_{i=0}^{n-1} h_i^2 \|f''\|_{[x_i, x_{i+1}], 1}. \end{cases}$$



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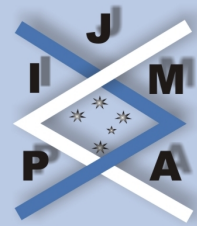
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Now, we observe that

$$\sum_{i=0}^{n-1} h_i^3 \|f''\|_{[x_i, x_{i+1}], \infty} \leq \|f''\|_{[a, b], \infty} \sum_{i=0}^{n-1} h_i^3.$$

Using Hölder's discrete inequality, we may write that

$$\begin{aligned} \sum_{i=0}^{n-1} h_i^{2+\frac{1}{q}} \|f''\|_{[x_i, x_{i+1}], p} &\leq \left(\sum_{i=0}^{n-1} h_i^{(2+\frac{1}{q})q} \right)^{\frac{1}{q}} \left(\sum_{i=0}^{n-1} \|f''\|_{[x_i, x_{i+1}], p}^p \right)^{\frac{1}{p}} \\ &= \left(\sum_{i=0}^{n-1} h_i^{2q+1} \right)^{\frac{1}{q}} \left(\sum_{i=0}^{n-1} \int_{x_i}^{x_{i+1}} |f''(t)|^p dt \right)^{\frac{1}{p}} \\ &= \left(\sum_{i=0}^{n-1} h_i^{2q+1} \right)^{\frac{1}{q}} \|f''\|_{[a, b], p}. \end{aligned}$$

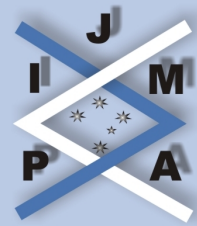
Also, we note that

$$\begin{aligned} \sum_{i=0}^{n-1} h_i^2 \|f''\|_{[x_i, x_{i+1}], 1} &\leq \max_{0 \leq i \leq n-1} \{h_i^2\} \sum_{i=0}^{n-1} \|f''\|_{[x_i, x_{i+1}], 1} \\ &= [\nu(I_n)]^2 \|f''\|_{[a, b], 1}. \end{aligned}$$

Consequently, by the use of (3.5), we deduce the desired result (3.3). □

For the particular case where the division I_n is equidistant, i.e.,

$$I_n := x_i = a + i \cdot \frac{b-a}{n}, \quad i = 0, \dots, n,$$



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we may consider the quadrature rule:

$$(3.6) \quad Q_n(f) := \frac{b-a}{2n} \sum_{i=0}^{n-1} \left\{ f \left[a + \left(\frac{4i+1}{4n} \right) (b-a) \right] \right. \\ \left. + f \left[a + \left(\frac{4i+3}{4n} \right) (b-a) \right] \right\}.$$

The following corollary will be more useful in practice.

Corollary 3.2. *With the assumption of Theorem 3.1, we have*

$$\int_a^b f(t) dt = Q_n(f) + R_n(f),$$

where $Q_n(f)$ is defined by (3.6) and the remainder $R_n(f)$ satisfies the estimate:

$$|R_n(I_n, f)| \leq \begin{cases} \frac{1}{96} \|f''\|_{[a,b],\infty} \frac{(b-a)^3}{n^2}; \\ \frac{1}{32(2q+1)^{\frac{1}{q}}} \|f''\|_{[a,b],p} \frac{(b-a)^{2+\frac{1}{q}}}{n^2}, & p > 1, \frac{1}{p} + \frac{1}{q} = 1; \\ \frac{1}{32} \|f''\|_{[a,b],1} \frac{(b-a)^2}{n^2}. \end{cases}$$

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