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Relativistic electromagnetism in rotating media

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Abstract

This work concerns relativistic electromagnetism in a cylindrical Frenet-Serret frame. The tensor formalism of Maxwell's equations and electromagnetic fields in a vacuum is first developed in terms of cylindrical coordinates and afterwards applied to a rotating frame using the relativistic Trocheris-Takeno description of rotations. The metric $ds^2 = g_{\mu\nu}\ dx^\mu dx^\nu$ *of this frame is then obtained to find the determinant g of the* $g_{\mu\nu}$ *matrix intervening in the relativistic Maxwell's equations, where the Greek indices take on the values 1,2,3,4. The propagation of harmonic cylindrical waves in rotating media is analyzed and it is shown that these waves can propagate only in some regions of spacetime. Geometrical optics and its paraxial approximation in rotating frames are also investigated in terms of a scalar field. Finally, the last section is devoted to electromagnetism in a rotating material medium with the use of covariant constitutive relations.*

Key Words: *Cylindrical frame, rotating medium, relativistic electromagnetism, metric tensor.*

1. Introduction

Relativistic electromagnetism in rotating media has been the subject of some publications in the past [1-5], with in particular the objective to explain the Wilsons' experiment [4,5]. We depart from these works, first, since most [1-4] use Galilean rotations, and second, because we consider relativistic electromagnetism in a rotating vacuum as a particular case of relativistic electromagnetism in a cylindrical frame. Thus, since tensors are the basic tools of relativistic theories, we first summarize the well known tensor formalism of fields and Maxwell's equations in an arbitrary cartesian frame [6-8]. Afterwards, this formalism is generalized to a cylindrical frame, and finally applied to a rotating medium in terms of the relativistic Trocheris-Takeno description of rotations [9,10]. The propagation of harmonic cylindrical waves in rotating media is carefully analyzed, and it is shown that waves can propagate only in some regions of space and time. Because of some recent controversy in nonrelativistic theory, we also discuss geometrical optics in a rotating frame, and its paraxial approximation in terms of a scalar field. The last section is devoted to electromagnetism in rotating material media with constitutive relations, a generalization to a cylindrical space-time of the Post [11] covariant constitutive relations. Four Appendices complete this paper.

2. Relativistic electromagnetism

2.1. Tensor formalism in cartesian frames

Let a cartesian frame endowed with the metric $ds^2 = g_{\mu\nu}\ dx^\mu\ dx^\nu$, in which $x^1 = x$, $x^2 = y$, $x^3 = z$, and $x^4 = ct$, for the Minkowski space-time and cartesian coordinates $ds^2 = c^2dt^2(dx^2+dy^2+dz^2)$. Then, the Maxwell equations for a vacuum, in the Heaviside-Lorentz system

$$\nabla.\mathbf{H} = 0, \qquad \nabla \wedge \mathrm{E} + 1/c\partial_t \mathrm{H} = 0 \tag{1a}$$

$$\nabla.\mathbf{E} = 0, \qquad \nabla \wedge \mathbf{H} - 1/c\partial_t \mathbf{E} = 0(1b) \tag{1b}$$

have the following tensor representation [12-14] in the MKSA system :

$$\partial_\sigma F_{\mu\nu} + \partial_\mu F_{\nu\sigma} + \partial_\nu F_{\sigma\mu} = 0 \tag{2a}$$

$$\partial_\nu (|g|^{1/2} F^{\mu\nu}) = 0, g = det.g_{\mu\nu} \tag{2b}$$

The greek indices σ , μ, and ν take the values 1,2,3, and 4, with $\partial_\nu = \partial / \partial x^\nu$. These equations correspond, respectively, to (1a) and (1b) with the electromagnetic field tensors $F_{\mu\nu}$ and $F^{\mu\nu}$:

$$F_{\mu\nu} = \begin{vmatrix} 0 & -H_z & H_y & -E_x \\ \mathrm{H}_z & 0 & -H_x & -E_y \\ -\mathrm{H}_y & \mathrm{H}_x & 0 & -E_z \\ \mathrm{E}_x & \mathrm{E}_y & \mathrm{E}_z & 0 \end{vmatrix}, \quad F^{\mu\nu} = \begin{vmatrix} 0 & H_z & -H_y & -E_x \\ -\mathrm{H}_z & 0 & H_x & -E_y \\ \mathrm{H}_y & -\mathrm{H}_x & 0 & -E_z \\ \mathrm{E}_x & \mathrm{E}_y & \mathrm{E}_z & 0 \end{vmatrix} \tag{3}$$

Using the dual tensor field [12,13] in which $\varepsilon^{\mu\nu\alpha\beta}$ is the Levi-Civita tensor,

$$F*^{\mu\nu} = |g|^{-1/2} \varepsilon^{\mu\nu\alpha\beta} F_{\alpha\beta,}(4) \tag{4}$$

the Maxwell equations (1a) have the alternative representation

$$\partial_\nu (|g|^{1/2} F*^{\mu\nu}) = 0 \tag{5}$$

$$\text{with } |g|^{1/2} F*^{\mu\nu} = \begin{vmatrix} 0 & -\mathrm{E}_z & E_y & -H_x \\ E_z & 0 & E_x & -H_y \\ -\mathrm{E}_y & -\mathrm{E}_x & 0 & -H_z \\ H_x & H_y & H_z & 0 \end{vmatrix} \tag{6}$$

It follows from Eq. (2a) and Eq. (6) that the Maxwell's equations in Eq. (1a) do not depend on the specific properties of the frame that is on the $g_{\mu\nu}$. This noticeable feature does not hold for the second set of Maxwell's equations.

2.2. Tensor formalism in cylindrical frames

Using cylindrical coordinates so that $x^1 = r$ (to avoid later confusion, r is used rather than the conventional ρ for x^1), $x^2 = \phi$, $x^3 = z$, and $x^4 = ct$, , we write the Maxwell equations in Eq. (1a) and Eq. (1b) in a form slightly different from than usual:

$$\begin{aligned}\partial_r(r\mathrm{H}_r)+\partial_\phi \mathrm{H}_\phi+\partial_z(r\mathrm{H}_z)=0\\ \partial_\phi \mathrm{E}_z-\partial_z(r\mathrm{E}_\phi)+1/c\partial_t(r\mathrm{H}_r)=0\\ \partial_z \mathrm{E}_r-\partial_r \mathrm{E}_z+1/c\partial_t \mathrm{H}_\phi=0\\ \partial_r(r\mathrm{E}_\phi)-\partial_\phi \mathrm{E}_r+1/c\partial_t(r\mathrm{H}_z)=0\end{aligned} \tag{7a}$$

$$\begin{aligned}\partial_r(rE_r)+\partial_\phi E_\phi+\partial_z(rE_z)=0\\ \partial_\phi H_z-\partial_z(rH_\phi)-1/c\partial_t(rE_r)=0\\ \partial_z H_r-\partial_r H_z-1/c\partial_t E_\phi=0\\ \partial_r(rH_\phi)-\partial_\phi H_r-1/c\partial_t(rE_z)=0\end{aligned} \tag{7b}$$

