NOTE ON AN OPEN PROBLEM

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Received: 23 August, 2007

Accepted: 13 March, 2008

Communicated by: F. Qi

2000 AMS Sub. Class.: 26D15.

Key words: Integral inequality.

Abstract: In this paper we give an affirmative answer to an open problem proposed by Quôc Anh Ngô, Du Duc Thang, Tran Tat Dat, and Dang Anh Tuan [6].

Acknowledgement: I am grateful to the referee for his comments, especially for Theorem 2.3.

journal of inequalities in pure and applied mathematics
issn: 1443-5756

Contents

journal of inequalities in pure and applied mathematics

issn: 1443-5756

1. Introduction

In [6] the authors proved some integral inequalities and proposed the following question:

Let f be a continuous function on $[0,1]$ satisfying

$$\int_x^1 f(t)dt \geq \frac{1-x^2}{2}, \qquad (0 \leq x \leq 1). \tag{1.1}$$

Under what conditions does the inequality

$$\int_0^1 f^{\alpha+\beta}(x)dx \geq \int_0^1 x^\alpha f^\beta(x)dx$$

hold for α, β?

In [1] the author has given an answer to this open problem, but there is a clear gap in the proof of Lemma 1.1, so that the other results of the paper break down too. In this paper we give an affirmative answer to this problem by presenting stronger results. First we prove the following two essential lemmas.

Throughout this paper, we always assume that f is a non-negative continuous function on $[0,1]$, satisfying (1.1).

Lemma 1.1. *If (1.1) holds, then for each $x \in [0,1]$ we have*

$$\int_x^1 t^k f(t)dt \geq \frac{1-x^{k+2}}{k+2} \quad (k \in \mathbb{N}).$$

journal of inequalities in pure and applied mathematics
issn: 1443-5756

Proof. By our assumptions, we have

$$\int_x^1 y^{k-1}\left(\int_y^1 f(t)dt\right)dy \geq \int_x^1 y^{k-1}\frac{1-y^2}{2}dy$$
$$= \frac{1}{2}\int_x^1 (y^{k-1}-y^{k+1})dy$$
$$= \frac{1}{k(k+2)} - \frac{1}{2k}x^k + \frac{1}{2(k+2)}x^{k+2}.$$

On the other hand, integrating by parts, we also obtain

$$\int_x^1 y^{k-1}\left(\int_y^1 f(t)dt\right)dy = \frac{1}{k}y^k\int_y^1 f(t)dt\bigg|_x^1 + \frac{1}{k}\int_x^1 y^k f(y)dy$$
$$= -\frac{1}{k}x^k\int_x^1 f(t)dt + \frac{1}{k}\int_x^1 y^k f(y)dy.$$

Thus

$$-\frac{1}{k}x^k\int_x^1 f(t)dt + \frac{1}{k}\int_x^1 y^k f(y)dy \geq \frac{1}{k(k+2)} - \frac{1}{2k}x^k + \frac{1}{2(k+2)}x^{k+2}$$

$$\Longrightarrow \int_x^1 y^k f(y)dy \geq x^k\int_x^1 f(t)dt + \frac{1}{k+2} - \frac{1}{2}x^k + \frac{k}{2(k+2)}x^{k+2}$$
$$\geq x^k\left(\frac{1}{2}-\frac{1}{2}x^2\right) + \frac{1}{k+2} - \frac{1}{2}x^k + \frac{k}{2(k+2)}x^{k+2}$$
$$= \frac{1-x^{k+2}}{k+2}.$$

□

journal of inequalities in pure and applied mathematics
issn: 1443-5756

Remark 1. By a similar argument, we can show that Lemma 1.1 also holds when k is a real number in $[1,\infty)$. That is

$$\int_x^1 t^\alpha f(t)dt \geq \frac{1-x^{\alpha+2}}{\alpha+2} \quad (\forall \alpha \geq 1).$$

It is also interesting to note that the result of [5, Lemma 1.3] holds if we take $x=0$ in Lemma 1.1.

