Cosmological Models of Universe with Variable Deceleration Parameter in Lyra's Manifold

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FRW models of the universe have been studied in the cosmological theory based on Lyra's manifold. A new class of exact solutions has been obtained by considering a time dependent displacement field for variable deceleration parameter from which three models of the universe are derived (i) exponential (ii) polynomial and (iii) sinusoidal form respectively. The behaviour of these models of the universe are also discussed. Finally some possibilities of further problems and their investigations have been pointed out.

Keywords: Cosmology; FRW universe; Lyra geometry

I. INTRODUCTION

Einstein proposed his general theory of relativity, in which gravitation is described in terms of geometry, and it motivated the geometrization of other physical fields. One of the first attempts in this direction was made by Weyl [1] who proposed a more general theory in which gravitation and electromagnetism is also described geometrically. However, this theory was never considered seriously as it was based on the non-integrability of length transfer. Later Lyra [2] suggested a modification of Riemannian geometry by introducing a gauge function which removes the non-integrability condition of the length of a vector under parallel transport. Subsequently, Sen [3] & Sen and Dunn [4] proposed a new scalar tensor theory of gravitation. They constructed an analog of the Einstein field equations based on Lyra's geometry which in normal gauge may be written as

$$R_{ij}-\frac{1}{2}g_{ij}R+\frac{3}{2}\phi_i\phi_j-\frac{3}{4}g_{ij}\phi_k\phi^k=-8\pi GT_{ij}, \qquad (1)$$

where ϕ_i is the displacement vector and other symbols have their usual meaning as in Riemannian geometry.

Halford [5] pointed out that the constant displacement vector field ϕ_i in Lyra's geometry plays the role of a cosmological constant in the normal general relativistic treatment. Halford [6] showed that the scalar-tensor treatment based on Lyra's geometry predicts the same effects, within observational limits, as in Einstein's theory. Several authors [7] have studied cosmological models based on Lyra's geometry with a constant displacement field vector. However, this restriction of the displacement field to be a constant is a coincidence and there is no a *priori* reason for it. Singh et al. [8] have studied Bianchi type I, III, Kantowski-Sachs and a new class of models with a time dependent displacement field and have made a comparative study of Robertson-Walker models with a constant deceleration parameter in Einstein's theory with a cosmological terms and in the cosmological theory based on Lyra's geometry. Recently Pradhan et al. [9] and Rahaman et al. [10] have studied cosmological models based on Lyra's geometry with constant and time time dependent displacement field in different context.

The Einstein's field equations are a coupled system of highly nonlinear differential equations and we seek physical solutions to the field equations for their applications in cosmology and astrophysics. In order to solve the field equations we normally assume a form for the matter content or that space-time admits killing vector symmetries [11]. Solutions to the field equations may also be generated by applying a law of variation for Hubble's parameter which was proposed by Berman [12]. In simplest case the Hubble law yields a constant value for the deceleration parameter. It is worth observing that most of the well-known models of Einstein's theory and Brans-Dicke theory with curvature parameter $k=0$, including inflationary models, are models with constant deceleration parameter. In earlier literature cosmological models with a constant deceleration parameter have been studied by several authors [9, 13]. But redshift magnitude test has had a chequered history. During the 1960s and the 1970s, it was used to draw very categorical conclusions. The deceleration parameter q_0 was then claimed to lie between 0 and 1 and thus it was claimed that the universe is decelerating. Today's situation, we feel, is hardly different. Observations (Knop et al. [14]; Riess et al. [15]) of Type Ia Supernovae (SNe) allow to probe the expansion history of the universe. The main conclusion of these observations is that the expansion of the universe is accelerating. So we can consider the cosmological models with variable cosmological term and deceleration parameter. The readers are advised to see the papers by Vishwakarma and Narlikar [16] and Virey et al. [17] and references therein for a review on the determination of the deceleration parameter from Supernovae data.

Recently Pradhan et al. [18] have studied the universe with time dependent deceleration parameter in presence of perfect fluid. Motivated by the recent results on the BOOMERANG experiment on Cosmic Microwave Background Radiation (Bernardis [20]), we wish to study a spatially flat cosmological model. In this paper, we have investigated spatially nonflat and flat cosmological models with a time dependent displacement field within the framework of Lyra's geometry. We have obtained exact solutions of the field equations of Sen [3] by taking the deceleration parameter to be variable. This paper is organized as follows. The metric and the field equations are presented in Section II. In Section III we deal with a general solution. The Sections IV, V, and VI deal with the three different cases for the solutions in exponential, polynomial and sinusoidal forms respectively. Finally in Section VII concluding remarks are given.