Let us now consider a cylindrical frame with the Minkowski metric

$$ds^2=c^2dt^2-(dr^2+r^2d\phi^2+dz^2). \tag{8}$$

To simplify calculations somewhat, from now on we require that c = 1 and $g_0 = r^2$ is the determiant of $g_{\mu\nu}$ in (8). We prove in Appendix A that the tensor equations in Eq. (2a) and Eq. (2b) are valid for the tensor fields $G_{\mu\nu}$ and $G^{\mu\nu}$

$$G_{\mu\nu}=\begin{vmatrix} 0 & -rH_z & H_\phi & -E_r\\ r\mathrm{H}_z & 0 & -rH_r & -rE_\phi\\ -\mathrm{H}_\phi & r\mathrm{H}_r & 0 & -E_z\\ \mathrm{E}_r & r\mathrm{E}_\phi & \mathrm{E}_z & 0\end{vmatrix} \tag{9a}$$

$$G^{\mu\nu}=\begin{vmatrix} 0 & H_z/r & -H_\phi & -E_r\\ -\mathrm{H}_z/r & 0 & H_r/r & -E_\phi/r\\ \mathrm{H}_\phi & -\mathrm{H}_r/r & 0 & -E_z\\ \mathrm{E}_r & \mathrm{E}_\phi/r & \mathrm{E}_z & 0\end{vmatrix} \tag{9b}$$

and for the covariant derivative tensor

$$\partial_1=\partial_r,\partial_2=\partial_\phi,\partial_3=\partial_z,\partial_4=\partial_t \tag{10}$$

So, the tensorial Maxwell equations in the cylindrical natural frame are

$$\partial_\sigma G_{\mu\nu}+\partial_\mu G_{\nu\sigma}+\partial_\nu G_{\sigma\mu}=0 \tag{11a}$$

$$\partial_\nu(|g|^{1/2}G^{\mu\nu})=0 \tag{11b}$$

corresponding to (7a) and (7b), respectively.

Then, in a cylindrical frame with the metric $ds^2 = g_{\mu\nu}\, dx^\mu\, dx^\nu$ and the determinant g, Eq. (11a) and consequently (7a) are left unchanged, as already noticed for cartesian frames, and writing $|g|^{1/2} = rq$, Eq. (11b) becomes

$$\partial_\nu(rqG^{\mu\nu})=0 \tag{12}$$

Taking Eq. (9b) into account , Eq. (7b) is transformed into

$$\begin{aligned}\partial_r(qrE_r)+\partial_\phi(qE_\phi)+\partial_z(qrE_z)=0\\ \partial_\phi(qH_z)-\partial_z(qrH_r)-\partial_t(qrE_r)=0\\ \partial_z(qH_r)-\partial_r(qH_z)-\partial_t(qE_r)=0\\ \partial_r(qrH_\phi)-\partial_\phi(qH_r)-\partial_t(qrE_z)=0\end{aligned} \tag{12a}$$

Remark: The components of the contravariant derivative operator are

$$\partial^1 = -\partial_r, \partial^2 = -1/r^2\partial_\phi, \partial^3 = -\partial_z, \partial^4 = \partial_t \tag{13}$$

giving the invariant

$$\partial^\mu \partial_\mu = \partial_t^2 - (\partial_r^2 + r^{-2}\partial_\phi^2 + \partial_z^2) \tag{13a}$$

3. Electromagnetism in a rotating Frenet-Serret frame

We consider a frame rotating with a constant angular velocity Ω around the o-z axis. The following relations, also used in previous work [5,15,16] and in which $\beta = \Omega$R (c = 1) exists,

between the cylindrical coordinates R ,Φ, Z, T and r, ϕ, z, t in the rotating and fixed frames [9,10] hold:

$$\begin{aligned} r &= R, \quad \phi = \Phi \cosh\beta - T/R \sinh\beta \\ z &= Z, \quad t = T\cosh\beta - R\Phi \sinh\beta \end{aligned} \tag{14}$$

Since the first set of Maxwell's equations does not depend on g we find at once from Eq. (11a) that

$$\begin{aligned} \partial_R(RH_r) + \partial_\Phi \,\mathrm{H}_\Phi + \partial_Z \sinh(RH\sinh_z)\sinh = 0 \\ \sinh\partial_\Phi \mathrm{E}_z - \partial_Z \sinh(R\sinh \mathrm{E}_\phi) + \partial_T \sinh(R\sinh \mathrm{H}_r)\sinh = 0 \\ \sinh\partial_Z \mathrm{E}_r - \partial_R \mathrm{E}_z \sinh + \sinh\partial_T \mathrm{H}_\phi \sinh = 0 \\ \sinh\partial_R \sinh(RE\sinh_\phi) - \partial_\Phi \sinh E\sinh_r + \partial_T \sinh(RH\sinh_z)\sinh = 0 \end{aligned} \tag{15}$$

Now, according to Eq. (11b), we need the determinant g of the metric tensor $g_{\mu\nu}$ to get the second set of Mawxell equations. The metric ds^2 of the rotating frame is obtained in Appendix B.

$$\begin{aligned} &ds^2 = dT^2 - dZ^2 - R^2 d\Phi^2 - (1 + B^2 - A^2)dR^2 - 2(A\sinh ß + B\cosh ß)dTdR- \\ &2(A\cosh ß + B\sinh ß)RdRd\Phi \end{aligned} \tag{16}$$

in which

$$\begin{aligned} A = ß\sinh ßT/R + ß\cosh ß\Phi + \sinh ß\Phi \\ B = ß\sinh ß\Phi + ß\cosh ßT/R - \sinh ßT/R \end{aligned} \tag{16a}$$

Using the notation X^4 = T, X^3 = Z, $X^2 = \Phi$, X^1 = R, we find from Eq. (16) that $ds^2 = g_{\mu\nu}\, dx^\mu\, dx^\nu$ with

$$\begin{aligned} g_{44} = 1, g_{33} = -1, g_{22} = -R^2, g_{11} = -(1 + B^2 - A^2) \\ g_{14} = g_{41} = 2(A\sinh ß + B\cosh ß), g_{12} = g_{21} = 2(A\cosh ß + B\sinh ß) \end{aligned} \tag{17}$$

So, the determinant g is

$$g = g_{33}[g_{11}g_{22}g_{44} - g_{12}^2 g_{44} - g_{14}^2 g_{22}] = R[g_{11} - g_{12}^2 R^{-2} - g_{14}^2] \tag{18}$$

