

# NOTE ON AN OPEN PROBLEM

**GHOLAMREZA ZABANDAN**

Department of Mathematics  
Faculty of Mathematical Science and Computer Engineering  
Teacher Training University  
599 Taleghani Avenue  
Tehran 15618, IRAN  
EMail: [Zabandan@saba.tmu.ac.ir](mailto:Zabandan@saba.tmu.ac.ir)

*Received:* 23 August, 2007

*Accepted:* 13 March, 2008

*Communicated by:* **F. Qi**

*2000 AMS Sub. Class.:* 26D15.

*Key words:* Integral inequality.

*Abstract:* In this paper we give an affirmative answer to an open problem proposed by Quốc Anh Ngô, Du Duc Thang, Tran Tat Dat, and Dang Anh Tuan [6].

*Acknowledgement:* I am grateful to the referee for his comments, especially for Theorem 2.3.



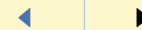
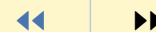
---

Note On An Open Problem  
Gholamreza Zabandan  
vol. 9, iss. 2, art. 37, 2008

---

[Title Page](#)

[Contents](#)



Page 1 of 11

[Go Back](#)

[Full Screen](#)

[Close](#)

journal of **inequalities**  
in pure and applied  
mathematics

issn: 1443-5756

# Contents

<a href="#">1 Introduction</a>	<a href="#">3</a>
<a href="#">2 Main Results</a>	<a href="#">8</a>



---

## Note On An Open Problem

Gholamreza Zabandan

vol. 9, iss. 2, art. 37, 2008

---

[Title Page](#)

[Contents](#)



Page **2** of 11

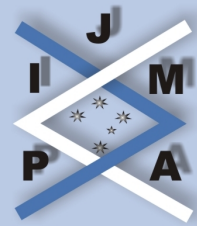
[Go Back](#)

[Full Screen](#)

[Close](#)

journal of **inequalities**  
in pure and applied  
mathematics

issn: 1443-5756



## 1. Introduction

In [6] the authors proved some integral inequalities and proposed the following question:

Let  $f$  be a continuous function on  $[0, 1]$  satisfying

$$(1.1) \quad \int_x^1 f(t)dt \geq \frac{1-x^2}{2}, \quad (0 \leq x \leq 1).$$

Under what conditions does the inequality

$$\int_0^1 f^{\alpha+\beta}(x)dx \geq \int_0^1 x^\alpha f^\beta(x)dx$$

hold for  $\alpha, \beta$ ?

In [1] the author has given an answer to this open problem, but there is a clear gap in the proof of Lemma 1.1, so that the other results of the paper break down too. In this paper we give an affirmative answer to this problem by presenting stronger results. First we prove the following two essential lemmas.

Throughout this paper, we always assume that  $f$  is a non-negative continuous function on  $[0, 1]$ , satisfying (1.1).

**Lemma 1.1.** *If (1.1) holds, then for each  $x \in [0, 1]$  we have*

$$\int_x^1 t^k f(t)dt \geq \frac{1-x^{k+2}}{k+2} \quad (k \in \mathbb{N}).$$

Note On An Open Problem

Gholamreza Zabandan

vol. 9, iss. 2, art. 37, 2008

Title Page

Contents



Page 3 of 11

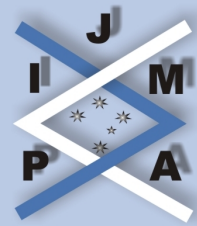
Go Back

Full Screen

Close

journal of **inequalities**  
in pure and applied  
mathematics

issn: 1443-5756



Note On An Open Problem

Gholamreza Zabandan

vol. 9, iss. 2, art. 37, 2008

Title Page

Contents



Page 4 of 11

Go Back

Full Screen

Close

*Proof.* By our assumptions, we have

$$\begin{aligned}\int_x^1 y^{k-1} \left( \int_y^1 f(t) dt \right) dy &\geq \int_x^1 y^{k-1} \frac{1-y^2}{2} dy \\ &= \frac{1}{2} \int_x^1 (y^{k-1} - y^{k+1}) dy \\ &= \frac{1}{k(k+2)} - \frac{1}{2k} x^k + \frac{1}{2(k+2)} x^{k+2}.\end{aligned}$$