II. FIELD EQUATIONS

The time-like displacement vector ϕ_i in the equation (1) is given by

$$\phi_i = (0,0,0,\beta(t)). \tag{2}$$

The energy-momentum tensor in the presence of a perfect fluid has the form

$$T_{ij} = (\rho + p)u_i u_j - p g_{ij} \tag{3}$$

together with co-moving coordinates $u^i u_i = 1$, where $u_i = (0,0,0,1)$. The metric for FRW spacetime is

$$ds^2 = dt^2 - R^2(t)\left[\frac{dr^2}{1-kr^2} + r^2(d\theta^2 + \sin^2\theta d\phi^2)\right], \tag{4}$$

where, $k = 1,-1,0$. For this metric, the field equations (1) with the equations (2) and (3) take the form

$$3H^2 + \frac{3k}{R^2} - \frac{3\beta^2}{4} = \chi\rho, \tag{5}$$

$$2\dot{H} + 3H^2 + \frac{k}{R^2} + \frac{3\beta^2}{4} = -\chi p, \tag{6}$$

where $\chi = 8\pi G$ and $H = \dot{R}/R$ is the Hubble's function. Equations (5) and (6) lead to the continuity equation

$$\chi\dot{\rho} + \frac{3}{2}\beta\dot{\beta} + 3\left[\chi(\rho + p) + \frac{3}{2}\beta^2\right]H = 0. \tag{7}$$

Assuming an equation of state

$$p = \gamma\rho, \;\; 0 \leq \gamma \leq 1. \tag{8}$$

Eliminating $\rho(t)$ from (5) and (6) we obtain

$$2\dot{H} + 3(1+\gamma)H^2 + (1+3\gamma)\frac{k}{R^2} + \frac{3}{4}(1-\gamma)\beta^2 = 0. \tag{9}$$

Here β^2 plays the role of a variable cosmological term $\Lambda(t)$. We have two independent equations in three unknowns viz $R(t)$, $\rho(t)$ and β. Therefore we need one more relation among the variables in order to obtain a unique solution. Hence, we consider the deceleration parameter to be time dependent.

III. SOLUTIONS OF THE FIELD EQUATIONS

We consider the deceleration parameter to be variable

$$q = -\frac{R\ddot{R}}{\dot{R}^2} = -\left(\frac{\dot{H} + H^2}{H^2}\right) = b \text{ (variable)}. \tag{10}$$

The equation (10) may be rewritten as

$$\frac{\ddot{R}}{R} + b\frac{\dot{R}^2}{R^2} = 0. \tag{11}$$

The general solution of Eq. (11) is given by

$$\int e^{\int \frac{b}{R}dR} dR = t + m, \tag{12}$$

where m is an integrating constant.

In order to solve the problem completely, we have to choose $\int \frac{b}{R}dR$ in such a manner that Eq. (12) be integrable. Without any loss of generality, we consider

$$\int \frac{b}{R}dR = \ln L(R), \tag{13}$$

which does not effect the nature of generality of solution. Hence from Eqs. (12) and (13), we obtain

$$\int L(R)dR = t + m. \tag{14}$$

Ofcourse, the choice of $L(R)$, in Eq. (14), is quite arbitrary but, since we are looking for physically viable models of the universe consistent with observations, we consider the following cases:

IV. SOLUTION IN THE EXPONENTIAL FORM

Let us consider $L(R) = \frac{1}{k_1 R}$, where k_1 is arbitrary constant. In this case, on integration of Eq. (14) gives the exact solution

$$R(t) = k_2 e^{k_1 t}, \tag{15}$$

where k_2 is an arbitrary constant. Using Eqs. (8) and (15) in (9) and (5) or (6), we obtain expressions for displacement field β, pressure p and energy density ρ as

$$\beta^2 = -\frac{4(1+\gamma)k_1^2}{(1-\gamma)} - \frac{4(1+3\gamma)k}{3(1-\gamma)k_2^2 e^{2k_1 t}}, \tag{16}$$

$$\chi p = \chi\gamma\rho = \frac{2\gamma}{(1-\gamma)}\left[3k_1^2 + \frac{2k}{k_2^2 e^{2k_1 t}}\right]. \tag{17}$$

From Eq. (15), since scale factor can not be negative, we find $R(t)$ is positive if $k_2 > 0$. From Fig. 1, it can be seen that in the early stages of the universe, i.e., near $t = 0$, the scale factor of the universe had been approximately constant and had increased very slowly. At specific time the universe had exploded suddenly and expanded to large scale. This is consistent with Big Bang scenario.

