Turk J Elec Eng & Comp Sci, Vol.18, No.5, 2010, © TÜBİTAK
doi:10.3906/elk-0906-3

Fractional order sliding mode control with reaching law approach

Mehmet Önder EFE
TOBB Economics and Technology University, Department of Electrical and Electronics Engineering, Söğütözü, Ankara-TURKEY
e-mail: onderefe@ieee.org

Abstract

Fractional order sliding mode control is studied in this paper. The control objectives are achieved by adopting the reaching law approach of sliding mode control. The main contribution of this work is to show that the philosophy of integer order sliding mode control is valid also for the systems represented by fractional order operators. A sufficient condition and its implications for stability are given. Matched and unmatched uncertainties are studied. The attractor nature of the switching manifold is analyzed together with a stable sliding subspace design condition. The claims are justified through a set of simulations and the results obtained are found promising.

Key Words: *Sliding mode control, fractional order control, reaching law approach*

1. Introduction

The fractional order derivative has its inception in an exchange of letters between L′Hôpital and Leibniz in 1695. The question was what would happen if the order of a derivative is not an integer. Though the use of non-integer orders in systems theory is not a new concept, the last two decades has witnessed many successful applications of fractional order differentiation and integration, i.e. *differintegration*. The field of automatic control systems is influenced from this *new* perspective of operators and significant number of contributions was reported on the fractional order variants of Proportional Integral Derivative (PID) controllers. Many approaches to feedback control theory have been reformulated within the context of fractional order control, [1–3]. In spite of the possibility of adapting the techniques of integer order to fractional order case, the fact that the mathematical representation of some processes, e.g. viscoelasticity, electrochemistry, multi pole magnetism, membrane conductance in biology, heat flows, fractional capacitors and lossy transmission lines, are involved with the fractional order derivatives, indicating that the processing of information contained in the *fractional order* appropriately may be unavoidable, [2].

In essence, just like the derivative operator for integer order denoted by $\mathbf{D}=\mathrm{d}/\mathrm{d}t$, the fractional order differintegration operator is denoted by $\mathbf{D}^{\beta}=\mathrm{d}^{\beta}/\mathrm{d}t^{\beta}$ where $\beta \in \Re$. In [1–3], the calculus of fractional order operators and the relevance of systems theory are investigated thoroughly with all preliminary facts.

In Oldham and Spanier [1], the fundamental mathematical aspects are focused on, whereas Podlubny [2] presents the differential equations of fractional orders and Das [3] discusses the concept with the perspective of an engineer.

Considering the fractional order system within the context of feedback control, some fundamental definitions and analogies are presented in [4–5], which consider both the continuous and the discrete time cases. Stabilization of a fractional order system with a fractional order PID is presented in [6], the parameter selection issues utilizing the rules inspired from the Ziegler-Nichols approach is presented in [7]. Time domain realizations of fractional order systems are studied in [8] and finite impulse response type differentiators are reported in [9]. The fractional order counterpart of the Kalman filter, a critically important tool in automatic control systems, is discussed in [10]. In addition to these resources, the principal constituents of linear systems theory such as controllability, observability, the computation of the state transition matrix, stability conditions, steady state performance and frequency response analysis are all addressed in [11]. Observer based control system design is considered in [12].

It is possible to extend the cited literature; however, the overall view stresses that the results are mainly applied to linear control concepts. Although very elegant results have been deduced so far, the field of automatic control with the fractional order system and controllers is an active research area to which the current paper aims to contribute.

Sliding mode control is implemented with the reaching law approach and is shown that the derived control law forces the emergence of the sliding regime along with the sliding manifold followed by a reaching mode. The choice of sliding mode control technique is deliberate as there is a significant volume of research on this particular robust control approach. An introductory work on the application of this technique in fractional order case is reported in [13], where the double integrator is considered as the plant under control. In [14], the classical integer order reaching condition is checked when the switching function is formed by the output of a fractional order PID module. Output feedback based sliding mode control is outlined in [15], and a fractional integral form of switching function is considered in [16] to obtain integer order expressions to explain stability. Solution of the state equation is built and conclusions involving Mittag-Leffler function are derived for stability, [16].

Delavari et al. consider an integer order plant and propose a switching function containing fractional order derivative in the velocity terms, [17], where classical reaching law is obtained after mathematical manipulations and the stability is explained utilizing the classical Lyapunov formalism. In [18–21], the integer order sliding mode is achieved via a number of fractional order adaptation laws operating on neural and fuzzy models, and in [22], preliminary results of the reaching law based fractional order control are considered on linear systems. The current study extends the concept with an in depth discussion of the attractive nature of the sliding subspace and presents a simple rule to check the attractiveness of the switching subspace without requiring manipulations to transform the fractional order switching dynamics to integer order domain. Having this picture in front, it is beneficial to describe the contributions of this paper as detailed below:

- The implementation of reaching law approach for sliding mode control with an illustrative example involving matched and unmatched uncertainties,

- The stability of the reaching dynamics characterized by the constant rate reaching $\mathbf{D}^{\beta}\sigma = -ksgn(\sigma)$ with $k > 0$, and constant plus proportional rate reaching $\mathbf{D}^{\beta}\sigma = -ksgn(\sigma) - p\sigma$, with $k,\ p > 0$, with $\mathbf{D}$ being the differentiation operator.
- Stability conclusion based on the definition of fractional differentiation.

Clearly, these will make it possible to postulate stabilizing closed loop control laws for fractional order *nonlinear* systems.

This paper is organized as follows. The second section summarizes the fractional order systems and the necessary definitions. Section three describes the design of fractional order sliding mode control. Observer design is presented in the fourth section; the issues of numerical realization are described in the section five. Two design examples are studied in the sixth section and concluding remarks are given at the end of the study.

2. A Summary of fractional order systems for feedback control

Caputo's definition of the fractional order differentiation is

$$\frac{d^{\beta}}{dt^{\beta}}\sigma(t) = \frac{1}{\Gamma(1-\beta)}\int_0^t \frac{\mathbf{D}\sigma(\tau)}{(t-\tau)^{\beta}}d\tau, \tag{1}$$

where we choose $0 < \beta < 1$ and β is the order of the differentiation. According to this definition, let m be an integer and $m-1 < \beta < m$ is satisfied. With such a value of m, the β-th order derivative of a function of time, say $\sigma(t)$, has the Laplace transform

$$\int_0^{\infty} e^{-st}\mathbf{D}^{\beta}\sigma(t)dt = s^{\beta}S(s) - \sum_{k=0}^{m-1} s^{\beta-k-1}\mathbf{D}^{(k)}\sigma(0), \tag{2}$$

where $\Gamma(\beta) = \int_0^t e^{-t}t^{\beta-1}dt$ is the Gamma function and $S(s) := \int_0^{\infty} e^{-st}\sigma(t)dt$.

From a control engineer's perspective, if a system is at rest initially, i.e. all initial conditions are zero, the operator $\mathbf{D}^{\beta}$ acting in time domain has a counterpart s^{β} in s-domain and the transfer function of a system described by a fractional order differential equation

$$(a_n\mathbf{D}^{\alpha_n} + a_{n-1}\mathbf{D}^{\alpha_{n-1}} + \cdots + a_1\mathbf{D}^{\alpha_1} + a_0)y(t) = (b_m\mathbf{D}^{\beta_m} + b_{m-1}\mathbf{D}^{\beta_{m-1}} + \cdots + b_1\mathbf{D}^{\beta_1} + b_0)u(t) \tag{3}$$

can be obtained as given by

$$\frac{Y(s)}{U(s)} = \frac{b_m s^{\beta_m} + b_{m-1}s^{\beta_{m-1}} + \cdots + b_1 s^{\beta_1} + b_0}{a_n s^{\alpha_n} + a_{n-1}s^{\alpha_{n-1}} + \cdots + a_1 s^{\alpha_1} + a_0}, \tag{4}$$

where $a_k,\ b_k \in \Re$ and $\alpha_k,\ \beta_k \in \Re^+$. Similarly, one could define the affine nonlinear systems of fractional order in state space as

$$\mathbf{D}^{\beta}\mathbf{x} = \mathbf{f}(\mathbf{x}) + \mathbf{g}(\mathbf{x})u, \tag{5}$$

where u is the control input, $\mathbf{f}$ and $\mathbf{g}\neq\mathbf{0}$ are the vector functions of the system state $\mathbf{x}$.