But

$$g_{12}^2 R^{-2} + g_{14}^2 = 4(A^2 - B^2) \tag{18a}$$

Taking into account the expression in Eq. (17) for g_{11} , we finally find that

$$g = R^2[5(A^2 - B^2) - 1] \tag{19}$$

in which, according to Eq. (16a),

$$A^2 - B^2 = (\Phi^2 - T^2/R^2)(ß^2 + \sinh^2 ß + 2ß \sinh ß \cosh ß) \tag{19a}$$

Then, the second set of Maxwell's equations is supplied by Eq. (12), in which $G^{\mu\nu}$ is the tensor in Eq. (9b) with r changed into R and $|g|^{1/2} = Rq$, with

$$q = [5(A^2 - B^2) - 1]^{1/2} \tag{19b}$$

We obtain in Appendix C the explicit form of these equations:

$$\begin{aligned} \partial_R(qRE_r) + \partial_\Phi(qE_\phi) + \partial_Z(qRE_z) = 0 \\ \partial_\Phi(qH_z) - \partial_Z(qRH_\phi) - \partial_T(qRE_r) = 0 \\ \partial_Z(qH_r) - \partial_R(qH_z) - \partial_T(qE_\phi) = 0 \\ \partial_R(qRH_\phi) - \partial_\Phi(qH_r) - \partial_T(qRE_z) = 0 \end{aligned} \tag{20}$$

These equations are rather intricate because q does not have a simple expression in terms of R, Φ, and T.

4. Harmonic cylindrical waves in a rotating frame

In this Section, we use the lower-case coordinates, $x^1 = r$, $x^2 = \phi$, $x^3 = z$, and $x^4 = t$ instead of R, Φ, Z, and T, and we suppose that the electromagnetic field made of cylindrical waves is

$$\psi(x_\nu) = \exp(ip^\nu x_\nu), p^\nu p_\nu = 0, i = \sqrt{-1} \tag{21}$$

for the phase function with $\nu = 1,2,3,4$.

Then, with

$$p^4 = \omega, p^1 = k_r, p^2 = k_\phi/r, p^3 = k_z \tag{21a}$$

the relations $p^\nu x_\nu$ and $p^\nu p_\nu = 0$ become

$$p^\nu x_\nu = \omega t - (k_r r + k_\phi \phi/r + k_z z) \tag{22a}$$

$$k_r^2 + r^{-2}k_\phi^2 + k_z^2 = \omega^2 \tag{22b}$$

Then, according to relation in Eq. (A.1) of Appendix A and using Eq. (21), the components of the tensor field $G_{\mu\nu}$ become $\exp(ip^\nu x_\nu)$, where exp(.) is used to denote the exponential, while the lower case fields (**e**,**h**) denote constant amplitudes

$$\begin{aligned} rE_\phi = re_\phi \exp(.), rH_r = -rh_r \exp(.), rH_z = -rh_z \exp(.) \\ E_r = e_r \exp(.), E_z = e_z \exp(.), H_\phi = -h_\phi \exp(.) \end{aligned} \tag{23}$$

So, taking into account Eq. (21), Eq. (22a), and Eq. (23), we find from Eq. (15) the equations

$$(rk_r - i)h_r + k_\phi h_\phi + rk_z h_z = 0 \tag{24}$$

$$k_\phi e_z - rk_z e_\phi + \omega r h_r = 0 \tag{25a}$$

$$k_z e_r - k_r e_z + \omega h_\phi = 0 \tag{25b}$$

$$(rk_r - i)e_\phi - k_\phi e_r + \omega r h_z = 0 \tag{25c}$$

Multiplying Eq. (25a) by ($rk_r -$ i), Eq. (25b) by k_ϕ , Eq. (25c) by k_z, and summing these expressions, it is at once checked that the divergence equation in Eq. (24) implies $e_z = 0$, while the Maxwell equations in Eq. (25) reduce to

$$rk_z e_\phi - \omega r h_r = 0 \tag{26a}$$

$$k_z e_r + \omega h_\phi = 0 \tag{26b}$$

$$(rk_r - i)e_\phi - k_\phi e_r + \omega r h_z = 0 \tag{26c}$$

Similarly, according to the relation in Eq. (A.3) of Appendix A, the components of the tensor field $rG^{\mu\nu}$ are

$$\begin{gathered} rE_r = -re_r \exp(.), rH_\phi = rh_\phi \exp(.), -rE_z = re_z \exp(.) \\ H_z = h_z \exp(.), H_r = h_r \exp(.), E_\phi = -e_\phi \exp(.) \end{gathered} \tag{27}$$

Let α_μ denote the derivatives of q. Then, they are are function of t,r,ϕ:

$$\alpha_t = \partial_t q, \alpha_r = \partial_r q, \alpha_\phi = \partial_\phi q, \alpha_z = \partial_z q = 0 \tag{28}$$

Then, we get from Eq. (20)

$$\begin{aligned} r\alpha_r e_r + \alpha_\phi e_\phi + r\alpha_z e_z + q[(rk_r - i)e_r + k_\phi e_\phi + rk_z e_z] = 0 \\ \alpha_\phi h_z - r\alpha_z h_\phi + r\alpha_t e_r + q[k_\phi h_z - rk_z h_\phi - \omega r e_r] = 0 \\ \alpha_z h_r - \alpha_r h_z + \alpha_t e_\phi + q[k_z h_r - k_r h_z - \omega e_\phi] = 0 \\ r\alpha_r h_\phi - \alpha_\phi h_r + r\alpha_t e_z + q[(rk_r - i)h_\phi - k_\phi h_r - \omega r e_z] = 0 \end{aligned} \tag{29}$$

With the parameters

$$\chi_r = \alpha_r + qk_r, \chi_\phi = \alpha_\phi + qk_\phi, \chi_z = \alpha_z + qk_z, \varpi = \alpha_t + q\omega \tag{30}$$

these equations become

$$(r\chi_r - iq)e_r + \chi_\phi e_\phi + r\chi_z e_z] = 0 \tag{31}$$

and

$$\chi_\phi h_z - r\chi_z h_\phi - \varpi r e_r = 0 \tag{32a}$$

$$\chi_z h_r - \chi_r h_z - \varpi e_\phi = 0 \tag{32b}$$

$$(r\chi_r - iq)h_\phi - \chi_\phi h_r - \varpi r e_z = 0 \tag{32c}$$

As for the first set of Maxwell's equations, the divergence equation in Eq. (31) implies $h_z = 0$ and Eqs. (32a,b,c) reduce to

$$-r\chi_z h_\phi - \varpi r e_r = 0 \tag{33a}$$

$$\chi_z h_r - \varpi e_\phi = 0 \tag{33b}$$

$$(r\chi_r - iq)h_\phi - \chi_\phi h_r - \varpi r e_z = 0 \tag{33c}$$

So, as it could have been expected, the cylindrical electromagnetic field is perpendicular to the axis of rotation, taking the form of a doughnut.The equations (26a,b,c) give the amplitudes e_r, h_r, h_ϕ, in terms of e_ϕ. Since $h_z = 0$, we get

