

ON SOME FENG QI TYPE q -INTEGRAL INEQUALITIES

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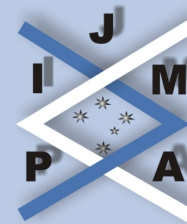
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Abstract: In this paper, we provide some Feng Qi type q -Integral Inequalities, by using analytic and elementary methods in Quantum Calculus.



**Feng Qi Type q -Integral
Inequalities**

Kamel Brahim, Néji Bettaibi
and Mouna Sellami

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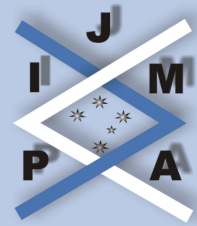
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1. Introduction

In [9], F. Qi studied an interesting integral inequality and proved the following result:

Theorem 1.1. For a positive integer n and an n^{th} order continuous derivative function f on an interval $[a, b]$ such that $f^{(i)}(a) \geq 0$, $0 \leq i \leq n - 1$ and $f^{(n)}(a) \geq n!$, we have

$$(1.1) \quad \int_a^b [f(t)]^{n+2} dt \geq \left[\int_a^b f(t) dt \right]^{n+1}.$$

Then, he proposed the following open problem:

Under what condition is the inequality (1.1) still true if n is replaced by any positive real number p ?

In view of the interest in this type of inequality, much attention has been paid to the problem and many authors have extended the inequality to more general cases (see [1, 8]). In this paper, we shall discuss a q -analogue of the Feng Qi problem and we will generalize the inequalities given in [1], [7] and [8].

This paper is organized as follows: In Section 2, we present definitions and facts from q -calculus necessary for understanding this paper. In Section 3, we discuss some generalizations of the so-called Feng Qi inequality.

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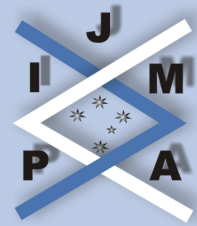
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2. Notations and Preliminaries

Throughout this paper, we will fix $q \in (0, 1)$. For the convenience of the reader, we provide a summary of the mathematical notations and definitions used in this paper (see [3] and [5]). We write for $a \in \mathbb{C}$,

$$[a]_q = \frac{1 - q^a}{1 - q}, \quad (a; q)_n = \prod_{k=0}^{n-1} (1 - aq^k), \quad n = 1, 2, \dots, \infty,$$

$$[0]_q! = 1, \quad [n]_q! = [1]_q [2]_q \dots [n]_q, \quad n = 1, 2, \dots$$

and

$$(x - a)_q^n = \begin{cases} 1 & \text{if } n = 0 \\ (x - a)(x - qa) \dots (x - q^{n-1}a) & \text{if } n \neq 0 \end{cases} \quad x \in \mathbb{C}, \quad n \in \mathbb{N}.$$

The q -derivative $D_q f$ of a function f is given by

$$(2.1) \quad (D_q f)(x) = \frac{f(x) - f(qx)}{(1 - q)x}, \quad \text{if } x \neq 0,$$

$(D_q f)(0) = f'(0)$ provided $f'(0)$ exists.

The q -Jackson integral from 0 to a is defined by (see [4])

$$(2.2) \quad \int_0^a f(x) d_q x = (1 - q)a \sum_{n=0}^{\infty} f(aq^n) q^n,$$

provided the sum converges absolutely.