When the system under interest is linear, as in

$$\mathbf{D}^{\beta}\mathbf{x} = \mathbf{A}\mathbf{x} + \mathbf{B}u, \tag{6}$$

$$y = \mathbf{C}\mathbf{x} + Du, \tag{7}$$

the transfer function characterizing the relation between $Y(s)$ and $U(s)$ is

$$H(s) = \mathbf{C}(s^{\beta}\mathbf{I} - \mathbf{A})^{-1}\mathbf{B} + D. \tag{8}$$

Solution to the homogeneous case (u=0) is obtained as

$$\mathbf{x}(t) = \left(\sum_{k=0}^{\infty}\frac{\mathbf{A}^{k}t^{k\beta}}{\Gamma(1+k\beta)}\right)\mathbf{x}(0) = E_{\beta}(\mathbf{A}t^{\beta})\mathbf{x}(0) = \mathbf{\Phi}(t)\mathbf{x}(0), \tag{9}$$

in which $E_{\beta}(\mathbf{A}t^{\beta})$ is the Mittag-Leffler function defined as $E_{\beta}(\mathbf{A}t^{\beta}) := \sum_{k=0}^{\infty}\frac{(\mathbf{A}t^{\beta})^{k}}{\Gamma(1+\beta k)}$ and denoted by $\Phi(t)$, [11]. Full solution of the fractional state equation in (6) and the output equation in (7) is

$$y(t) = \mathbf{C}\mathbf{\Phi}(t-t_0)\mathbf{x}(t_0) + \mathbf{C}\int_{t_0}^{t}\mathbf{\Phi}(t-\tau)\mathbf{B}u(\tau)d\tau + Du(t). \tag{10}$$

Controllability and observability conditions are similar to the integer-order case and are, respectively, given by

$$\mathbf{W_c} = \begin{pmatrix} \mathbf{B} & \mathbf{AB} & \cdots & \mathbf{A}^{n-1}\mathbf{B} \end{pmatrix},\ \text{rank}(\mathbf{W}_c) = n \tag{11}$$

$$\mathbf{W_o} = \begin{pmatrix} \mathbf{C} \\ \mathbf{CA} \\ \vdots \\ \mathbf{CA}^{n-1} \end{pmatrix},\ \text{rank}(\mathbf{W}_o) = n. \tag{12}$$

Lastly in this section, it is useful to define the stability conditions for fractional order dynamic system representations. Denoting λ_i as an eigenvalue of the matrix $\mathbf{A}$, the system in (6)–(7) is said to be stable if the condition

$$|\arg(\lambda_i)| > \beta\frac{\pi}{2} \tag{13}$$

is satisfied by all eigenvalues of $\mathbf{A}$. For the transfer function representation in (8), $\lambda_i s$ correspond to the poles and the same condition applies for stability.

It is straightforward to see that in the integer order case ($\beta = 1$), the stability condition above describes the open left half s-plane for the stability. An in depth discussion on these issues can be found in [11].

3. Fractional order sliding mode control

Consider the n-th order fractional dynamic system given in (5) and define the switching function as

$$\sigma = \mathbf{\Lambda}(\mathbf{x} - \mathbf{r}), \tag{14}$$

where Λ is a design parameter making the sliding manifold defined by $\sigma = 0$ a stable subspace whose stability can be determined by using (13). Practically this means that the nominal plant model is a linear while the process is indeed nonlinear. Let $\mathbf{r}$ be the vector of differentiable command signals and the reaching law approach is to obtain $\mathbf{D}^{\beta}\sigma = -ksgn(\sigma)$ with $k>0$. When $\beta = 1$, this corresponds to $\dot{\sigma} = -ksgn(\sigma)$, and ensures $\sigma\dot{\sigma} < 0$ if $\sigma \neq 0$. Clearly this is the time derivative of a Lyapunov function $V = \sigma^2/2$ and the physical meaning of enforcing such a subdynamics is to render the sliding manifold an attractor and, once the error vector gets trapped to it, the motion thereafter takes place in the vicinity of the sliding hypersurface. Before generalizing such a result, one has to prove that the mechanism works also for the cases where the order of differentiation is not an integer. To show this, differentiate $\mathbf{D}^{\beta}\sigma = -ksgn(\sigma)$ at the order $-\beta$, this would leave σ alone, and differentiate at order unity to obtain $\dot{\sigma}$. These steps are given as

$$\mathbf{D}^1\left(\mathbf{D}^{-\beta}\left(\mathbf{D}^{\beta}\sigma\right)\right) = -k\mathbf{D}^1\left(\mathbf{D}^{-\beta}sgn(\sigma)\right) \tag{15}$$

and the resulting expression is

$$\dot{\sigma} = -k\mathbf{D}^{1-\beta}sgn(\sigma). \tag{16}$$

Since $0<\beta<1$, $sgn\left(\mathbf{D}^{1-\beta}sgn(\sigma)\right) = sgn(\sigma)$ and forcing $\mathbf{D}^{\beta}\sigma = -ksgn(\sigma)$ makes the locus described by $\sigma = 0$ a global attractor. To demonstrate this, the reaching dynamics $\mathbf{D}^{\beta}\sigma = -ksgn(\sigma)$ is solved numerically and two sample trajectories are shown within the axes σ and $\dot{\sigma}$. This is deliberate as it is straightforward to interpret the prescribed motion in the integer order axis settings. In Figure 1, the solution of $\mathbf{D}^{\beta}\sigma = -ksgn(\sigma)$ is shown for $\beta = 0.5$, $\sigma(0) = 1$ and $\sigma(0) = -1$. Clearly, the initial push toward the switching manifold is excessive and it gradually decreases as σ gets closer to zero.

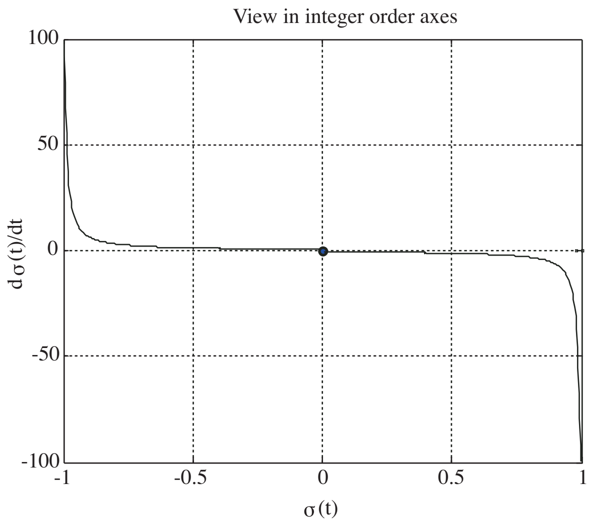

Figure 1. For $\mathbf{D}^{\beta}\sigma = -ksgn(\sigma)$, reaching the sliding manifold from both sides.