On the other hand, integrating by parts, we also obtain

$$\begin{aligned}\int_x^1 y^{k-1} \left( \int_y^1 f(t) dt \right) dy &= \frac{1}{k} y^k \int_y^1 f(t) dt \Big|_x^1 + \frac{1}{k} \int_x^1 y^k f(y) dy \\ &= -\frac{1}{k} x^k \int_x^1 f(t) dt + \frac{1}{k} \int_x^1 y^k f(y) dy.\end{aligned}$$

Thus

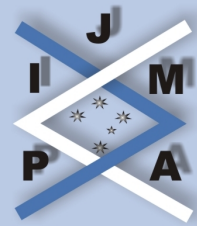
$$\begin{aligned}-\frac{1}{k} x^k \int_x^1 f(t) dt + \frac{1}{k} \int_x^1 y^k f(y) dy &\geq \frac{1}{k(k+2)} - \frac{1}{2k} x^k + \frac{1}{2(k+2)} x^{k+2} \\ \implies \int_x^1 y^k f(y) dy &\geq x^k \int_x^1 f(t) dt + \frac{1}{k+2} - \frac{1}{2} x^k + \frac{k}{2(k+2)} x^{k+2} \\ &\geq x^k \left( \frac{1}{2} - \frac{1}{2} x^2 \right) + \frac{1}{k+2} - \frac{1}{2} x^k + \frac{k}{2(k+2)} x^{k+2} \\ &= \frac{1 - x^{k+2}}{k+2}.\end{aligned}$$



journal of **inequalities**  
in pure and applied  
mathematics

issn: 1443-575b

© 2007 Victoria University. All rights reserved.



*Remark 1.* By a similar argument, we can show that Lemma 1.1 also holds when  $k$  is a real number in  $[1, \infty)$ . That is

$$\int_x^1 t^\alpha f(t) dt \geq \frac{1 - x^{\alpha+2}}{\alpha + 2} \quad (\forall \alpha \geq 1).$$

It is also interesting to note that the result of [5, Lemma 1.3] holds if we take  $x = 0$  in Lemma 1.1.

**Lemma 1.2.** *Let  $f$  be a non-negative continuous function on  $[0, 1]$  such that  $\int_x^1 f(t) dt \geq \frac{1-x^2}{2}$  ( $0 \leq x \leq 1$ ). Then for each  $x \in [0, 1]$  and  $k \in \mathbb{N}$ , we have*

$$\int_x^1 f^k(t) dt \geq \frac{1 - x^{k+1}}{k + 1}.$$

*Proof.* Since

$$\begin{aligned} 0 &\leq \int_x^1 (f(t) - t)(f^k(t) - t^k) dt \\ &= \int_x^1 f^{k+1}(t) dt - \int_x^1 t^k f(t) dt - \int_0^1 t f^k(t) dt + \int_x^1 t^{k+1} dt \end{aligned}$$

it follows that

$$\int_x^1 f^{k+1}(t) dt \geq \int_x^1 t^k f(t) dt + \int_x^1 t f^k(t) dt - \frac{1}{k+2}(1 - x^{k+2}).$$

By using Lemma 1.1, we get

$$(1.2) \quad \int_x^1 f^{k+1}(t) dt \geq \int_x^1 t f^k(t) dt.$$

Note On An Open Problem

Gholamreza Zabandan

vol. 9, iss. 2, art. 37, 2008

Title Page

Contents

◀▶

◀▶

Page 5 of 11

Go Back

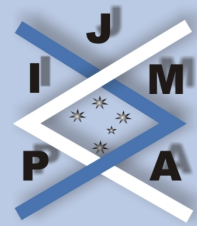
Full Screen

Close

journal of **inequalities**  
in pure and applied  
mathematics

issn: 1443-5756

© 2007 Victoria University. All rights reserved.