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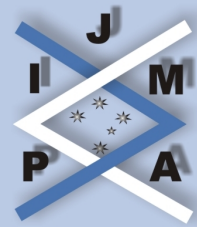
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The q -Jackson integral in a generic interval $[a, b]$ is given by (see [4])

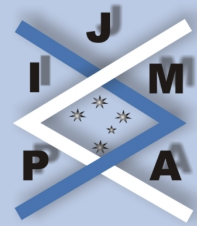
$$(2.3) \quad \int_a^b f(x) d_q x = \int_0^b f(x) d_q x - \int_0^a f(x) d_q x.$$

We recall that for any function f , we have (see [5])

$$(2.4) \quad D_q \left(\int_a^x f(t) d_q t \right) = f(x).$$

Finally, for $b > 0$ and $a = bq^n$, n a positive integer, we write

$$[a, b]_q = \{bq^k : 0 \leq k \leq n\} \quad \text{and} \quad (a, b]_q = [q^{-1}a, b]_q.$$



3. q -Integral Inequalities of Feng Qi type

Let us begin with the following useful result:

Lemma 3.1. *Let $p \geq 1$ be a real number and g be a nonnegative and monotone function on $[a, b]_q$. Then*

$$pg^{p-1}(qx)D_qg(x) \leq D_q[g(x)]^p \leq pg^{p-1}(x)D_qg(x), \quad x \in (a, b]_q.$$

Proof. We have

$$(3.1) \quad D_q[g^p](x) = \frac{g^p(x) - g^p(qx)}{(1-q)x} = \frac{1}{(1-q)x^p} \int_{g(qx)}^{g(x)} t^{p-1} dt.$$

Since g is a nonnegative and monotone function, we have

$$g^{p-1}(qx) [g(x) - g(qx)] \leq \int_{g(qx)}^{g(x)} t^{p-1} dt \leq g^{p-1}(x) [g(x) - g(qx)].$$

Therefore, according to the relation (3.1), we obtain

$$pg^{p-1}(qx)D_qg(x) \leq D_q[g^p](x) \leq pg^{p-1}(x)D_qg(x).$$

□

Proposition 3.2. *Let f be a function defined on $[a, b]_q$ satisfying*

$$f(a) \geq 0 \quad \text{and} \quad D_qf(x) \geq (t-2)(x-a)^{t-3} \quad \text{for} \quad x \in (a, b]_q \quad \text{and} \quad t \geq 3.$$

Then

$$\int_a^b [f(x)]^t d_qx \geq \left(\int_a^b f(qx) d_qx \right)^{t-1}.$$

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Proof. Put $g(x) = \int_a^x f(qu)d_q u$ and

$$F(x) = \int_a^x [f(u)]^t d_q u - \left(\int_a^x f(qu)d_q u \right)^{t-1}.$$

We have

$$D_q F(x) = f^t(x) - D_q[g^{t-1}](x).$$

Since f and g increase on $[a, b]_q$, we obtain from Lemma 3.1,

$$\begin{aligned} D_q F(x) &\geq f^t(x) - (t-1)g^{t-2}(x)f(qx) \\ &\geq f^t(x) - (t-1)g^{t-2}(x)f(x) = f(x)h(x), \end{aligned}$$

where $h(x) = f^{t-1}(x) - (t-1)g^{t-2}(x)$.

On the other hand, we have

$$D_q h(x) = D_q[f^{t-1}](x) - (t-1)D_q[g^{t-2}](x).$$

By using Lemma 3.1, we obtain

$$(3.2) \quad D_q h(x) \geq (t-1)f^{t-2}(qx)D_q f(x) - (t-1)(t-2)g^{t-3}(x)D_q g(x)$$

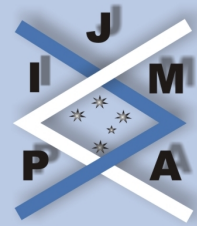
$$(3.3) \quad \geq (t-1)f(qx) [f^{t-3}(qx)D_q f(x) - (t-2)g^{t-3}(x)].$$

Since the function f increases, we have

$$\int_a^x f(qu)d_q u \leq f(qx)(x-a).$$

Then, from the conditions of the proposition and inequalities (3.2) and (3.3), we get

$$D_q h(x) \geq (t-1)f^{t-2}(qx) [D_q f(x) - (t-2)(x-a)^{t-3}] \geq 0,$$



and from the fact $h(a) = f^{t-1}(a) \geq 0$, we get $h(x) \geq 0$, $x \in [a, b]_q$.