It is straightforward to demonstrate that choosing $\mathbf{D}^{\beta}\sigma = -ksgn(\sigma) - p\sigma$ with $p>0$ has the same effect in the reaching dynamics of that in integer order design. Since $p\sigma = p\left|\sigma\right| sgn(\sigma)$, we have the following relation in between the quantity $\dot{\sigma}$ and sgn(σ):

$$\dot{\sigma} = -k\mathbf{D}^{1-\beta} sgn(\sigma) - p\mathbf{D}^{1-\beta}\left(\left|\sigma\right| sgn(\sigma)\right) = -\mathbf{D}^{1-\beta}\left(\left(k + p\left|\sigma\right|\right) sgn(\sigma)\right). \tag{17}$$

Due to the relation $sgn\left(\mathbf{D}^{1-\beta} sgn(\sigma)\right) = sgn(\sigma)$, the reaching dynamics governed by the above expression will create a stronger push from both sides of the switching manifold. In other words, the attraction strength of the switching manifold is higher for any σ with $p \neq 0$ than that with $p=0$. This is shown comparatively in Figure 2.

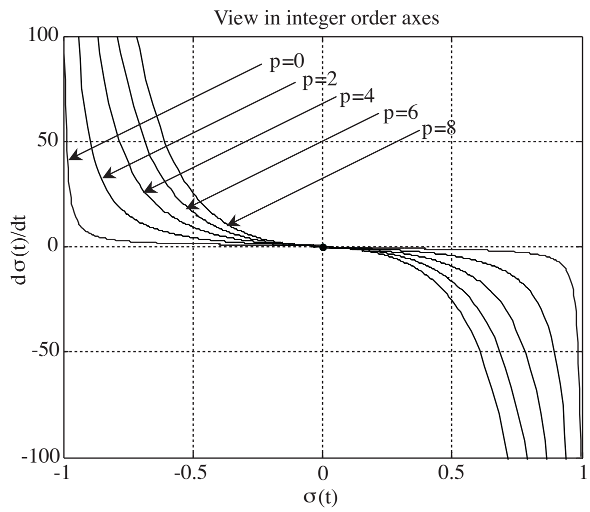

Figure 2. For $\mathbf{D}^{\beta}\sigma = -ksgn(\sigma) - p\sigma$, reaching the sliding manifold from both sides.

For a fixed σ, the increasing values of p creates larger $\dot{\sigma}$ values and this leads to quicker approach to the locus characterized by $\sigma = 0$.

If one chooses a Lyapunov function such as $V = \sigma^2/2$, and takes the β-th order derivative of it, according to the Leibniz's rule of differentiation, the result is $\mathbf{D}^{\beta}V = \sum_{k=0}^{\infty}\frac{\Gamma(1+\beta)}{\Gamma(1+k)\Gamma(1-k+\beta)}\mathbf{D}^{k}\sigma\mathbf{D}^{\beta-k}\sigma$, which requires the manipulation of infinitely many terms. This clearly does not allow inferring the attractiveness of $\sigma = 0$ deduced either from $\sigma\left(\mathbf{D}^{\beta}\sigma\right) < 0$ or from $\mathbf{D}^{\beta}\sigma = -ksgn(\sigma) - p\sigma$.

Due to the definition in (1), we have the equality $\sigma\left(\mathbf{D}^{\beta}\sigma\right) = \frac{\sigma}{\Gamma(1-\beta)}\int_{0}^{t}\frac{\mathbf{D}\sigma(\tau)}{(t-\tau)^{\beta}}d\tau$. This relation stipulates that for $\sigma > 0$, $\mathbf{D}\sigma$ (the first derivative of σ) must be negative to have $\sigma\left(\mathbf{D}^{\beta}\sigma\right) < 0$, or alternatively, for $\sigma < 0$, $\mathbf{D}\sigma$ (the first derivative of σ) must be positive to have $\sigma\left(\mathbf{D}^{\beta}\sigma\right) < 0$. Therefore for closed loop stability forcing $\sigma\left(\mathbf{D}^{\beta}\sigma\right) < 0$ via an appropriately designed control law is sufficient. This explanation of inferring the stability shows that the stability requirement $\sigma\dot{\sigma} < 0$ (or $\sigma\mathbf{D}\sigma < 0$) of the integer order design is forced naturally if $\sigma\left(\mathbf{D}^{\beta}\sigma\right) < 0$ is forced. This is a major contribution of the current paper. In short, a control law ensuring $\sigma\left(\mathbf{D}^{\beta}\sigma\right) < 0$ also ensures $\sigma\dot{\sigma} < 0$ and the closed loop system becomes stable.

Having the result supported graphically in mind, taking the β-th order derivative of (14) yields

$$\mathbf{D}^{\beta}\sigma = \mathbf{\Lambda}(\mathbf{D}^{\beta}\mathbf{x} - \mathbf{D}^{\beta}\mathbf{r}) = \mathbf{\Lambda}(\mathbf{f}(\mathbf{x}) + \mathbf{g}(\mathbf{x})u - \mathbf{D}^{\beta}\mathbf{r}). \tag{18}$$

Equating the above expression to $-ksgn(\sigma) - p\sigma$ and solving for u lets us have the control signal

$$u = \frac{-\mathbf{\Lambda}\mathbf{f}(\mathbf{x}) + \mathbf{\Lambda}\mathbf{D}^{\beta}\mathbf{r} - ksgn(\sigma) - p\sigma}{\mathbf{\Lambda}\mathbf{g}(\mathbf{x})}, \tag{19}$$

where it is necessary to have $\mathbf{\Lambda}\mathbf{g}(\mathbf{x}) \neq 0$. With such a control law deduced from a nominal model, what would be the response if the model in (5) is a nominal representation of a plant containing uncertainties $\Delta\mathbf{f}(\mathbf{x})$ and $\Delta\mathbf{g}(\mathbf{x})$ such as

$$\mathbf{D}^{\beta}\mathbf{x} = (\mathbf{f}(\mathbf{x}) + \Delta\mathbf{f}(\mathbf{x})) + (\mathbf{g}(\mathbf{x}) + \Delta\mathbf{g}(\mathbf{x}))\,u? \tag{20}$$

Inserting (19) into (20) and computing $\mathbf{D}^{\beta}\sigma$ we get the dynamics

$$\mathbf{D}^{\beta}\sigma = -\left(1 + \frac{\mathbf{\Lambda}\Delta\mathbf{g}(\mathbf{x})}{\mathbf{\Lambda}\mathbf{g}(\mathbf{x})}\right)(ksgn(\sigma) + p\sigma) + \frac{\mathbf{\Lambda}\Delta\mathbf{g}(\mathbf{x})}{\mathbf{\Lambda}\mathbf{g}(\mathbf{x})}\mathbf{\Lambda}\left(\mathbf{D}^{\beta}\mathbf{r} - \mathbf{f}(\mathbf{x})\right) + \mathbf{\Lambda}\Delta\mathbf{f}(\mathbf{x}) \tag{21}$$

1. If there are no uncertainties, i.e. $\Delta\mathbf{f}(\mathbf{x})=\mathbf{0}$ and $\Delta\mathbf{g}(\mathbf{x})=\mathbf{0}$, then we have $\mathbf{D}^{\beta}\sigma = -ksgn(\sigma) - p\sigma$, which is desired to observe the sliding regime after hitting the sliding hypersurface.