From Eq. (16), it is observed that β^2 is a decreasing function of time. As mentioned earlier the constant vector displacement field ϕ_i in Lyra's geometry plays the role of cosmological constant Λ in the normal general relativistic treatment

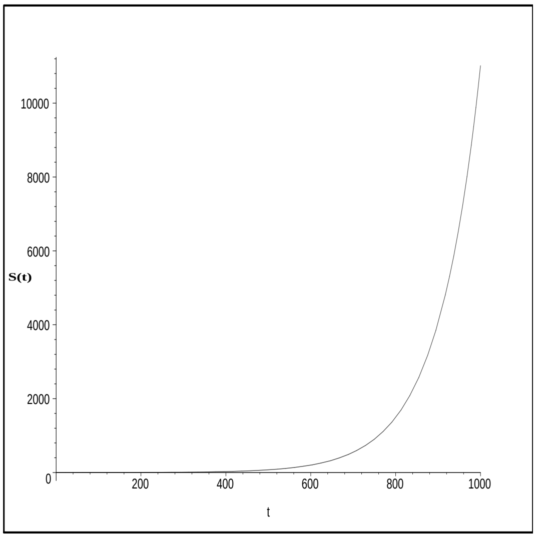

FIG. 1: The plot of scale factor $R(t)$ vs time with parameters $k_1 = 0.01$, $k_2 = 0.5$, and $\gamma = 0.5$

TABLE I: Values of β^2 and ρ for dust and radiation exponential models.

γ	β^2	ρ
0	$-4[k_1^2 + \frac{k}{3k_2^2 e^{2k_1 t}}]$	$\frac{2}{\chi}[3k_1^2 + \frac{2k}{k_2^2 e^{2k_1 t}}]$
$\frac{1}{3}$	$-4[2k_1^2 + \frac{k}{k_2^2 e^{2k_1 t}}]$	$\frac{3}{\chi}[3k_1^2 + \frac{2k}{k_2^2 e^{2k_1 t}}]$

and the scalar-tensor treatment based on Lyra's geometry predicts the same effects, within observational limits, as the Einstein's theory. Recent cosmological observations (Garnavich et al. [20], Perlmutter et al. [21], Riess et al. [22], Schmidt et al. [23]) suggest the existence of a positive cosmological constant Λ with the magnitude $\Lambda(G\hbar/c^3 \approx 10^{-123}$. These observations on magnitude and red-shift of type Ia supernova suggest that our universe may be an accelerating one with induced cosmological density through the cosmological Λ-term. In our model, it is seen that β plays the same role as cosmological constant and preserves the same character as Λ-term, in turn with recent observations.

From Eq. (17), we observe that $p > 0$ and $\rho > 0$ for $k > 0$. We also see that the energy density decreases to a small positive value and remains constant thereafter. The expressions for β^2 and ρ cannot be determined for the stiff matter $(p = \rho)$ models. The expressions for β^2 and ρ corresponding to $\gamma = 0, 1/3$ are given in Table 1.

We can obtain the values of β^2 and ρ for **flat FRW** model if we set $k = 0$ in Eqs. (16) and (17).

V. SOLUTION IN THE POLYNOMIAL FORM

Let $L(R) = \frac{1}{2k_3\sqrt{R+k_4}}$, where k_3 and k_4 are constants. In this case, on integrating, Eq. (14) gives the exact solution

$$R(t) = \alpha_1 t^2 + \alpha_2 t + \alpha_3, \tag{18}$$

where α_1, α_2 and α_3 are arbitrary constants. Using Eqs. (8) and (18) in (9) and (5) or (6), we obtain the expressions for displacement field β, pressure p and energy density ρ as

$$\beta^2 = \frac{16\alpha_1[\alpha_3 + (2+3\gamma)\alpha_1 t^2 + (2+3\gamma)\alpha_2 t] + 4(1+3\gamma)(\alpha_2^2 + k)}{3(\gamma-1)(\alpha_1 t^2 + \alpha_2 t + \alpha_3)^2}, \tag{19}$$

$$\chi p = \chi\gamma\rho = \frac{4\gamma[5\alpha_1^2 t^2 + 5\alpha_1\alpha_2 t + \alpha_2^2 + \alpha_1\alpha_3 + k]}{(1-\gamma)(\alpha_1 t^2 + \alpha_2 t + \alpha_3)^2}. \tag{20}$$