$$e_r = (rk_r - i)k_\phi e_\phi, h_r = k_z e_\phi/\omega, h_\varphi = -k_z(rk_r - i)e_\phi/k_\varphi\omega \tag{34}$$

while the equations in Eq. (33a,b,c) impose some constraints on the directions along which these cy-lindrical waves can propagate. Substituting Eq. (26a) into Eq.(33b), and Eq.(26b) into Eq.(33c) gives in both cases

$$k_z\chi_z - \omega\varpi = 0. \tag{35}$$

With the expressions in Eq. (26a,b) of h_r and h_ϕ, the equation in Eq. (33c) becomes

$$(r\chi_r - iq)e_r + \chi_\phi e_\phi = 0 \tag{36a}$$

since $e_z = 0$, while (26c) gives

$$(rk_r - i)e_\phi - k_\phi e_r = 0 \tag{36b}$$

with $h_z = 0$.

The homogeneous linear system in Eq. (36 a,b) has a nontrivial solution if its determinant is null, implying

$$(r\chi_r - iq)(rk_r - i) + \chi_\phi k_\phi = 0 \tag{37}$$

So, for a given ω and $\mathbf{k}(k_r, k_\phi, k_z)$that satisfy Eq. (22b), the relations in Eq. (35) and Eq. (37) must be satisfied to make possible the propagation of cylindrical waves.

We illustrate this situation in the near universe for a low angular velocity Ω, assumed to be positive, so that ß << 1. The following 0(ß^4) approximation

$$\sinh ß = ß + ß^3/6 + 0(ß^4), \cosh ß = 1 + ß^2/2 + 0(ß^4) \tag{38}$$

is applied so that we find from Eq. (19a)

$$A^2 - B^2 = 4ß^2(\phi^2 - t^2/r^2) + 0(ß^4) \tag{39}$$

Neglecting ϕ^2 with respect to c^2t^2/r^2 gives

$$A^2 - B^2 = 4\Omega^2 t^2 + 0(ß^4, \phi^2) \tag{39a}$$

Now, according to Eq. (19b), $q = |5(A^2 - B^2) - 1|^{1/2}$. Neglecting the - 1 term, we find

$$q = \sqrt{20}\Omega|t|, \tag{40}$$

and, taking into account Eq. (40), the α parameters in Eq. (28) become

$$\alpha_r = \alpha_\phi = \alpha_z = 0, \alpha_t = \pm\sqrt{20}\Omega \tag{41}$$

Here, the plus/minus sign in α_t corresponds to $t > 0/t < 0$ so that $\alpha_t/q = 1/t$.

Then, according to Eq. (30) and Eq. (41), the constraint in Eq. (35) gives

$$k_z^2 - \omega^2 - \omega/t = 0, \tag{42}$$

or in other terms, $t = \omega/(k_z^2 - \omega^2)$, which taking into account Eq. (22) can be found to be negative.

Still, using the relations in Eq. (30) and Eq. (41), the constraint supplied by the equation in Eq. (37) becomes

$$k_\phi^2 = (1 + irk_r)^2. \tag{43}$$

But, according to Eq. (22) : $r^2 k_r^2 + k_\phi^2 = (\omega^2 - k_z^2) r^2$. Substituting this relation into Eq. (43) gives

$$(\omega^2 - k_z^2) r^2 - 2irk_r - 1 = 0. \tag{44}$$

Now, Eq.(43) implies that $k_r = 0$and $k_\phi = 1$; consequently, from Eq. (44), it can be found that $r^2 = 1/(\omega^2 - k_z^2)$.

To summarize, the $0(ß^4)$ approximation makes propagation possible for negative time $t = \omega/(k_z^2 - \omega^2)$ in the regions $r^2 = 1/(\omega^2 - k_z^2)$ provided that $k_r = 0$ and $k_\phi = 1$.

The main virtue of this illustration, to be taken *cum grano salis*, is to show the difficulties of visualizing cylindrical waves in a cylindrical space-time frame.

5. Geometrical optics in a rotating frame

5.1. General theory

In a cylindrical frame with the metric $ds^2 = g_{\mu\nu}\, dx^\mu dx^\nu$, ($x^1 = r$, $x^2 = \phi$, $x^3 = z$, $x^4 = t$) and the determinant g, the optics field is described by a scalar $\Psi(r,\phi,z,t)$ solution of the wave equation (relativistic d'Alembertian [12]), is

$$\partial_\mu (|g|^{1/2} g^{\mu\nu} \partial_\nu \Psi) = 0 \tag{45}$$

where ∂_ν is the derivative operator in Eq. (10) and $\partial_1 = \partial_r$, $\partial_2 = \partial_\phi$, $\partial_3 = \partial_z$, $\partial_4 = \partial_t$. Ψ is the eikonal function characterizing the electromagnetic geometrical wave fronts. The components $g^{\mu\nu}$ of the conjugate tensor are divided by g, the cofactors of the components $g_{\mu\nu}$ in the matrix $g_{\mu\nu}$ [17].

In the natural cylindrical frame, the nonnull $g^{\mu\nu}$ can be found to be

$$-g^{11} = -g^{33} = g^{44} = 1, g^{22} = -r^2 \tag{46}$$

Since $|g|^{1/2} = r$, the wave equation in Eq. (45) becomes

$$\Delta_a \Psi \equiv (\partial_r^2 + r^{-1}\partial_r + r^{-2}\partial_\phi^2 - \partial_t^2)\Psi = 0 \tag{47}$$

with the solutions [18]

$$\Psi(r, \phi, z, t) = J_n(k_r r) \exp(in\phi + ik_z z + i\omega t), k_r^2 = \omega^2 - k_z^2 \tag{48}$$

where J_n is an n^{th} order Bessel function of the first kind.

In a rotating frame, first the components $g^{\mu\nu}$ must be found. According to Eq. (17), the matrix $g_{\mu\nu}$ with the determinant as shown in Eq. (19), is

$$\begin{vmatrix} g_{11} & g_{12} & 0 & g_{14} \\ g_{21} & g_{22} & 0 & 0 \\ 0 & 0 & g_{33} & 0 \\ g_{41} & 0 & 0 & sg_{44} \end{vmatrix} \tag{49}$$

and a simple calculation yields $g^{11} = g^{-1} g_{22} g_{33} g_{44}$, $g^{22} = g^{-1} g_{33}\,(g_{11} g_{44} - g_{14} g_{41})$, $g^{44} = g^{-1} g_{33}\,(g_{11} g_{22} - g_{12} g_{21})$

$$\begin{gathered} g^{33} = g^{-1}g_{44}(g_{11}g_{22} - g_{12}g_{21}) - g^{-1}g_{22}g_{14}g_{41} \\ g^{14} = g^{41} = g^{-1}g_{33}g_{22}g_{41,}g^{12} = g^{21} = g^{-1}g_{33}g_{44}g_{21} \end{gathered} \tag{50}$$

for $g^{\mu\nu} = g^{-1}\gamma_{\mu\nu}$, where $\gamma_{\mu\nu}$ is the cofactor of $g_{\mu\nu}$ in Eq. (49).