2. If $\Delta\mathbf{g}(\mathbf{x})$ is zero and the columns of $\Delta\mathbf{f}(\mathbf{x})$ are in the range space of $\mathbf{g}(\mathbf{x})$, then $\mathbf{D}^{\beta}\sigma = -ksgn(\sigma) - p\sigma + \mathbf{\Lambda}\Delta\mathbf{f}(\mathbf{x})$. This case further requires the hold of the condition in (22) for maintaining $\sigma\left(\mathbf{D}^{\beta}\sigma\right) < 0$:

$$k > |\mathbf{\Lambda}\Delta\mathbf{f}(\mathbf{x})|\,. \tag{22}$$

3. If there are nonzero uncertainty terms, then (21) is valid and the designer needs to set k and p carefully to maintain the attractiveness of the subspace defined by $\sigma = 0$. The conditions needed to maintain $\sigma\left(\mathbf{D}^{\beta}\sigma\right) < 0$ are

$$\left|\frac{\mathbf{\Lambda}\Delta\mathbf{g}(\mathbf{x})}{\mathbf{\Lambda}\mathbf{g}(\mathbf{x})}\right| < 1 \tag{23}$$

$$\left(1 + \frac{\mathbf{\Lambda}\Delta\mathbf{g}(\mathbf{x})}{\mathbf{\Lambda}\mathbf{g}(\mathbf{x})}\right)k > \left|\frac{\mathbf{\Lambda}\Delta\mathbf{g}(\mathbf{x})}{\mathbf{\Lambda}\mathbf{g}(\mathbf{x})}\mathbf{\Lambda}\left(\mathbf{D}^{\beta}\mathbf{r} - \mathbf{f}(\mathbf{x})\right) + \mathbf{\Lambda}\Delta\mathbf{f}(\mathbf{x})\right|. \tag{24}$$

Here, we assume that columns of $\Delta\mathbf{f}(\mathbf{x})$ and $\Delta\mathbf{g}(\mathbf{x})$ are in the range space of $\mathbf{g}(\mathbf{x})$, i.e. the uncertainties are matched uncertainties. If the matching conditions are not satisfied, the closed loop performance will be degraded to some extent and the degree of this is determined by the functional details embodying the plant dynamics.

As a last issue, it is useful to emphasize that the existence of a finite time hitting is an open question in fractional order setting of the sliding mode control problem.

4. Observer design in fractional order

In general the system in (5) has an output equation $y = h(\mathbf{x})$ and if the state information is to be extracted from the output, then one needs the observer defined as

$$\mathbf{D}^{\beta}\hat{\mathbf{x}} = \mathbf{f}(\hat{\mathbf{x}}) + \mathbf{g}(\hat{\mathbf{x}})u + \mathbf{L}(y - \hat{y}) \tag{25}$$

$$\hat{y} = h(\hat{\mathbf{x}}). \tag{26}$$

The problem above is to design the vector gain $\mathbf{L}$ such that $\|\mathbf{x} - \hat{\mathbf{x}}\| \to 0$ emerges in some acceptable manner. There is not a formal way to describe a stable observer for an arbitrary triple of $\mathbf{f}$, $\mathbf{g}$ and $\mathbf{h}$. The design process for the nonlinear case is typically peculiar to the mathematical descriptions of these functions. For the linear system in (6)–(7), we have the observer

$$\mathbf{D}^{\beta}\hat{\mathbf{x}} = \mathbf{A}\hat{\mathbf{x}} + \mathbf{B}u + \mathbf{L}(y - \hat{y}) \tag{27}$$

$$\hat{y} = \mathbf{C}\hat{\mathbf{x}} + Du. \tag{28}$$

Defining $\xi := \mathbf{x} - \hat{\mathbf{x}}$ as the discrepancy between the system state and the observer's estimate, the dynamics of ξ is governed by $\mathbf{D}^{\beta}\xi = (\mathbf{A} - \mathbf{LC})\,\xi$. An appropriate choice of $\mathbf{L}$ via Bass-Gura or Ackermann formulas could place the eigenvalues of the matrix $\mathbf{A} - \mathbf{LC}$ such that the stability condition in (13) is met. Using the state information constructed by the observer as if it was the true state, the control law in (19) needs to be modified as

$$u = \frac{-\mathbf{\Lambda}\mathbf{f}(\hat{\mathbf{x}}) + \mathbf{\Lambda}\mathbf{D}^{\beta}\mathbf{r} - ksgn(\hat{\sigma}) - p\hat{\sigma}}{\mathbf{\Lambda}\mathbf{g}(\hat{\mathbf{x}})}, \tag{29}$$

where $\hat{\sigma} := \mathbf{\Lambda}\,(\hat{\mathbf{x}} - \mathbf{r})$ is the switching function defined by the use of $\hat{\mathbf{x}}$. It should here be noted that meeting the stability condition in (13) forces the observer states toward the system states at a rate characterized by $t^{-\beta}$, which is slower than any exponential convergence as highlighted by Das, [3]. However, the behavior for small t is very fast and this property strengthens the attractor nature of the sliding manifold by causing a fast reaching mode.

5. Issues of numerical realization

A fundamental problem with the fractional order systems is to realize the fractional differintegration operators in real time. Some results on this issue are reported in [8]. A frequently followed approach is to approximate these operators via integer order components as defined as

$$\mathbf{D}^{\beta} := s^{\beta} \approx K\frac{\prod_{k=1}^{N} 1 + s/w_{p_k}}{\prod_{k=1}^{N} 1 + s/w_{z_k}} \tag{30}$$

A widely used approximation is the Crone method approximating the s^{β} term as given above. Crone method adjusts the gain K such that, when w=1 rad/sec, the magnitude of the expression coincides to the 0 dB level. Here, N is the order of the approximation and the Crone algorithm determines the pole and zero locations in such a way that the approximation is optimum over the chosen frequency band. In the left subplots of Figure

3, the approximation order N is equal to 5 and in the right subplots, N is 38. The magnitude and phase plots are given and it is seen that as N increases a better fit is obtained at the cost of increasing the computational intensity. For both cases, the pole and zero locations prescribed by the Crone algorithm are marked along the frequency axis.