Note On An Open Problem

Gholamreza Zabandan

vol. 9, iss. 2, art. 37, 2008

Title Page

Contents



Page 6 of 11

Go Back

Full Screen

Close

We continue the proof by mathematical induction. The assertion is obvious for  $k =$

1. Let  $\int_x^1 f^k(t)dt \geq \frac{1-x^{k+1}}{k+1}$ , we show that  $\int_x^1 f^{k+1}(t)dt \geq \frac{1-x^{k+2}}{k+2}$ . We have

$$\begin{aligned} \int_x^1 \left( \int_y^1 f^k(t)dt \right) dy &\geq \int_x^1 \frac{1-y^{k+1}}{k+1} dy \\ &= \frac{1}{k+1} \left( y - \frac{1}{k+2} y^{k+2} \right) \Big|_x^1 \\ &= \frac{1}{k+2} - \frac{1}{k+1} x + \frac{1}{(k+1)(k+2)} x^{k+2}. \end{aligned}$$

On the other hand, integrating by parts, we also obtain

$$\begin{aligned} \int_x^1 \left( \int_y^1 f^k(t)dt \right) dy &= y \int_y^1 f^k(t)dt \Big|_x^1 + \int_x^1 y f^k(y) dy \\ &= -x \int_x^1 f^k(t)dt + \int_x^1 y f^k(y) dy. \end{aligned}$$

Thus

$$-x \int_x^1 f^k(t)dt + \int_x^1 y f^k(y) dy \geq \frac{1}{k+2} - \frac{1}{k+1} x + \frac{1}{(k+1)(k+2)} x^{k+2}$$

and hence

$$\begin{aligned} \int_x^1 y f^k(y) dy &\geq x \int_x^1 f^k(t)dt + \frac{1}{k+2} - \frac{1}{k+1} x + \frac{1}{(k+1)(k+2)} x^{k+2} \\ &\geq x \frac{1-x^{k+1}}{k+1} + \frac{1}{k+2} - \frac{1}{k+1} x + \frac{1}{(k+1)(k+2)} x^{k+2} \\ &= \frac{1-x^{k+2}}{k+2}. \end{aligned}$$

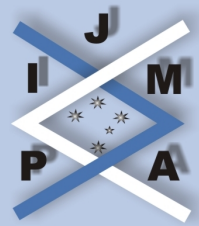
journal of **inequalities**  
in pure and applied  
mathematics

issn: 1443-575b

So by (1.2) we get

$$\int_x^1 f^{k+1}(t)dt \geq \int_x^1 t f^k(t)dt \geq \frac{1-x^{k+2}}{k+2},$$

which completes the proof. □



---

**Note On An Open Problem**

Gholamreza Zabandan

vol. 9, iss. 2, art. 37, 2008

---

Title Page

Contents



Page 7 of 11

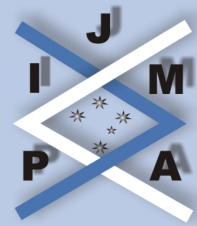
Go Back

Full Screen

Close

journal of **inequalities**  
in pure and applied  
mathematics

issn: 1443-5756



## 2. Main Results

**Theorem 2.1.** Let  $f$  be a non-negative and continuous function on  $[0, 1]$ . If  $\int_x^1 f(t)dt \geq \frac{1-x^2}{2}$  ( $0 \leq x \leq 1$ ), then for each  $m, n \in \mathbb{N}$ ,

$$\int_0^1 f^{m+n}(x)dx \geq \int_0^1 x^m f^n(x)dx.$$

*Proof.* By using the general Cauchy inequality [5, Theorem 3.1], we have

$$\frac{n}{m+n} f^{m+n}(x) + \frac{m}{m+n} x^{m+n} \geq x^m f^n(x),$$

which implies

$$\frac{n}{m+n} \int_0^1 f^{m+n}(x)dx + \frac{m}{m+n} \int_0^1 x^{m+n} dx \geq \int_0^1 x^m f^n(x)dx.$$