Then, since according to Eq. (19b) $|g|^{1/2} = rq$, we find from Eq. (45) and Eq. (50):

$$\begin{aligned} &\partial_r[rq(g^{11}\partial_r\Psi + g^{12}\partial_\phi\Psi + g^{14}\partial_t\Psi)] + r\partial_\phi[q(g^{22}\partial_\phi\Psi + g^{21}\partial_r\Psi)] + \\ &r\partial_z(qg^{33}\partial_z\Psi) + r\partial_t[q(g^{44}\partial_t\Psi + g^{41}\partial_r\Psi)] = 0 \end{aligned} \tag{51}$$

Obtaining analytical solutions of this equation is a challenge without the use of drastic approximations.

5.2. Paraxial approximation

We assume that $ß = \Omega r/c$ is small enough to justify an $0(ß^2)$ approximation, identifying paraxial and nonrelativistic approximations [19]. First, Eq. (14) yield

$$r = R, z = Z, \phi = \Phi - \Omega T + 0(ß^2), t = T - \Omega\Phi + 0(ß^2) \tag{52}$$

for $\sinh ß = ß + 0(ß^2)$, $\cosh ß = 1 + 0(ß^2)$, while according to Eq. (16a) and Eq. (19a)

$$A = 2ß\Phi + 0(ß^2), B = 0 + 0(ß^2), A^2 - B^2 = 0 + 0(ß^2) \tag{53}$$

Since according to Eq. (12b) $|g|^{1/2} = r + 0(ß^2)$, the components in Eq. (17) of $g_{\mu\nu}$ (using lower case coordinates), reduce to

$$\begin{gathered} g_{44} = 1, g_{33} = -1, g_{22} = -r^2, g_{11} = -1 + 0(ß^2), \\ g_{14} = g_{41} = 0 + 0(ß^2), g_{12} = g_{21} = -4\beta\phi \end{gathered} \tag{54}$$

Substituting Eq. (54) into Eq. (50) gives to the $0(ß^2)$ order

$$g^{11} = g^{33} = -g^{44} = 1, g^{22} = 1/r^2, g^{14} = g^{41} = 0, g^{12} = g^{21} = 4ß\phi/r \tag{55}$$

Taking into account Eq. (55), the wave equation Eq. (51) becomes

$$r\Delta_a\Psi + 4\phi\partial_r(ß\partial_\phi\Psi) + 4ß\partial_\phi(\phi\partial_r\Psi) = 0 \tag{56}$$

with q = 1, where $\Delta_a\Psi$ is the d'Alembertian in Eq. (47).

Considering optic fields, which do not depend on ϕ, we look for the solutions of Eq. (56) in the form

$$\Psi(r,z,t,) = \Psi^{\dagger}(r)\exp(ik_z z + i\omega t), \tag{57}$$

implying that $\Psi^{\dagger}$ (r) satisfies the equation

$$(r\partial_r^2 + \partial_r + k_r^2 r + \Omega r\partial_r)\Psi^{\dagger}(r) = 0, k_r^2 = \omega^2 - k_z^2 \tag{58}$$

With $\Psi^{\dagger}$ (r) represented by a power series expansion $\sum a_n r^n$, we for the first five terms in which a_0 is an arbitrary constant (see Appendix D)

$$\Psi^{\dagger}(r) = a_0 - a_0 k_r^2 r^2/4 + a_0 k_r^2 \Omega r^3/18 + a_0(k_r^4/64 - k_r^2\Omega^2/96)r^4 \tag{59}$$

We observe at once that the terms that do not depend on Ω are a truncated series expansion of J_0(kz). This result is not unexpected since we could have looked for the solutions of Eq. (56) in the form $\Psi^{\dagger}$ (r) $= a_0 J_0(k_r r)$ $+\sum b_n r^n$. So, taking into account Eq. (59), we may write this approximation as

$$\Psi^{\dagger}(r) = a_0 J(k_r r) + a_0 k_r^2 \Omega r^3/18 - a_0 k_r^2 \Omega^2 r^4/96 \tag{60}$$

The paraxial theory of light propagation is usually displayed in cartesian frames [19], thereby making comparison with the present analysis difficult. Recently , Tiwari [20] has discussed a paraxial approximation for rotating light using the Galilean rotation transformation that ne-glects $\Omega\phi$ in the fourth relation of Eq. (52). Eq. (38) of Tiwari's work differs from Eq. (56) in the terms that depend on the angular frequency, Ω.Eq. (38) contains the component -$2\Omega\partial_t\partial_\phi\Psi$, while in Eq. (56) we have $\Omega(1/r\partial_\phi\Psi + 8\partial_\phi\partial_t\Psi + 4\partial_r\Psi - 2\partial_t\partial_\phi\Psi)$, including three supplementary terms that are hard to neglect.

The angular momentum of a rotating light beam is also discussed in [20, 21].

6. Rotating material medium

In a cartesian material frame,the electromagnetic field is described by the two tensors $F_{\mu\nu}$ and $F^{\mu\nu}$ (which should not be confused with the notation in Sec.2.1):

$$F_{\mu\nu} = \begin{vmatrix} 0 & -B_z & B_y & E_x \\ B_z & 0 & -B_x & E_y \\ -B_y & B_x & 0 & E_z \\ -E_x & -E_y & -E_z & 0 \end{vmatrix}, \quad F^{\mu\nu} = \begin{vmatrix} 0 & H_z & -H_y & D_x \\ -H_z & 0 & H_x & D_y \\ H_y & -H_x & 0 & D_z \\ -D_x & -D_y & -D_z & 0 \end{vmatrix} \tag{61}$$

and the Maxwell equations are

$$\partial_\sigma F_{\mu\nu} + \partial_\mu F_{\nu\sigma} + \partial_\nu F_{\sigma\mu} = 0a), \partial_\nu(|g|^{1/2}F^{\mu\nu}) = 0b) \tag{62}$$

while the constitutive relations between (**D,H**) and (**E,B**) are, according to Post [11],

$$F^{\mu\nu} = {}^{1}/{}_{2}\chi^{\mu\nu\alpha\beta}F_{\alpha\beta} \tag{63}$$

where $\chi^{\mu\nu\alpha\beta}$ is a fourth rank tensor with many symmetry properties that reduce the number of its independent components to twenty. For a linear, anisotropic, nondispersive medium, we get from Eq. (63) (Table 6.23 in [11])

$$F_{ij} = {}^1\!/_2\mu_{ijkl}F^{kl} \tag{64a}$$

$$F^{4j} = \varepsilon^{jk}F_{4k} \tag{64b}$$

where the latin indices take on the values of 1,2,3, and the permittivity and permeability tensors are symmetric.