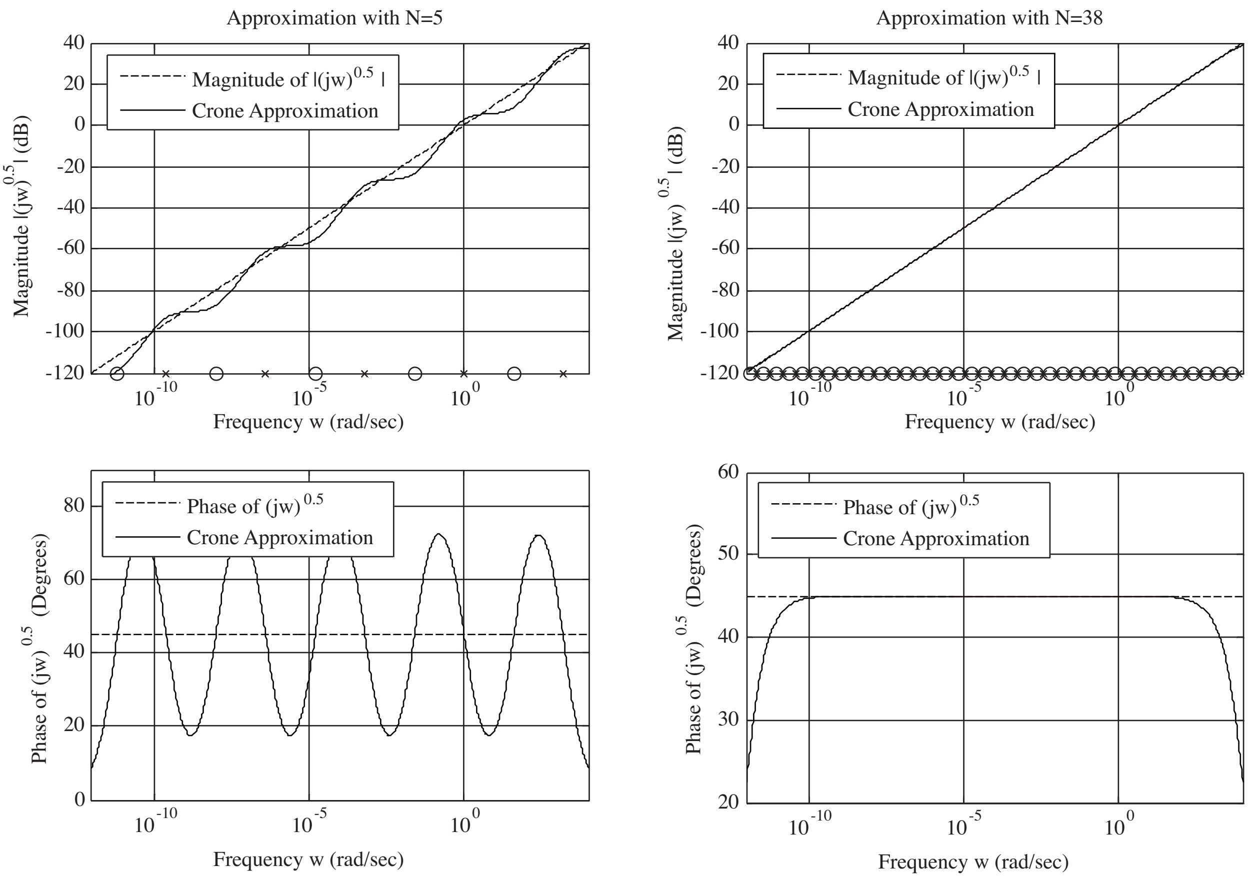

Figure 3. Left subplots: Crone approximation to the operator $\mathbf{D}^{\beta}$, $\beta = 0.5$ w_{min}=1et12 rad/sec, w_{max}=1e+4 rad/sec and N=5. Right subplots: Crone approximation to the operator $\mathbf{D}^{\beta}$, $\beta = 0.5$ w_{min}=1et12 rad/sec, w_{max}=1e+4 rad/sec and N=38.

In the literature, there are other approximations postulating alternative algorithms to distribute the poles and zeros; Carlson, Matsuda, high/low frequency continued fraction approximations are just to name a few. A detailed discussion is given in Valerio, [23].

6. Simulation studies

In the simulations, we consider a third order plant model given by (31), where $\beta = 0.5$. It is possible to consider other values of β, however, since $\beta = 0.5$ case is known as semidifferentiation, tables for this case are readily available, [1]. Further, choosing $\beta = 0.5$ makes the operator equally away from the no-differentiation case and first order differentiation, hence

$$\begin{aligned} \mathbf{D}^{\beta}x_1 &= x_2 + \Delta f_1 \\ \mathbf{D}^{\beta}x_2 &= x_3 + \Delta f_2 \\ \mathbf{D}^{\beta}x_3 &= -x_1 - 2x_2 - x_3 + \Delta f_3 + (1+\Delta g)\,u \end{aligned} \tag{31}$$

Compactly, we have the following state space representation:

$$\mathbf{D}^{\beta}\mathbf{x} = (\mathbf{A} + \Delta\mathbf{A})\,\mathbf{x} + (\mathbf{B} + \Delta\mathbf{B})\,u \tag{32}$$

$$\mathbf{y} = \mathbf{C}\mathbf{x}, \tag{33}$$

where we assume that the nominal system ($\mathbf{A}$, $\mathbf{B}$, $\mathbf{C}$) satisfies the controllability and observability conditions in (11) and (12), respectively, and we assume that we have the linear nominal part of the plant for designing the controller. In the simulations, we choose $\mathbf{A}$, $\mathbf{B}$ and $\mathbf{C}$ as

$$\mathbf{A} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & -2 & -1 \end{bmatrix},\ \mathbf{B} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix},\ \mathbf{C} = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}. \tag{34}$$

If the matrix $\mathbf{L}$ is chosen as $\mathbf{L} = [32\ 328\ 927]^T$, regions of stability are shown in Figure 4. This choice places the eigenvalues of **A-LC** to locations -10, -11 and -12 and these values were chosen arbitrarily. The only condition we consider in setting these eigenvalues is to obtain a reasonably fast observer response. The unstable region is the right of the lines shown in Figure 4.

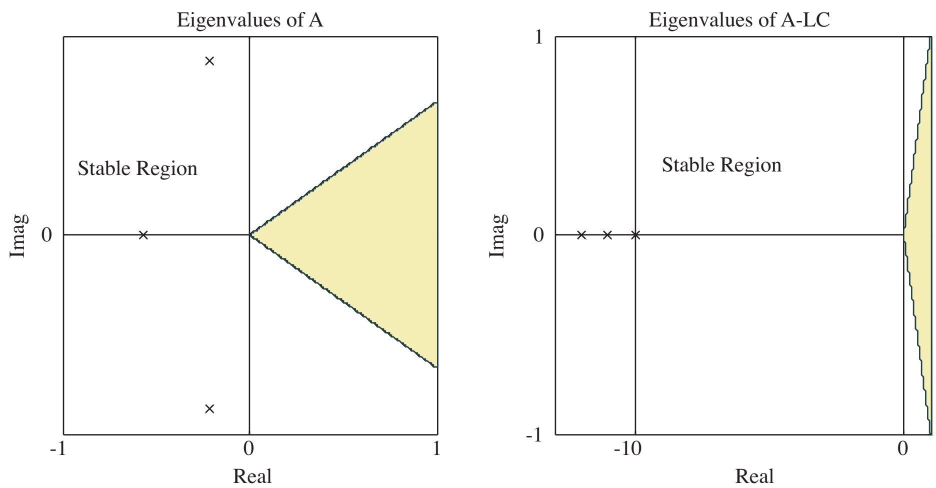

Figure 4. Left: The locations of the eigenvalues of **A.** Right: The locations of the eigenvalues of **A-LC** in the complex plane. The unstable region is shaded in both plots.

In the simulations we consider two cases. In the first set, the uncertainties are matched uncertainties, and performance has been analyzed. The second case considers the unmatched uncertainties, which are, by definition, not in the range space of $\mathbf{B}$.

6.1. Matched uncertainties

The uncertainty terms for the first case are described as in (31), where the columns of $\Delta\mathbf{A}$ and $\Delta\mathbf{B}$ are all in the span of $\mathbf{B}$, and we choose $k = 0.5$, p=0.5 and $\Lambda = [1\ 2\ 1]$, hence the uncertainty terms:

$$\Delta\mathbf{A} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0.1\sin 2\pi t & 0.2\sin 6\pi t & 0.15\sin 10\pi t \end{bmatrix}, \ \Delta\mathbf{B} = \begin{bmatrix} 0 \\ 0 \\ 0.02\sin 14\pi t \end{bmatrix}. \tag{35}$$

Note the parameters are perturbed and vary with time. Since $\mathbf{B}=[0\ 0\ 1]^T$, these perturbations are in the span of $\mathbf{B}$ and the uncertainties are matched uncertainties.

The results of the first set of simulations are given in Figures 5–7. A sinusoidal trajectory for the first state is chosen. The state information from the observer is used as if it was the true state knowledge, and the