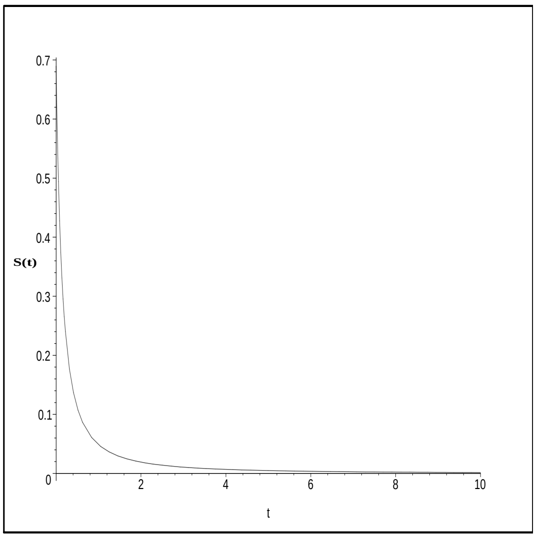

FIG. 2: The plot of scale factor $R(t)$ vs time with parameters $\alpha_1 = 1.00$, $\alpha_2 = 4.00$, $\alpha_3 = 1.00$ and $\gamma = 0.5$

From Eq. (18), we note that $R(t) > 0$ for $0 \le t < \infty$ if α_1, α_2 and α_3 are positive constants. Figure 2, shows that the scale factor is a decreasing function of time, implying that our universe is expanding.

Eq. (19) shows that $\beta^2 < 0$ for all times as $\gamma - 1 < 0$ and is a decreasing function of time, characteristically similar to Λ in Einstein's theory of gravitation. In this model, β plays the role as cosmological constant and it preserves the same character as Λ-term. This is consistent with recent observations (Garnavich et al. [20], Perlmutter et al. [21], Riess et al. [22], Schmidt et al. [23]). A negative cosmological constant adds to the attractive gravity of matter; therefore, universe with a negative cosmological constant is invariably doomed to re-collapse. A positive cosmological constant resists the attractive gravity of matter due to its negative pressure. For most of the time, the positive cosmological constant eventually dominates over the attraction of matter and drives the universe to expand exponentially.

The expressions for β^2 and ρ cannot be determined for the stiff matter ($p=\rho$) models. For dust model ($\gamma=0$), β^2 and $\rho(t)$ are given by

$$\beta^2 = -\frac{16\alpha_1[2\alpha_1 t^2+2\alpha_2 t+\alpha_3]+4\alpha_2^2+4k}{(\alpha_1 t^2+\alpha_2 t+\alpha_3)^2}, \quad (21)$$

$$\chi\rho = \frac{4[5\alpha_1^2 t^2+5\alpha_1\alpha_2 t+\alpha_2^2+\alpha_1\alpha_3+k]}{(\alpha_1 t^2+\alpha_2 t+\alpha_3)^2} \quad (22)$$

For radiative model ($\gamma=\frac{1}{3}$), β^2 and $\rho(t)$ are given by

$$\beta^2 = -\frac{8\alpha_1[2\alpha_1 t^2+2\alpha_2 t+\alpha_3]+8\alpha_2^2+8k}{(\alpha_1 t^2+\alpha_2 t+\alpha_3)^2}, \quad (23)$$

$$\chi\rho = \frac{6[5\alpha_1^2 t^2+5\alpha_1\alpha_2 t+\alpha_2^2+\alpha_1\alpha_3+k]}{(\alpha_1 t^2+\alpha_2 t+\alpha_3)^2} \quad (24)$$

If we set $k=0$, in above equations (19) - (24), we get solutions for **flat FRW** universe.