We find from Eq. (62a) and Eq. (64a)

$$B_x = -F_{23} = {}^1\!/_2\mu_{23ij}F^{ij}, B_y = -F_{13} = {}^1\!/_2\mu_{13ij}F^{ij}, B_z = -F_{12} = {}^1\!/_2\mu_{12ij}F^{ij} \tag{65}$$

yielding $B_x = \mu_{1323}\ H_x + \mu_{1312}\ H_y + \mu_{1313}\ H_z$ and similar expressions for the other components.

We also get from Eq. (62b) and Eq. (64b)

$$D_x = F^{14} = \varepsilon^{1j}F_{4j}, D_y = F^{24} = \varepsilon^{2j}F_{4j}, D_z = F^{34} = \varepsilon^{3j}F_{4j} \tag{66}$$

yielding $D_x = \varepsilon^{11}E_x + \varepsilon^{12}E_y + \varepsilon^{13}E_z$ and similar expressions for D_y and D_z.

In a cylindrical material frame, for the greek indices taking on the values 1,2,3,4, and related to the coordinates r, ϕ, z, t, respectively, the tensors $G_{\mu\nu}$ and $G^{\mu\nu}$ become

$$G_{\mu\nu} = \begin{vmatrix} 0 & -rB_z & B_\phi & -E_r \\ -\mathrm{H}_z/r & 0 & H_r/r & -D_\phi \\ -B_\phi & rB_r & 0 & -E_z \\ E_r & r\mathrm{E}_\phi & \mathrm{E}_z & 0 \end{vmatrix} \qquad G^{\mu\nu} = \begin{vmatrix} 0 & H_z/r & -H_\phi & -D_r \\ rB_z & 0 & -rB_r & -rE_\phi \\ \mathrm{H}_\phi & -\mathrm{H}_r/r & 0 & -D_z \\ D_r & D_\phi/r & D_z & 0| \end{vmatrix} \tag{67}$$

According to (A.2), the Maxwell equations in Eq. (62), with the derivatives ∂_r, ∂_ϕ, ∂_z, ∂_t, and $|g|^{1/2} = rq$, are

$$\begin{aligned} \partial_r(rB_r) + \partial_\phi(B_\phi) + \partial_z(rB_z) = 0 \\ -\partial_t(rB_r) - \partial_\phi(E_z) + \partial_z(rE_\phi) = 0 \\ \partial_t(B_\phi) - \partial_r(E_z) + \partial_z(E_r) = 0 \\ -\partial_t(rB_z) - \partial_r(rE_\phi) + \partial_\phi(E_r) = 0 \end{aligned} \tag{68a}$$

Similarly, according to (A.4), using rq instead of r yields

$$\begin{aligned} \partial_r(qrD_r) - \partial_\phi(qD_\phi) - \partial_z(qrD_z) = 0 \\ \partial_\phi(qH_z) - \partial_z(qrH_\phi) - \partial_t(qrD_r) = 0 \\ -\partial_\phi(qH_r) + \partial_r(qrH_\phi) - \partial_t(qrD_z) = 0 \\ -\partial_r(qH_z) + \partial_z(qH_r) - \partial_t(qD_\phi) = 0 \end{aligned} \tag{68b}$$

Suppose now a linear, anisotropic, non-dispersive medium, cylindrical so that instead of the constitutive relations in Eq. (64a) and Eq. (64b) we find from Eq. (67) $G^{41} = D_r$, $G^{42} = D_\phi/r$, $G^{43} = D_z$,

$$G_{41} = E_r, G_{42} = rE_\phi, G_{43} = E_z, \tag{69}$$

where the latin indices take on the values 1,2,and 3, corresponding to the coordinates r, ϕ, and z, respectively. Thus, the constitutive relations in Eq. (64b) have the matrix representation $\mathbf{D} = \underline{\underline{\varepsilon}}\ \mathrm{E}$:

$$(D_r, D_\phi/r, D_z) = \begin{vmatrix} \varepsilon_{11} & \varepsilon_{12}/r & \varepsilon_{13} \\ \varepsilon_{21}/r & \varepsilon_{22}/r^2 & \varepsilon_{23}/r \\ \varepsilon_{31} & \varepsilon_{32}/r & \varepsilon_{33} \end{vmatrix} \begin{vmatrix} E_r \\ rE_\phi \\ E_z| \end{vmatrix} \tag{70}$$

To justify Eq. (70), consider the divergence equation $\partial_r(\mathrm{qrD}_r) + \partial_\phi\ (\mathrm{qD}_\phi) + \partial_z(\mathrm{qrD}_z) = 0$. Then, according to Eq. (70)

$$\begin{aligned}\partial_r(qrD_r) &= \partial_r[qr(\varepsilon_{11}E_r + \varepsilon_{12}E_\phi + \varepsilon_{13}E_z)]\\ \partial_\phi(qD_\phi) &= \partial_\phi[q(\varepsilon_{21}E_r + \varepsilon_{22}E_\phi + \varepsilon_{23}E_z)]\\ \partial_z(qrD_z) &= \partial_z[qr(\varepsilon_{31}E_r + \varepsilon_{32}E_\phi + \varepsilon_{33}E_z)]\end{aligned} \tag{70a}$$

and permittivity intervenes in Maxwell's equation in Eq. (68b) through the relation $\mathrm{D}_i = \varepsilon_{ij}\mathrm{E}_j$. That is,

$$D_r = \varepsilon_{rr}E_r + \varepsilon_{r\phi}E_\phi + \varepsilon_{rz}E_z,$$

where similar expressions for the other two components are also found.