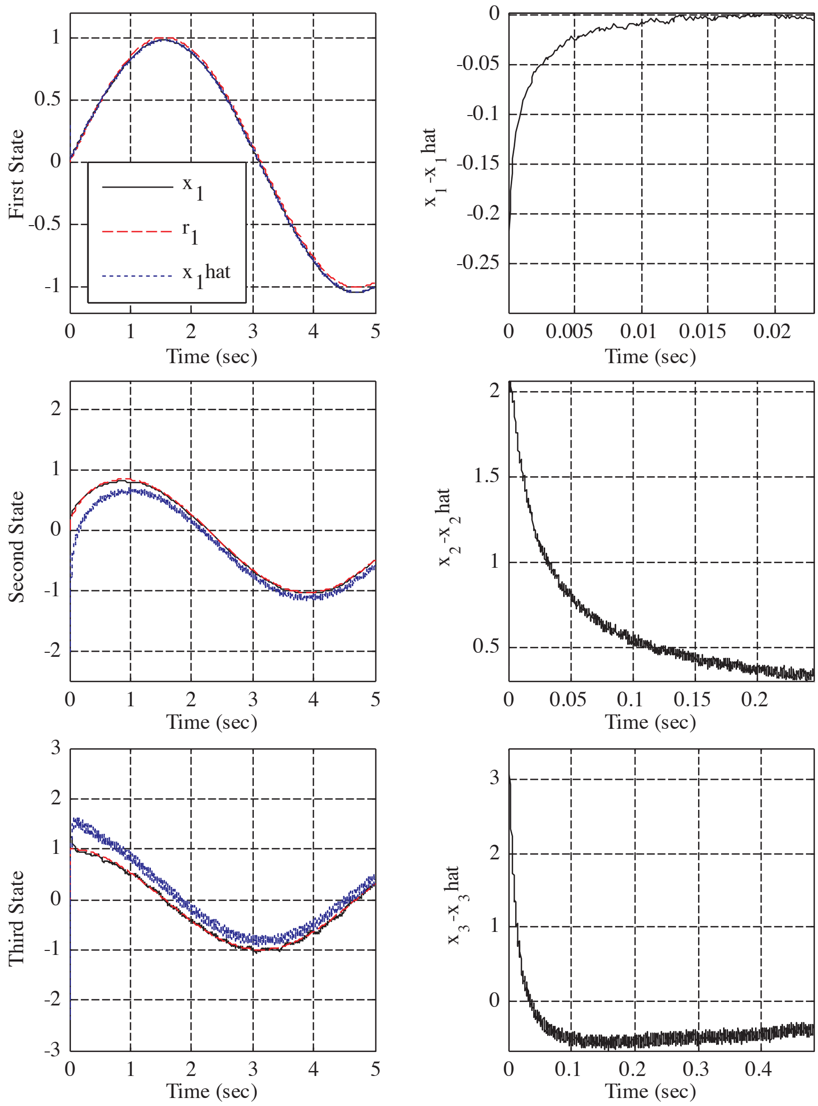

Figure 5. Left: System states, reference signal and the response of the observer. Right: The state estimation errors. The uncertainties are matched.

control law in (29) is implemented. It was observed from Figure 5 that a very fast transient is observed then a slow convergence emerged. Noted also by Matignon et al. [12], the observer catches the system states at a rate $t^{-\beta}$ characterized also by the Mittag-Leffler function $E_{\beta}(\mathbf{A}t^{\beta})$. Clearly such an approach should not be seen as an alternative to the exponential convergence, yet the fractional order mechanism truly works as seen also from the results in Figure 6. The phase space motion is depicted from four different viewpoints to illustrate the attracting nature of the switching subspace, the plane in this case. The error vector gets trapped to the sliding subspace and the motion thereafter takes place in the vicinity of this particular locus, which is stable by the design and the errors converge the origin. The applied control signal is depicted in Figure 7 and the switching nature of the control law is reflected to the signal $u(t)$ as fast fluctuations having reasonable magnitude. In sum, the control law in (29) with the observer designed according to the nominal linear model of the plant is able to display a certain degree of robustness against disturbances such as observation noise, nonzero initial conditions and time varying process parameters. The results in this sense are promising.

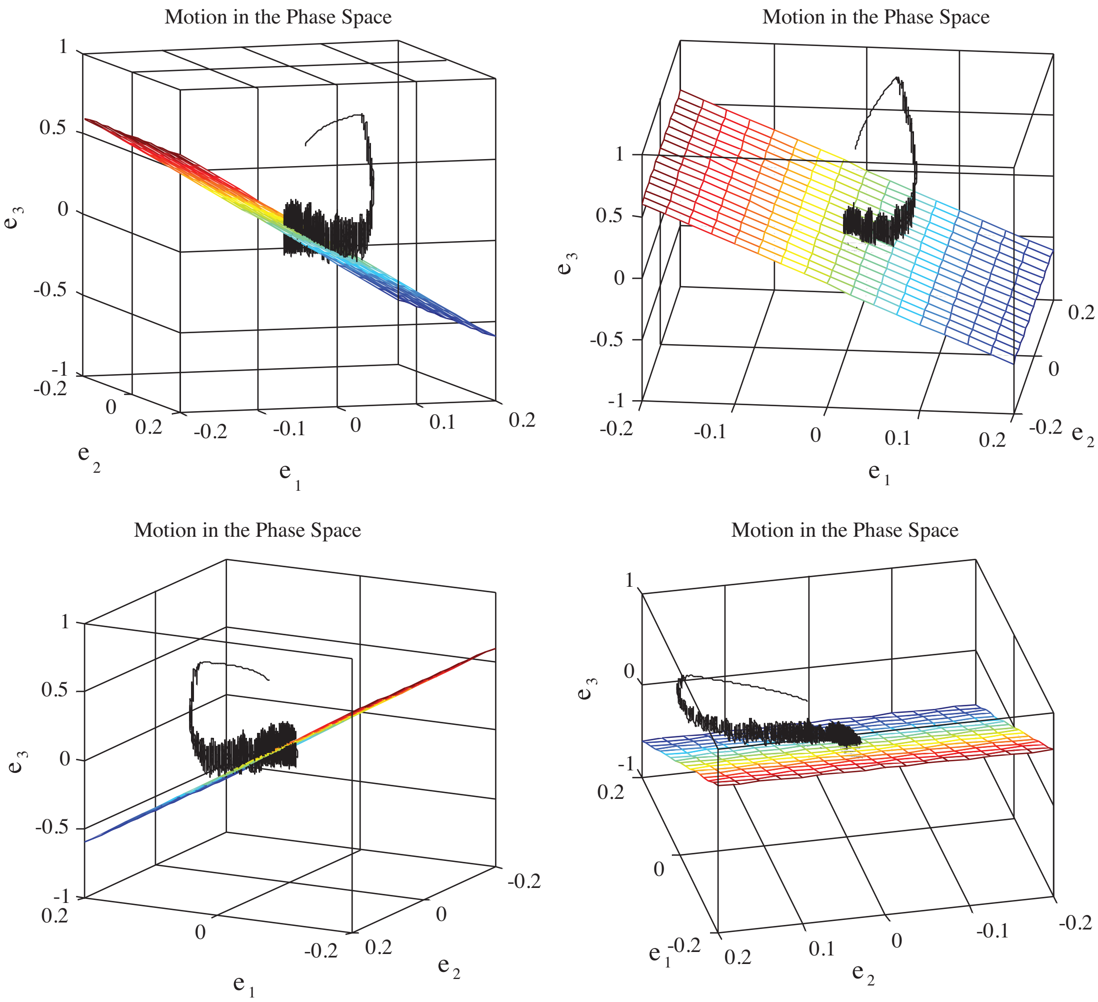

Figure 6. Phase space behavior from four different viewpoints. The uncertainties are matched.

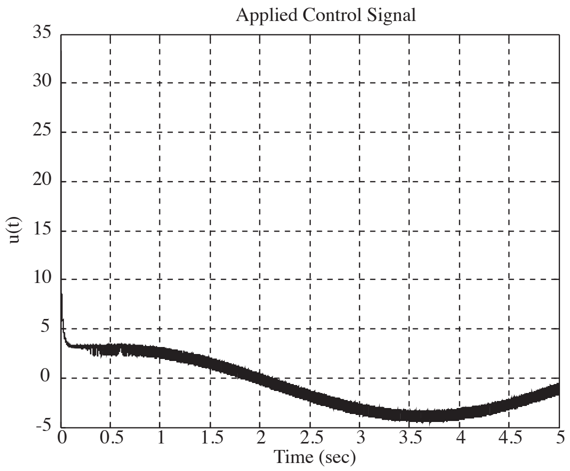

Figure 7. Applied control signal. The uncertainties are matched.