From $F(a) = 0$ and $D_q F(x) = f(x)h(x) \geq 0$, it follows that $F(x) \geq 0$ for all $x \in [a, b]_q$, in particular

$$F(b) = \int_a^b [f(u)]^t d_q u - \left(\int_a^b f(qu) d_q u \right)^{t-1} \geq 0.$$

□

Corollary 3.3. Let n be a positive integer and f be a function defined on $[a, b]_q$ satisfying

$$f(a) \geq 0 \quad \text{and} \quad D_q f(x) \geq n(x-a)^{n-1}, \quad x \in (a, b]_q.$$

Then,

$$\int_a^b (f(x))^{n+2} d_q x \geq \left(\int_a^b f(qx) d_q x \right)^{n+1}.$$

Proof. It suffices to take $t = n + 2$ in Proposition 3.2 and the result follows. □

Corollary 3.4. Let n be a positive integer and f be a function defined on $[a, b]_q$ satisfying

$$D_q^i f(a) \geq 0, \quad 0 \leq i \leq n-1 \quad \text{and} \quad D_q^n f(x) \geq n[n-1]_q! \quad x \in (a, b]_q.$$

Then,

$$\int_a^b (f(x))^{n+2} d_q x \geq \left(\int_a^b f(qx) d_q x \right)^{n+1}.$$

Proof. Since $D_q^n f(x) \geq n[n-1]_q!$, then by q -integrating $n-1$ times over $[a, x]$, we get

$$D_q f(x) \geq n(x-a)_q^{n-1} \geq n(x-a)^{n-1}.$$

The result follows from Corollary 3.3. □

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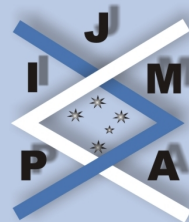
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Proposition 3.5. Let $p \geq 1$ be a real number and f be a function defined on $[a, b]_q$ satisfying

$$(3.4) \quad f(a) \geq 0, \quad D_q f(x) \geq p, \quad \forall x \in (a, b]_q.$$

Then we have

$$(3.5) \quad \int_a^b [f(x)]^{p+2} d_q x \geq \frac{1}{(b-a)^{p-1}} \left[\int_a^b f(qx) d_q x \right]^{p+1}.$$

Proof. Put $g(t) = \int_a^t f(qx) d_q x$ and

$$(3.6) \quad H(t) = \int_a^t [f(x)]^{p+2} d_q x - \frac{1}{(b-a)^{p-1}} \left[\int_a^t f(qx) d_q x \right]^{p+1}, \quad t \in [a, b]_q.$$

We have

$$D_q H(t) = [f(t)]^{p+2} - \frac{1}{(b-a)^{p-1}} D_q [g^{p+1}](t), \quad t \in (a, b]_q.$$

Since f and g increase on $[a, b]_q$, we obtain, according to Lemma 3.1, for $t \in (a, b]_q$,

$$\begin{aligned} D_q H(t) &\geq [f(t)]^{p+2} - \frac{1}{(b-a)^{p-1}} (p+1) g^p(t) f(qt) \\ &\geq [f(t)]^{p+2} - \frac{1}{(b-a)^{p-1}} (p+1) g^p(t) f(t) \\ &\geq \left([f(t)]^{p+1} - \frac{1}{(b-a)^{p-1}} (p+1) g^p(t) \right) f(t) = h(t) f(t), \end{aligned}$$