We also find from Eq. (67)

$$\begin{gathered}G_{23} = -rB_r, G_{31} = -B_\phi, G_{12} = -rB_z\\ G^{12} = H_z/r, G^{23} = H_r/r, G^{31} = H_\phi\end{gathered} \tag{71}$$

so that the constitutive relations in Eq. (64a) become $\mathbf{B} = \underset{=}{\mu}\ \mathbf{H}$:

$$(rB_r, B_\phi, rB_z) = r^2 \begin{vmatrix} r\mu_{11} & \mu_{12}r & \mu_{13}\\ \mu_{21} & \mu_{22}/r & \mu_{23}\\ r\mu_{31} & \mu_{32}r & \mu_{33}\end{vmatrix} \begin{vmatrix} H_r\\ rH_\phi\\ H_z\end{vmatrix} \tag{72}$$

Using the divergence equation $\partial_r(\mathrm{rB}_r) + \partial_\phi\ (\mathrm{B}_\phi) + \partial_z(\mathrm{rB}_z) = 0$, we find from Eq. (72)

$$\begin{aligned}\partial_r(rB_r) &= \partial_r[r(\mu_{11}H_r + \mu_{12}H_\phi + \mu_{13}H_z)]\\ \partial_\phi(B_\phi) &= \partial_\phi[\mu_{21}H_r + \mu_{22}H_\phi + \mu_{23}H_z]\\ \partial_z(rB_z) &= \partial_z[r(\mu_{31}H_r + \mu_{32}H_\phi + \mu_{33}H_z)]\end{aligned} \tag{73}$$

so that the Maxwell equations in Eq. (68a) depend on permeability through the relation

$\mathrm{B}_i = \mu_{ij}\mathrm{H}_j$, with $\mathrm{B}_r = \mu_{rr}\mathrm{H}_r + \mu_{r\phi}\ \mathrm{H}_\phi + \mu_{rz}\mathrm{H}_z$ and similar expressions for the other two components.

Note that D_i and B_i are not tensors. The covariant permittivity and permeability tensors have the matrix representations shown in Eq. (70) and Eq. (72), from which it follows that no isotropic medium exists in a rotating frame, as expected considering the centrifugal effect.

Remark: Post [11] has succinctly analyzed the wave equation for a medium with cylindrical symmetry in cylindrical coordinates. Assuming a rotation symmetric around the z axis, the permittivity and permeability have the matrix representations (Tables 7.34 and 7.35 in [11])

$$\begin{vmatrix} r\varepsilon_t & 0 & 0\\ 0 & \varepsilon_t/r & 0\\ 0 & 0 & r\varepsilon_a\end{vmatrix} \qquad \begin{vmatrix} \mu_t/r & 0 & 0\\ 0 & r\mu_t & 0\\ 0 & 0 & \mu_r/r\end{vmatrix} \tag{74}$$

These matrices differ from Eq. (70) and Eq. (72),which are written in the principal axes of the medium. Additionaly, Post assumes an isotropic cylindrical medium is possible, which is also inconsistent with Eq. (70) and Eq. (72).

7. Conclusions

The relativistic electromagnetism in rotating media or, more generally, in cylindrical Frenet-Serret frames, has important features in comparison with electromagnetism in cartesian frames. In mechanics, an inertial force must be introduced in Newton's law to take into account the effects of the centrifugal force arising from circular motion. In electromagnetism, the consequences are more subtle: they appear in the radial components of the magnetic field and in the azimuthal components of the electric field, which depends explicitly on the radial coordinate, r. As in cartesian frames, the first set of Maxwell's equations is not disturbed by the motion, while the second set strongly depends on the specific properties of the medium under rotation.

As a result, for instance, cylindrical waves, analogous to plane waves for cartesian

spaces, can propagate only in some regions of space-time, provided that the wave vector sa-tisfies some rather drastic conditions (see [12] for an analysis of gravitational cylindrical wa-ves). So, it is difficult to conceive the behaviour of the electromagnetic field in rotating media.

Geometrical optics, playing an important role in modern technology due to the expansion of photonics, has been the object of some controversy [3 ,4, 20, 21]. It has been assumed that in rotating media, the geometrical optics field may be described by a scalar field representing t wavefront propagation; but, this assumption must be justified, since the propagation of elec-tromagnetic field discontinuities has yet to be investigated [12].

The most important differences between electromagnetic fields in cylindrical and cartesian

frames appear in material media, because, in the first case, the tensors associated with permittivity and permeability depend on the radial coordinate, r. Then, although Maxwell's equations in their 3D-form are similar in both cases, the covariance of the permittivity and permeability constitutive relations forbids the existence of isotropic media in rotating frame, a result in agreement with the effects of centrifugal motion. Further work is needed to investigate physical processes, such as the Fresnel-Fizeau effect and the Faraday rotation.

Is the Universe cylindrical? Although Einstein and Rosen have obtained exact cylindrical wave solutions of the general relativity equations [22], it is fortunate that this possibility was later discarded, because it would have made the Cosmos still more difficult to appre-hend.

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Appendix A: Maxwell equations in the natural cylindrical frame

According to Eq. (9a), the components of the tensor $G_{\mu\nu}$ are

$$G_{12} = -rH_z, G_{13} = H_\phi, G_{23} = -rH_r, G_{14} = -E_r, G_{24} = -rE_\varphi, G_{34} = -E_z \tag{A.1}$$

Substituting Eq. (A.1) into the tensorial equations of Eq. (2a), and using the derivative operator in Eq. (10), we get

$$0 = \partial_1 G_{23} + \partial_2 G_{31} + \partial_3 G_{12} = \partial_r(rH_r) + \partial_\phi(H_\phi) + \partial_z(rH_z)$$

$$0 = \partial_4 G_{23} + \partial_2 G_{34} + \partial_3 G_{42} = -\partial_t(rH_r) - \partial_\phi(E_z) + \partial_z(rE_\phi)$$

$$0 = \partial_4 G_{13} + \partial_1 G_{34} + \partial_3 G_{41} = \partial_t(H_\phi) - \partial_r(E_z) + \partial_z(E_r)$$

$$0 = \partial_4 G_{12} + \partial_1 G_{24} + \partial_2 G_{41} = -\partial_t(rH_z) - \partial_r(rE_\phi) + \partial_\phi(E_r) \quad (A.2)$$

which are exactly the Maxwell equations in Eq. (7a).

We similarly find from Eq. (9b)

$$rG^{12} = H_z, rG^{13} = -rH_\phi, rG^{23} = H_r, rG^{14} = -rE_r, rG^{24} = -E_\phi, rG^{34} = -rE_z \quad (A.3)$$

Taking into account Eq. (A.3), the second tensorial set in Eq. (2b), ∂_ν ($rG^{\mu\nu}$) = 0, gives

$$0 = \partial_1(rG^{41}) + \partial_2(rG^{42}) + \partial_3(rG^{43}) = -\partial_r(rE_r) - \partial_\phi(E_\phi) - \partial_z(rE_z)$$

$$0 = \partial_2(rG^{12}) + \partial_3(rG^{13}) + \partial_4(rG^{14}) = \partial_\phi H_z - \partial_z(rH_\phi) - \partial_t(rE_r)$$

$$0 = \partial_2(rG^{32}) + \partial_1(rG^{31}) + \partial_4(rG^{34}) = -\partial_\phi H_r + \partial_r(rH_\phi) - \partial_t(rE_z)$$

$$0 = \partial_1(rG^{21}) + \partial_3(rG^{23}) + \partial_4(rG^{24}) = -\partial_r H_z + \partial_z H_r - \partial_t E_\phi \quad (A.4)$$

which one can recognize at once as the Maxwell equations in Eq. (7b).