Lemma 1.2. *Let f be a non-negative continuous function on $[0,1]$ such that $\int_x^1 f(t)dt \geq \frac{1-x^2}{2}$ $(0 \leq x \leq 1)$. Then for each $x \in [0,1]$ and $k \in \mathbb{N}$, we have*

$$\int_x^1 f^k(t)dt \geq \frac{1-x^{k+1}}{k+1}.$$

Proof. Since

$$\begin{aligned} 0 &\leq \int_x^1 (f(t)-t)(f^k(t)-t^k)dt \\ &= \int_x^1 f^{k+1}(t)dt - \int_x^1 t^k f(t)dt - \int_0^1 t f^k(t)dt + \int_x^1 t^{k+1}dt \end{aligned}$$

it follows that

$$\int_x^1 f^{k+1}(t)dt \geq \int_x^1 t^k f(t)dt + \int_x^1 t f^k(t)dt - \frac{1}{k+2}(1-x^{k+2}).$$

By using Lemma 1.1, we get

$$\int_x^1 f^{k+1}(t)dt \geq \int_x^1 t f^k(t)dt. \tag{1.2}$$

journal of inequalities in pure and applied mathematics
issn: 1443-5756

We continue the proof by mathematical induction. The assertion is obvious for $k = 1$. Let $\int_x^1 f^k(t)dt \geq \frac{1-x^{k+1}}{k+1}$, we show that $\int_x^1 f^{k+1}(t)dt \geq \frac{1-x^{k+2}}{k+2}$. We have

$$\begin{aligned}\int_x^1 \left(\int_y^1 f^k(t)dt\right) dy &\geq \int_x^1 \frac{1-y^{k+1}}{k+1}dy \\ &= \frac{1}{k+1}\left(y - \frac{1}{k+2}y^{k+2}\right)\Big|_x^1 \\ &= \frac{1}{k+2} - \frac{1}{k+1}x + \frac{1}{(k+1)(k+2)}x^{k+2}.\end{aligned}$$

On the other hand, integrating by parts, we also obtain

$$\begin{aligned}\int_x^1 \left(\int_y^1 f^k(t)dt\right) dy &= y\int_y^1 f^k(t)dt\Big|_x^1 + \int_x^1 yf^k(y)dy \\ &= -x\int_x^1 f^k(t)dt + \int_x^1 yf^k(y)dy.\end{aligned}$$

Thus

$$-x\int_x^1 f^k(t)dt + \int_x^1 yf^k(y)dy \geq \frac{1}{k+2} - \frac{1}{k+1}x + \frac{1}{(k+1)(k+2)}x^{k+2}$$

and hence

$$\begin{aligned}\int_x^1 yf^k(y)dy &\geq x\int_x^1 f^k(t)dt + \frac{1}{k+2} - \frac{1}{k+1}x + \frac{1}{(k+1)(k+2)}x^{k+2} \\ &\geq x\frac{1-x^{k+1}}{k+1} + \frac{1}{k+2} - \frac{1}{k+1}x + \frac{1}{(k+1)(k+2)}x^{k+2} \\ &= \frac{1-x^{k+2}}{k+2}.\end{aligned}$$

journal of inequalities in pure and applied mathematics
issn: 1443-5756

So by (1.2) we get

$$\int_x^1 f^{k+1}(t)dt \geq \int_x^1 tf^k(t)dt \geq \frac{1-x^{k+2}}{k+2},$$

which completes the proof. □

Note On An Open Problem
Gholamreza Zabandan
vol. 9, iss. 2, art. 37, 2008

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2. Main Results

Theorem 2.1. *Let f be a non-negative and continuous function on $[0,1]$. If $\int_x^1 f(t)dt \geq \frac{1-x^2}{2}$ $(0 \leq x \leq 1)$, then for each $m, n \in \mathbb{N}$,*

$$\int_0^1 f^{m+n}(x)dx \geq \int_0^1 x^m f^n(x)dx.$$

Proof. By using the general Cauchy inequality [5, Theorem 3.1], we have

$$\frac{n}{m+n} f^{m+n}(x) + \frac{m}{m+n} x^{m+n} \geq x^m f^n(x),$$

which implies

$$\frac{n}{m+n} \int_0^1 f^{m+n}(x)dx + \frac{m}{m+n} \int_0^1 x^{m+n}dx \geq \int_0^1 x^m f^n(x)dx.$$