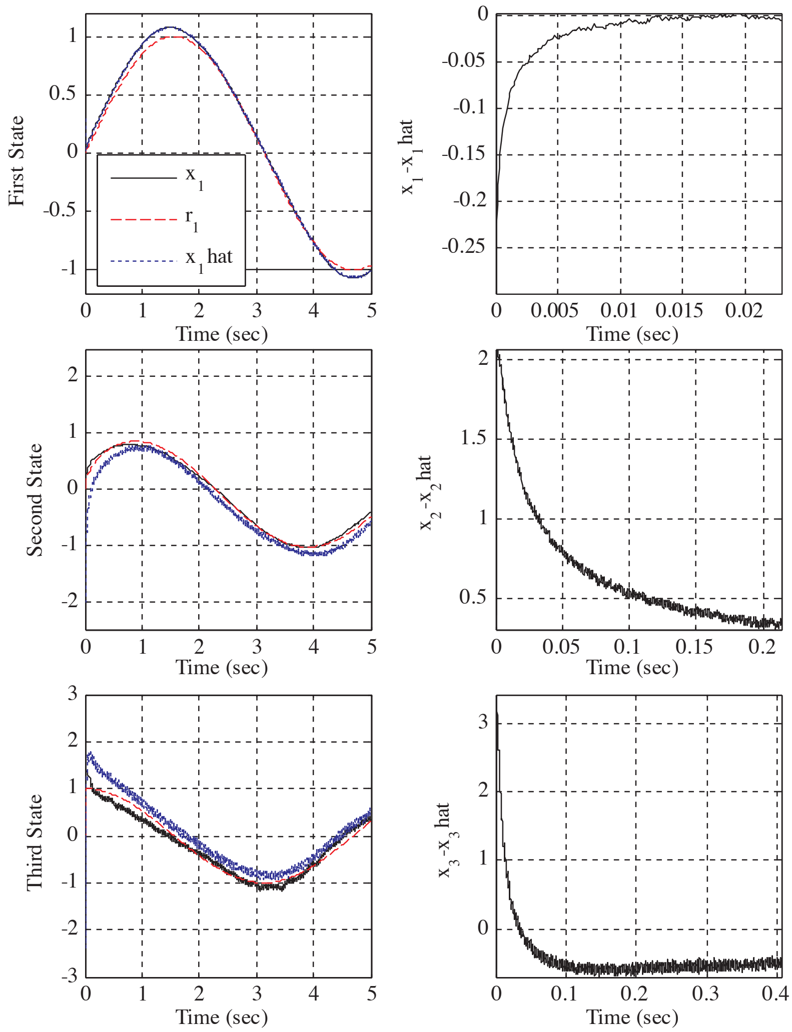

Figure 8. Left: System states, reference signal and the response of the observer. Right: The state estimation errors. The uncertainties are unmatched.

6.2. Unmatched uncertainties

In order to demonstrate the extent to which we could force the limits of robustness, we choose unmatched uncertainties described below and repeat the simulations. In this case, we choose p=1.5 and keep all other simulation conditions the same.

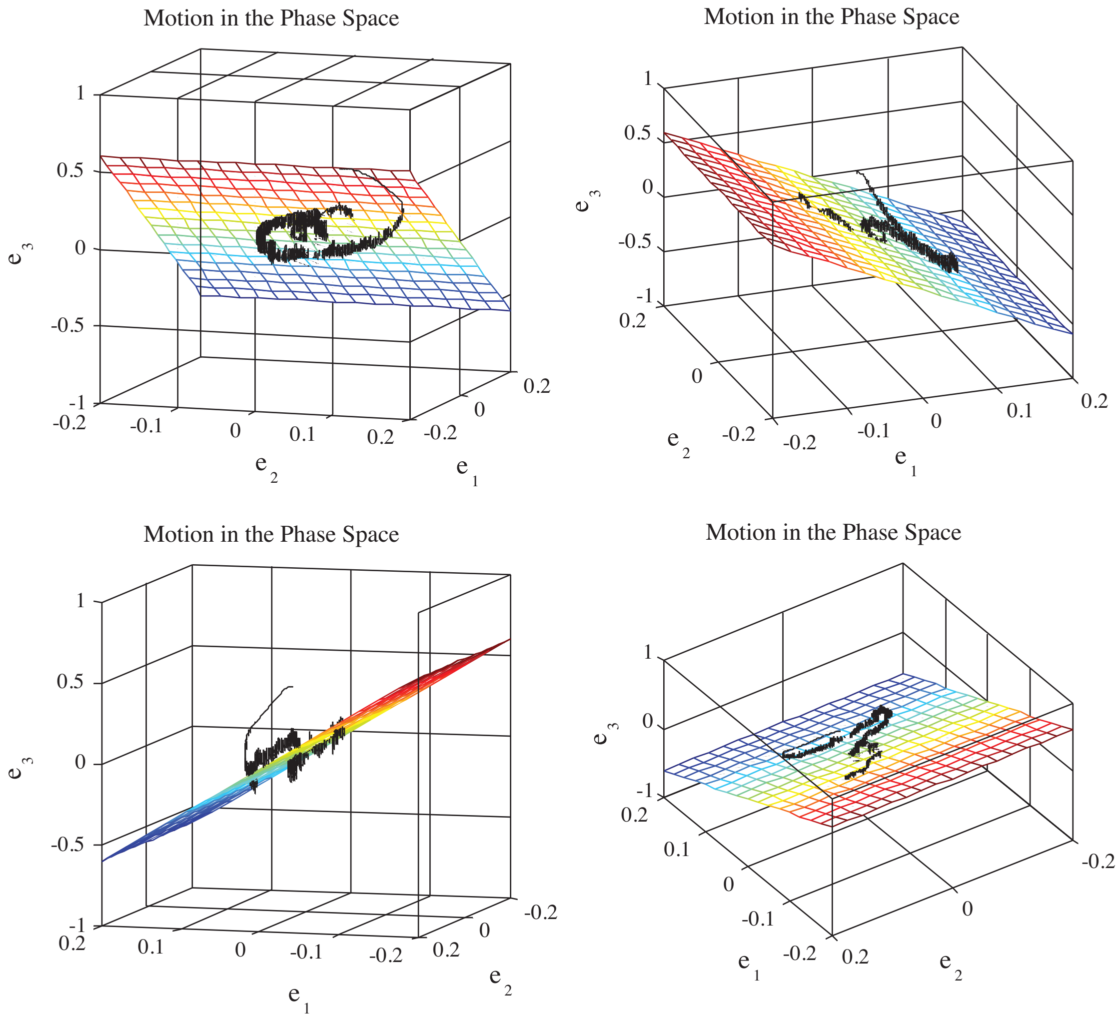

Figure 9. Phase space behavior from four different viewpoints. The uncertainties are unmatched.

$$\Delta\mathbf{A} = \begin{bmatrix} 0.1x_3 & 0.16x_1^2 & 0 \\ 0.13x_2^2 & 0 & 0.12x_2 \\ 0.02x_2 + 0.1\sin 2\pi t & 0.03x_3 + 0.2\sin 6\pi t & 0.14x_1 + 0.15\sin 10\pi t \end{bmatrix} \tag{36}$$

$$\Delta\mathbf{B} = \begin{bmatrix} 0 \\ 0 \\ 0.02\sin 14\pi t + 0.03x_1^2 x_2 x_3 \end{bmatrix}. \tag{37}$$

The results of the simulations are given in Figures 8–10. Cleary from the trajectories seen in the left column of Figure 8, some degradation in the performance is visible. States quickly come close to their desired values and they remain close to each other. The components of the observer error vector $\mathbf{x} - \hat{\mathbf{x}}$ have tendency toward zero and according to the four different viewpoints of the phase space motion, the error vector is attracted by the sliding manifold and it is guided toward some neighborhood of the origin. Since the uncertainties are unmatched, some deviations are visible in the result. Although there is not a visible change in the control signal shown in Figure 10, the closed loop control system is robust against the disturbances and uncertainties to a certain extent visible from the results.