Hence

$$\begin{aligned} \int_0^1 f^{m+n}(x)dx &\geq \int_0^1 x^m f^n(x)dx + \frac{m}{m+n} \int_0^1 f^{m+n}(x)dx - \frac{m}{(m+n)(m+n+1)} \\ &= \int_0^1 x^m f^n(x)dx + \frac{m}{m+n} \left( \int_0^1 f^{m+n}(x)dx - \frac{1}{m+n+1} \right). \end{aligned}$$

By Lemma 1.2, we have  $\int_0^1 f^{m+n}(x)dx \geq \frac{1}{m+n+1}$ . Therefore

$$\int_0^1 f^{m+n}(x)dx \geq \int_0^1 x^m f^n(x)dx.$$

□

Note On An Open Problem

Gholamreza Zabandan

vol. 9, iss. 2, art. 37, 2008

Title Page

Contents



Page 8 of 11

Go Back

Full Screen

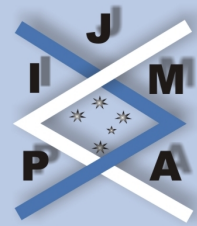
Close

journal of **inequalities**  
in pure and applied  
mathematics

issn: 1443-575b

© 2007 Victoria University. All rights reserved.





Note On An Open Problem

Gholamreza Zabandan

vol. 9, iss. 2, art. 37, 2008

[Title Page](#)

[Contents](#)

[◀◀](#) [▶▶](#)

[◀](#) [▶](#)

Page 9 of 11

[Go Back](#)

[Full Screen](#)

[Close](#)

journal of **inequalities**  
in pure and applied  
mathematics

issn: 1443-5756

© 2007 Victoria University. All rights reserved.

**Theorem 2.2.** Let  $f$  be a continuous function such that  $f(x) \geq 1$  ( $0 \leq x \leq 1$ ). If  $\int_x^1 f(t)dt \geq \frac{1-x^2}{2}$ , then for each  $\alpha, \beta > 0$ ,

$$(2.1) \quad \int_0^1 f^{\alpha+\beta}(x)dx \geq \int_0^1 x^\alpha f^\beta(x)dx.$$

*Proof.* By a similar method to that used in the proof of Theorem 2.1 the inequality (2.1) holds if  $\int_0^1 f^{\alpha+\beta}(x)dx \geq \frac{1}{\alpha+\beta+1}$ . So it is enough to prove that  $\int_0^1 f^\gamma(x)dx \geq \frac{1}{\gamma+1}$  ( $\gamma > 0$ ). Since  $f(x) \geq 1$  ( $0 \leq x \leq 1$ ) and  $[\gamma] \leq \gamma < [\gamma] + 1$ , we have

$$\int_0^1 f^\gamma(x)dx > \int_0^1 f^{[\gamma]}(x)dx.$$

By Lemma 1.2 we obtain

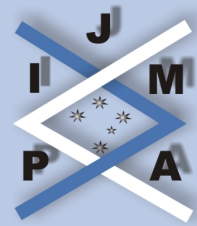
$$\int_0^1 f^\gamma(x)dx \geq \int_0^1 f^{[\gamma]}(x)dx \geq \frac{1}{[\gamma]+1} \geq \frac{1}{\gamma+1}.$$