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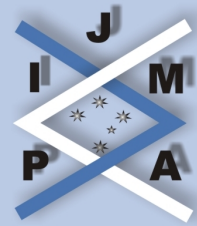
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where

$$h(t) = [f(t)]^{p+1} - \frac{1}{(b-a)^{p-1}}(p+1)g^p(t).$$

On the other hand, we have

$$D_q h(t) = D_q [f^{p+1}](t) - \frac{1}{(b-a)^{p-1}}(p+1)D_q [g^p](t).$$

By using Lemma 3.1, we obtain

$$\begin{aligned} D_q h(t) &\geq (p+1)f^p(qt)D_q f(t) - \frac{(p+1)p}{(b-a)^{p-1}}g^{p-1}(t)f(qt) \\ &\geq (p+1)f(qt) \left[f^{p-1}(qt)D_q f(t) - \frac{p}{(b-a)^{p-1}}g^{p-1}(t) \right]. \end{aligned}$$

Since f increases, then for $t \in [a, b]_q$,

$$(3.7) \quad \int_a^t f(qx)d_q x \leq (b-a)f(qt),$$

therefore,

$$(3.8) \quad D_q h(t) \geq (p+1)f^p(qt)[D_q f(t) - p].$$

We deduce, from the relation (3.4), that h increases on $[a, b]_q$.

Finally, since $h(a) = f^{p+1}(a) \geq 0$, then H increases and $H(b) \geq H(a) \geq 0$, which completes the proof. \square

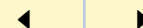
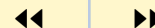
Corollary 3.6. *Let $p \geq 1$ be a real number and f be a nonnegative function on $[0, 1]$ such that $D_q f(x) \geq 1$. Then*

$$(3.9) \quad \int_0^1 [f(x)]^{p+2} d_q x \geq \frac{1}{p} \left[\int_0^1 f(qx) d_q x \right]^{p+1}.$$



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Proof. Replacing, in the previous proposition, $f(x)$ by $pf(x)$, b by 1 and a by q^N ($N = 1, 2, \dots$), we obtain then the result by tending N to ∞ . \square

In what follows, we will adopt the terminology of the following definition.

Definition 3.7. Let $b > 0$ and $a = bq^n$, where n is a positive integer. For each real number r , we denote by $E_{q,r}([a, b])$ the set of functions defined on $[a, b]_q$ such that

$$f(a) \geq 0 \quad \text{and} \quad D_q f(x) \geq [r]_q, \quad \forall x \in (a, b]_q.$$

Proposition 3.8. Let $f \in E_{q,2}([a, b])$. Then for all $p > 0$, we have

$$(3.10) \quad \int_a^b [f(x)]^{2p+1} d_q x > \left[\int_a^b (f(x))^p d_q x \right]^2.$$

Proof. For $t \in [a, b]_q$, we put

$$F(t) = \int_a^t [f(x)]^{2p+1} d_q x - \left[\int_a^t (f(x))^p d_q x \right]^2 \quad \text{and} \quad g(t) = \int_a^t [f(x)]^p d_q x.$$

Then, we have for $t \in [a, b]_q$,

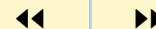
$$\begin{aligned} D_q F(t) &= [f(t)]^{2p+1} - [f(t)]^p (g(t) + g(qt)) \\ &= [f(t)]^p ([f(t)]^{p+1} - [g(t) + g(qt)]) \\ &= [f(t)]^p G(t), \end{aligned}$$

where $G(t) = [f(t)]^{p+1} - [g(t) + g(qt)]$.



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On the other hand, we have

$$\begin{aligned} D_q G(t) &= \frac{f^{p+1}(t) - f^{p+1}(qt)}{(1-q)t} - f^p(t) - qf^p(qt) \\ &= f^p(t) \frac{f(t) - (1-q)t}{(1-q)t} - f^p(qt) \frac{f(qt) + q(1-q)t}{(1-q)t}. \end{aligned}$$

By using the relation $D_q f(t) \geq [2]_q$, we obtain $f(t) \geq f(qt) + (1-q^2)t$, therefore

$$(3.11) \quad D_q G(t) \geq \frac{f^p(t) - f^p(qt)}{(1-q)t} [f(qt) + q(1-q)t] > 0, \quad t \in (a, b]_q.$$