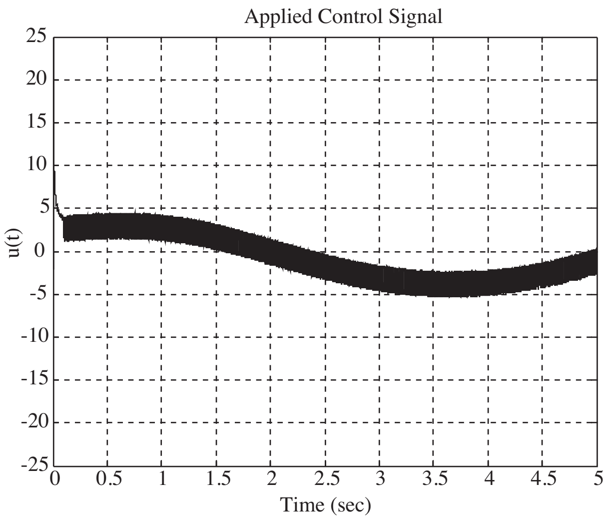

Figure 10. Applied control signal. The uncertainties are unmatched.

7. Conclusions

This paper considers the design of sliding mode control for fractional order systems. Since the system under control is represented via fractional order differintegration operators, the control law is likely to exploit the techniques of fractional order calculus. The goal of this work is not to claim that fractional order approach outperforms the integer order counterpart. Instead, the work demonstrates that the sliding mode control can be adapted to fractional order nonlinear dynamical systems and, robustness and invariance properties of sliding mode control are preserved provided that the matching conditions hold. Otherwise, the system performance is degraded to the extent determined by the effect of unmatched uncertainties. The contribution of this study is to demonstrate that reaching law approach forcing $\mathbf{D}^{\beta}\sigma = -ksgn(\sigma)$ in the closed loop creates an attracting subspace $\sigma = 0$; furthermore, choice of the reaching law as $\mathbf{D}^{\beta}\sigma = -ksgn(\sigma) - p\sigma$ further improves the reaching performance of the closed loop system. Since the Leibniz's rule for differentiation produces infinitely many terms, starting with a Lyapunov function is impractical, as its non-integer derivative will introduce infinitely many terms. Another contribution of the current work is to show that it is *sufficient* to have $\sigma\left(\mathbf{D}^{\beta}\sigma\right) < 0$ for stability in the closed loop and natural implication if this is shown to have $\sigma\dot{\sigma} < 0$. An observer based approach is followed and a process, whose nominal part is a linear system, is selected. The response with matched and unmatched uncertainties are studied and it is seen that the observer based closed loop sliding mode control system performs satisfactory. Based on the results obtained, the current paper advances the subject area towards the postulation of nonlinear control laws for fractional order systems.

Acknowledgments

This work was supported by TÜBİTAK, Grant No 107E137. The author gratefully acknowledges the facilities of the Unmanned Aerial Vehicles Lab. of TOBB ETU.

References

[1] K.B. Oldham, J. Spanier, The Fractional Calculus, Academic Press, 1974.

[2] I. Podlubny, Fractional Differential Equations, Elsevier Science & Technology Books, 1st Ed., 1998.

[3] S. Das, Functional Fractional Calculus for System Identification and Controls, Springer, 1st Ed., 2008.

[4] M.D. Ortigueira, "Introduction to Fractional Linear Systems. Part 1: Continuous Time Case," IEE Proceedings on Visual Image Signal Processing, v.147, n.1, pp.62-70, 2000.

[5] M.D. Ortigueira, "Introduction to Fractional Linear Systems. Part 2: Discrete Time Case," IEE Proc. Vis. Image Signal Processing, v.147, n.1, pp.71-78, 2000.

[6] S.E. Hamamci, "Stabilization Using Fractional Order PI and PID Controllers," Nonlinear Dynamics, v.51, no.1-2, pp.329-343, 2007.

[7] D. Valério, J. Sa da Costa, "Tuning of Fractional PID Controllers Ziegler-Nichols Type Rules," Signal Processing, v.86, pp.2771-2784, 2006.

[8] D. Valério, J. Sa da Costa, "Time Domain Implementation of Fractional Order Controllers," IEE Proc. Control Theory and Appls., v.152, n.5, pp.539-552, 2005.

[9] C.-C. Tseng, "Design of Fractional Order Digital FIR Differentiators," IEEE Signal Processing Letters, v.8, n.3, pp.77-79, 2001.

[10] D. Sierociuk, A. Dzielinski, "Fractional Kalman Filter Algorithm for the States, Parameters and Order of Fractional System Estimation," Int. J. Appl. Math. Comput. Sci., v.16, n.1, pp.129-140, 2006.

[11] B.M. Vinagre, C.A. Monje, A.J. Calderon, "Fractional Order Systems and Fractional Order Control Actions," 41st IEEE Int. Conf. on Decision and Control, Las Vegas, NV, 10-13 December, pp.15-38, 2002.

[12] D. Matignon, B. d'Andera-Novel, "Observer Based Controllers for Fractional Differential Systems," Int. Conf. on Decision and Control, San Diego, California, pp.4967-4972, 1997.

[13] B.M. Vinagre, A.J. Calderon, "On Fractional Sliding Mode Control," 7th Portuguese Conference on Automatic Control (CONTROLO'2006), Lisbon, Portugal, Sep. 11-13, 2006.

[14] A.J. Calderon, B.M. Vinagre, V. Feliu, "Fractional Order Control Strategies for Power Electronic Buck Converters," Signal Processing, v.86, pp.2803–2819, 2006.

[15] R. El-Khazali, "Output Feedback Sliding Mode Control of Fractional Systems," 12th IEEE International Conference on Electronics, Circuits and Systems, Dec. 11-14, pp.1-4, 2005.

[16] A. Si-Ammour, S. Djennoune, M. Bettayeb, "A Sliding Mode Control for Linear Fractional systems With Input and State Delays," Communications in Nonlinear Science and Numerical Simulation, v.14, pp.2310–2318, 2009.

[17] H. Delavari, R. Ghaderi, A. Ranjbar, S. Momani, "Fuzzy Fractional Order Sliding Mode Controller for Nonlinear Systems," Communications in Nonlinear Science and Numerical Simulation, v.15, pp.963–978, 2010.

[18] M.Ö. Efe, "A Fractional Order Adaptation Law for Integer Order Sliding Mode Control of a 2DOF Robot," New Trends in Nanotechnology and Fractional Calculus Applications, Ed. Z.B. Güvenç, D. Baleanu, J.A. Tenreiro Machado, 2010.

[19] M.Ö. Efe, "Fractional Fuzzy Adaptive Sliding Mode Control of a 2 DOF Direct Drive Robot Arm," IEEE Transactions on Systems, Man and Cybernetics Part B: Cybernetics, v.28, no.6, pp.1561-1570, December 2008.

[20] M.Ö. Efe, "ADALINE Based Robust Control in Robotics: A Riemann-Liouville Fractional Differintegration Based Learning Scheme," Soft Computing, v.13, no.1, pp.23-29, 2009.

[21] M.Ö. Efe and C. Kasnakoğlu, "A Fractional Adaptation Law for Sliding Mode Control," International Journal of Adaptive Control and Signal Processing, v.22, no.10, pp.968-986, December 2008.

[22] M.Ö. Efe, "Fractional Order Sliding Mode Controller Design for Fractional Order Dynamic Systems," New Trends in Nanotechnology and Fractional Calculus Applications, Ed. Z.B. Güvenç, D. Baleanu, J.A. Tenreiro Machado, 2010.

[23] D. Valério, Fractional control toolbox for MatLab, 2005.