Hence

$$\begin{aligned} \int_0^1 f^{m+n}(x)dx &\geq \int_0^1 x^m f^n(x)dx + \frac{m}{m+n} \int_0^1 f^{m+n}(x)dx - \frac{m}{(m+n)(m+n+1)} \\ &= \int_0^1 x^m f^n(x)dx + \frac{m}{m+n} \left(\int_0^1 f^{m+n}(x)dx - \frac{1}{m+n+1} \right). \end{aligned}$$

By Lemma 1.2, we have $\int_0^1 f^{m+n}(x)dx \geq \frac{1}{m+n+1}$. Therefore

$$\int_0^1 f^{m+n}(x)dx \geq \int_0^1 x^m f^n(x)dx.$$

□

Note On An Open Problem
Gholamreza Zabandan
vol. 9, iss. 2, art. 37, 2008

journal of inequalities in pure and applied mathematics
issn: 1443-5756

Theorem 2.2. *Let f be a continuous function such that $f(x) \geq 1$ $(0 \leq x \leq 1)$. If $\int_x^1 f(t)dt \geq \frac{1-x^2}{2}$, then for each $\alpha, \beta > 0$,*

$$\int_0^1 f^{\alpha+\beta}(x)dx \geq \int_0^1 x^\alpha f^\beta(x)dx. \tag{2.1}$$

Proof. By a similar method to that used in the proof of Theorem 2.1 the inequality (2.1) holds if $\int_0^1 f^{\alpha+\beta}(x)dx \geq \frac{1}{\alpha+\beta+1}$. So it is enough to prove that $\int_0^1 f^\gamma(x)dx \geq \frac{1}{\gamma+1}$ $(\gamma > 0)$. Since $f(x) \geq 1$ $(0 \leq x \leq 1)$ and $[\gamma] \leq \gamma < [\gamma]+1$, we have

$$\int_0^1 f^\gamma(x)dx > \int_0^1 f^{[\gamma]}(x)dx.$$

By Lemma 1.2 we obtain

$$\int_0^1 f^\gamma(x)dx \geq \int_0^1 f^{[\gamma]}(x)dx \geq \frac{1}{[\gamma]+1} \geq \frac{1}{\gamma+1}.$$

□

Remark 2. The condition $f(x) \geq 1$ $(0 \leq x \leq 1)$ in Theorem 2.2 is necessary for $\int_0^1 f^\gamma(x)dx \geq \frac{1}{\gamma+1}$ $(\gamma > 0)$. For example, let

$$f(x) = \begin{cases} 0 & 0 \leq x \leq \frac{1}{2} \\ 2(2x-1) & \frac{1}{2} < x \leq 1 \end{cases}$$

and $\gamma = \frac{1}{2}$, then f is continuous on $[0,1]$ and $\int_x^1 f(t)dt \geq \frac{1-x^2}{2}$, but $\int_0^1 f^{\frac{1}{2}}(x)dx = \frac{\sqrt{2}}{3} < \frac{2}{3}$.

journal of inequalities in pure and applied mathematics
issn: 1443-5756

In the following theorem, we show that the condition $f(x) \geq 1$ $(0 \leq x \leq 1)$ in Theorem 2.2 can be removed if we assume that $\alpha + \beta \geq 1$.

Theorem 2.3. *Let f be a non-negative continuous function on $[0,1]$. If $\int_x^1 f(t)dt \geq \frac{1-x^2}{2}$ $(0 \leq x \leq 1)$, then for each $\alpha, \beta > 0$ such that $\alpha + \beta \geq 1$, we have*

$$\int_0^1 f^{\alpha+\beta}(x)dx \geq \frac{1}{\alpha+\beta+1}.$$

Proof. By using Theorem A of [5] for $g(t) = t$, $\alpha = 1$, $a = 0$, and $b = 1$, the assertion is obvious. $\square$

journal of inequalities in pure and applied mathematics
issn: 1443-5756

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journal of inequalities in pure and applied mathematics
issn: 1443-5756