Appendix B: Metric of the Frenet-Serret rotating frame

We start with the metric of the natural cylindrical frame, in which $x^1 = r$, $x^2 = \phi$, $x^3 = z$, and $x^4 = t$:

$$ds^2 = dx_4^2 - (dx_1^2 + x_1^2 dx_2^2 + dx_3^2) \quad (B.1)$$

Using the coordinates $X^1 = R$, $X^2 = \Phi$, $X^3 = Z$, and $X^4 = T$ in the rotating frame, the relations in Eq. (14) between the coordinates in these two frames become, with ß = ΩX_1,

$$x^1 = X^1, x^3 = X^3 \quad (B.2)$$

$$x^4 = \cosh ßX^4 - \sinh ßX^1X^2, x^2 = \cosh ßX^2 - \sinh ßX^4/X^1 \quad (B.3)$$

A simple calculation gives

$$dx^4 = \cosh ßdX^4 - X^1 \sinh ßdX^2 - AdX^1 \quad (B.4)$$

$$A = ß\sinh ßX^4/X^1 + ß\cosh ßX^2 + \sinh ßX^2 \quad (B.4a)$$

and since $x^1 = X^1$:

$$x^1dx^2 = -\sinh ßdX^4 + X^1 \cosh ßdX^2 - BdX^1 \quad (B.5)$$

$$B = ß\sinh ßX^2 + ß\cosh ßX^4/X^1 - \sinh ßX^4/X^1 \quad (B.5a)$$

Henceforth, we write x_μ , X_μ instead of x^μ , X^μ (μ = 1,2,3,4) to avoid confusion with exponents. Then, we get from Eq. (B.4) and Eq. (B.5)

$$dx_4 - x_1dx_2 = exp ß(dX_4 - X_1dX_2) - (A-B)dX_1, dx_4 + x_1dx_2 = exp(-ß)(dX_4 + X_1dX_2) - (A+B)dX_1, \quad (B.6)$$

so that

$$dx_4^2 - x_1^2 dx_2^2 = dX_4^2 - X_1^2 dX_2^2 - (B^2 - A^2)dX_1^2 - 2(A \sinh ß + B \cosh ß)dX_1 dX_4 -$$
$$2(A \cosh ß + B \sinh ß)X_1 dX_1 dX_2 \quad (B.7)$$

Substituting Eq. (B.2) and Eq. (B.7) into Eq. (B.1) yields the metric of the rotating frame:

$$ds^2 = dX_4^2 - dX_3^2 - X_1^2 dX_2^2 - (1 + B^2 - A^2)dX_1^2 - 2(A \sinh ß + B \cosh ß)dX_1 dX_4 -$$
$$2(A \cosh ß + B \sinh ß)X_1 dX_1 dX_2 \quad (B.8)$$

Appendix C: Maxwell's equations in a rotating frame

According to Eq. (19), we have $|g|^{1/2}$ = Rq, with q = $|5(A^2 - B^2)$ -1$]^{1/2}$, so the Maxwell equations in Eq. (12) become

$$\partial_\nu (qRG^{\mu\nu}) = 0 \quad (C.1)$$

where ∂_ν is the covariant derivative vector

$$\partial_1 = \partial_R, \partial_2 = \partial_\Phi, \partial_3 = \partial_Z, \partial_4 = \partial_T \quad (C.2)$$

and $G^{\mu\nu}$ is the tensor in Eq. (9b), with r denoted by R, so that

$$RG^{12} = H_z, RG^{13} = -RH_\phi, RG^{23} = H_r, RG^{14} = -RE_r, RG^{24} = -E_\phi, RG^{34} = -RE_z \quad (C.3)$$

Then, proceeding as in Eq. (A.4), we get from Eq. (C.1), taking into account Eq. (C.2) and Eq. (C.3), the second set of Maxwell equations in the rotating frame:

$$0 = \partial_1(qRG^{41}) + \partial_2(qRG^{42}) + \partial_3(qRG^{43}) = -\partial_R(qRE_r) - \partial_\Phi(qE_\phi) - \partial_Z(qRE_z)$$
$$0 = \partial_2(qRG^{12}) + \partial_3(qRG^{13}) - \partial_4(qRG^{14}) = \partial_\Phi(qH_z) - \partial_Z(qRH_\phi) - \partial_T(qRE_r)$$
$$0 = \partial_2(qRG^{32}) + \partial_1(qRG^{31}) - \partial_4(qRG^{34}) = -\partial_\Phi(qH_r) + \partial_R(qRH_\phi) - \partial_T(qRE_z)$$
$$0 = \partial_1(qRG^{21}) + \partial_3(qRG^{23}) - \partial_4(qRG^{24}) = -\partial_R(qH_z) + \partial_Z(qH_r) - \partial_T(qE_\phi) \quad (C.4)$$

Appendix D: Power series expansion

We seek the solution of the equation

$$(r\partial_r^2 + \partial_r + k^2 r + \Omega r \partial_r)\Psi^\dagger(r) = 0 \quad (D.1)$$

in the form of a power series expansion limited to the first five terms :

$$\Psi^\dagger(r) = a_0 + a_1 r + a_2 r^2 + a_3 r^3 + a_4 r^4 \quad (D.2)$$

A simple calculation supplies the relations

$$\partial_r \Psi^\dagger = a_1 + 2a_2 r + 3a_3 r^2 + 4a_4 r^3$$

$$r\partial_r^2\Psi^\dagger = 2a_2 r + 6a_3 r^2 + 12a_4 r^3$$

$$k^2 r\Psi^\dagger = k^2 r(a_0 + a_1 r + a_2 r^2 + a_3 r^3)$$

$$\Omega r\partial_r\Psi^\dagger = \Omega r(a_1 + 2a_2 r + 3a_3 r^2 + 4a_4 r^3) \quad (D.3)$$

Substituting Eq. (D.3) into Eq. (D.1) and nullifying the coefficients of r^n, we find

$$a_1 = 0 \Rightarrow a_1 = 0$$

$$4a_2 + a_0 k^2 + \Omega a_1 = 0 a_2 = -a_0 k^2/4$$

$$9a_3 + a_1 k^2 + 2\Omega a_2 = 0 a_3 = a_0 k^2 \Omega/18$$

$$16a_4 + a_2 k^2 + 3\Omega a_3 = 0 a_4 = a_0 k^4/64 - a_0 k^2\Omega^2/96 \quad (D.4)$$

So, finally,

$$\Psi^\dagger(r) = a_0 - a_0 k^2 r^2/4 + a_0 k^2\Omega r^3/18 + a_0(k^4/64 - k^2\Omega^2/96)r^4 \quad (D.5)$$

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