VI. SOLUTION IN THE SINUSOIDAL FORM

Let $L(R)=\frac{1}{\beta\sqrt{1-R^2}}$, where β is constant.
In this case, on integrating, Eq. (14) gives the exact solution

$$R = M\sin(\beta t)+N\cos(\beta t)+\beta_1, \quad (25)$$

where M, N and β_1 are constants. Using Eqs. (8) and (25) in (9) and (5) or (6) , we obtain the expressions for displacement field β, pressure p and energy density ρ as

$$\beta^2 = \frac{4[2\beta^2(M^2+N^2)+2\beta^2\beta_1(P-\beta_1)-3(1+\gamma)Q-(1+3\gamma)k]}{3(1-\gamma)P^2}, \quad (26)$$

$$\chi p = \chi\gamma\rho = \frac{2\gamma[3Q+2k-\beta^2(M^2+N^2)-\beta^2\beta_1(P-\beta_1)]}{(1-\gamma)P^2}, \quad (27)$$

where

$$P = M\sin\beta t+N\cos\beta t+\beta_1,$$

$$Q = (M\cos\beta t-N\sin\beta t)^2.$$

From the Figure 3, we note that at early stage of the universe, the scale of the universe increases gently and then decreases sharply, and after wards it will oscillate for ever. We must mention here that the oscillation takes place in positive quadrant. This has physical meaning.

The expressions for β^2 and ρ cannot be determined for the stiff matter ($p=\rho$) models. For dust model ($\gamma=0$), β^2 and $\rho(t)$ are given by

$$\beta^2 = \frac{4[2\beta^2(M^2+N^2)+2\beta^2\beta_1(P-\beta_1)-3Q-k]}{3P^2}, \quad (28)$$

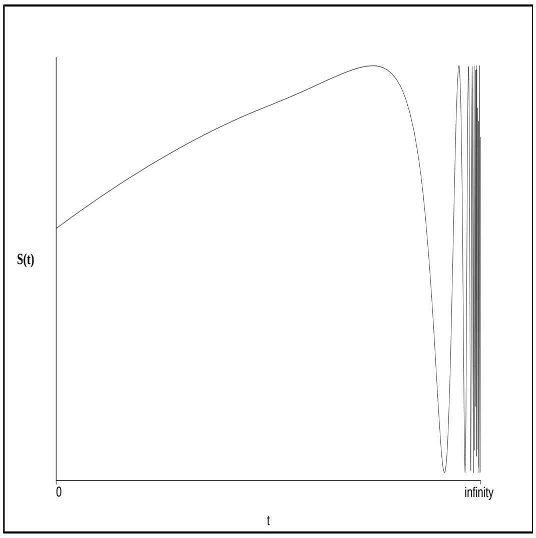

FIG. 3: The plot of scale factor $R(t)$ vs time with parameters $M=2.00$, $N=1.00$, $\beta=10.00$, $\beta_1=0.2$, and $\gamma=0.5$

$$\chi\rho = \frac{2[3Q+2k-\beta^2(M^2+N^2)-\beta^2\beta_1(P-\beta_1)]}{P^2}, \quad (29)$$

For radiative model ($\gamma=\frac{1}{3}$), β^2 and $\rho(t)$ are given by

$$\beta^2 = \frac{4[\beta^2(M^2+N^2)+\beta^2\beta_1(P-\beta_1)-2Q-k]}{P^2}, \quad (30)$$

$$\chi\rho = \frac{3[3Q+2k-\beta^2(M^2+N^2)-\beta^2\beta_1(P-\beta_1)]}{P^2}, \quad (31)$$

For **flat FRW** universe, we put $k=0$ in above results. Since, in these cases, we have many alternatives for choosing values of M, N, β, β_1, it is just enough to look for suitable values of these parameters, such that the physical initial and boundary conditions are satisfied.

VII. CONCLUSIONS

In this paper we have obtained exact solutions of Sen's equations in Lyra geometry for time dependent deceleration parameter in FRW spacetime. The nature of the displacement field $\beta(t)$ and the energy density $\rho(t)$ have been examined for three cases (i) exponential form (ii) polynomial form and (iii) sinusoidal form. The solutions obtained in Sections IV, V and VI are to our knowledge quite new. Here, it is found that the displacement field plays the role of a variable cosmological term Λ.

In recent past, there is an upsurge of interest in scalar fields in general relativity and alternative theories of gravitation in the context of inflationary cosmology (La and Steinhardt [24]; Ellis [25]; Barrow [26]). Therefore, the study of cosmological models in Lyra geometry may be relevant for inflationary models. Further, the space dependence of the displacement field β is important for inhomogeneous models for the early stages of the evolution of the universe. Besides, the implication of Lyra's geometry for astrophysical interesting bodies is still an open questions. The problem of equation of motion and gravitational radiation need investigation. Finally, in spite of very good possibility of Lyra's geometry to provide a theoretical foundation for relativistic gravitation, astrophysics and cosmology, the experimental point is yet to be undertaken. But still the theory needs a fair trial.

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