□

**Remark 2.** The condition  $f(x) \geq 1$  ( $0 \leq x \leq 1$ ) in Theorem 2.2 is necessary for  $\int_0^1 f^\gamma(x)dx \geq \frac{1}{\gamma+1}$  ( $\gamma > 0$ ). For example, let

$$f(x) = \begin{cases} 0 & 0 \leq x \leq \frac{1}{2} \\ 2(2x-1) & \frac{1}{2} < x \leq 1 \end{cases}$$

and  $\gamma = \frac{1}{2}$ , then  $f$  is continuous on  $[0, 1]$  and  $\int_x^1 f(t)dt \geq \frac{1-x^2}{2}$ , but  $\int_0^1 f^{\frac{1}{2}}(x)dx = \frac{\sqrt{2}}{3} < \frac{1}{3}$ .



In the following theorem, we show that the condition  $f(x) \geq 1$  ( $0 \leq x \leq 1$ ) in Theorem 2.2 can be removed if we assume that  $\alpha + \beta \geq 1$ .

**Theorem 2.3.** *Let  $f$  be a non-negative continuous function on  $[0, 1]$ . If  $\int_x^1 f(t)dt \geq \frac{1-x^2}{2}$  ( $0 \leq x \leq 1$ ), then for each  $\alpha, \beta > 0$  such that  $\alpha + \beta \geq 1$ , we have*

$$\int_0^1 f^{\alpha+\beta}(x)dx \geq \frac{1}{\alpha + \beta + 1}.$$

*Proof.* By using Theorem A of [5] for  $g(t) = t$ ,  $\alpha = 1$ ,  $a = 0$ , and  $b = 1$ , the assertion is obvious.  $\square$

---

**Note On An Open Problem**

Gholamreza Zabandan

vol. 9, iss. 2, art. 37, 2008

---

Title Page

Contents



Page 10 of 11

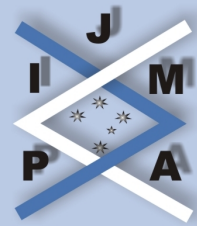
Go Back

Full Screen

Close

journal of **inequalities**  
in pure and applied  
mathematics

issn: 1443-5756



## References

- [1] L. BOUGOFFA, Note on an open problem, *J. Ineq. Pure and Appl. Math.*, **8**(2) (2007), Art. 58. [ONLINE: <http://jipam.vu.edu.au/article.php?sid=871>].
- [2] K. BOUKERRIOUA AND A. GUEZANE-LAKOUD, On an open question regarding an integral inequality, *J. Ineq. Pure and Appl. Math.*, **8**(3) (2007), Art. 77. [ONLINE: <http://jipam.vu.edu.au/article.php?sid=885>].
- [3] W.J. LIU, C.C. LI AND J.W. DONG, On an open problem concerning an integral inequality, *J. Ineq. Pure and Appl. Math.*, **8**(3) (2007), Art. 74. [ONLINE: <http://jipam.vu.edu.au/article.php?sid=882>].
- [4] F. QI, Several integral inequalities, *J. Ineq. Pure and Appl. Math.*, **1**(2) (2000), Art. 19. [ONLINE: <http://jipam.vu.edu.au/article.php?sid=113>].
- [5] QUÔC ANH NGÔ, On the inverse of an integral inequality, *RGMI A Research Report Collection*, **10**(4) (2007), Art. 10. [ONLINE: <http://www.staff.vu.edu.au/RGMIA/v10n4.asp>].
- [6] QUÔC ANH NGÔ, DU DUC THANG, TRAN TAT DAT, AND DANG ANH TUAN, Notes on an integral inequality, *J. Ineq. Pure and Appl. Math.*, **7**(4) (2006), Art. 120. [ONLINE: <http://jipam.vu.edu.au/article.php?sid=737>].

---

Note On An Open Problem

Gholamreza Zabandan

vol. 9, iss. 2, art. 37, 2008

---

Title Page

Contents



Page 11 of 11

Go Back

Full Screen

Close

journal of **inequalities**  
in pure and applied  
mathematics

issn: 1443-575b