Hence, G is strictly increasing on $[a, b]_q$. Moreover, we have

$$G(a) = [f(a)]^{p+1} + (1-q)af(a) \geq 0,$$

for all $t \in (a, b]_q$, $G(t) > G(a) \geq 0$, which proves that $D_q F(t) > 0$, for all $t \in (a, b]_q$. Thus, F is strictly increasing on $[a, b]_q$. In particular, $F(b) > F(a) = 0$. \square

Corollary 3.9. Let $\alpha > 0$ and $f \in E_{q,2}([a, b])$. Then for all positive integers m , we have

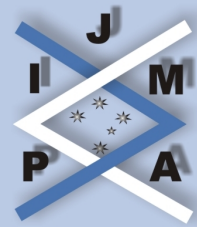
$$(3.12) \quad \int_a^b [f(x)]^{(\alpha+1)2^m-1} d_q x > \left[\int_a^b [f(x)]^\alpha d_q x \right]^{2^m}.$$

Proof. We suggest here a proof by induction. For this purpose, we put:

$$p_m(\alpha) = (\alpha + 1)2^m - 1.$$

We have

$$(3.13) \quad p_m(\alpha) > 0 \quad \text{and} \quad p_{m+1}(\alpha) = 2p_m(\alpha) + 1.$$



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From Proposition 3.8, we deduce that the inequality (3.12) is true for $m = 1$.
 Suppose that (3.12) holds for an integer m and let us prove it for $m + 1$.
 By using the relation (3.13) and Proposition 3.8, we obtain

$$(3.14) \quad \int_a^b [f(x)]^{(\alpha+1)2^{m+1}-1} d_q x > \left[\int_a^b [f(x)]^{(\alpha+1)2^m-1} d_q x \right]^2.$$

And, by assumption, we have

$$(3.15) \quad \int_a^b [f(x)]^{(\alpha+1)2^m-1} > \left[\int_a^b [f(x)]^\alpha d_q x \right]^{2^m}.$$

Finally, the relations (3.14) and (3.15) imply that the inequality (3.12) is true for $m + 1$. This completes the proof. \square

Corollary 3.10. *Let $f \in E_{q,2}([a, b])$ and $\alpha > 0$. For $m \in \mathbb{N}$, we have*

$$(3.16) \quad \left[\int_a^b [f(x)]^{(\alpha+1)2^{m+1}-1} d_q x \right]^{\frac{1}{2^{m+1}}} > \left[\int_a^b [f(x)]^{(\alpha+1)2^m-1} d_q x \right]^{\frac{1}{2^m}}.$$

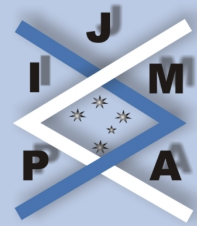
Proof. Since, from Proposition 3.8,

$$(3.17) \quad \int_a^b [f(x)]^{(\alpha+1)2^{m+1}-1} d_q x > \left[\int_a^b [f(x)]^{(\alpha+1)2^m-1} d_q x \right]^2,$$

then

$$(3.18) \quad \left[\int_a^b [f(x)]^{(\alpha+1)2^{m+1}-1} d_q x \right]^{\frac{1}{2^{m+1}}} > \left[\int_a^b [f(x)]^{(\alpha+1)2^m-1} d_q x \right]^{\frac{1}{2^m}}.$$

\square



Corollary 3.11. Let $f \in E_{q,2}([a, b])$. For all integers $m \geq 2$, we have

$$(3.19) \quad \int_a^b [f(x)]^{2^{m+1}-1} d_q x > \left[\int_a^b [f(x)]^3 d_q x \right]^{2^{m-1}}$$

$$(3.20) \quad > \left[\int_a^b f(x) d_q x \right]^{2^m}.$$

Proof. By using Proposition 3.8 and the two previous corollaries for $\alpha = 1$, we obtain the required result. \square

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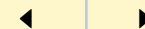
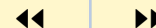
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