Universal quantum limits on single channel information, entropy and heat flow

Miles P. Blencowe*
Department of Physics and Astronomy, Dartmouth College, Hanover, New Hampshire 03755, USA

Vincenzo Vitelli†
The Blackett Laboratory, Imperial College, London SW7 2BZ, UK

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Abstract

We show that the recently discovered universal upper bound on the thermal conductance of a single channel comprising particles obeying arbitrary fractional statistics is in fact a consequence of a more general universal upper bound, involving the averaged entropy and energy currents of a single channel connecting heat reservoirs with arbitrary temperatures and chemical potentials. The latter upper bound in turn leads, via Holevo's theorem, to a universal (i.e., statistics independent) upper bound on the optimum capacity for classical information transmission down a single, wideband quantum channel.

I. INTRODUCTION

In a recent experiment [1], Schwab *et al.* succeeded in measuring for the first time the thermal conductance quantum for a suspended, dielectric wire of submicron cross section. In accordance with predictions [2], the thermal conductance was found to approach the limiting value $4 \times \pi k_B^2 T/6\hbar \approx 4 \times 10^{-12} T$ W K^{-1} as the wire thermal reservoirs were cooled such that the dominant phonon wavelength became comparable to the wire cross section. The factor of 4 is just the number of independent vibrational mode branches of the wire satisfying $\omega(k) \to 0$ as $k \to 0$ (see, e.g., Ref. [3]). Only such modes can have non-negligible phonon occupation numbers as $T \to 0$, giving four available channels for heat transport. The single channel thermal conductance can never exceed the thermal conductance quantum $\pi k_B^2 T/6\hbar$. The conductance quantum can only be attained for ballistic transport (i.e., no scattering) as was achieved in the experiment.

In common with the quantum limits for other single channel, linear transport coefficients, such as the electronic conductance quantum e^2/h [4], the thermal conductance quantum does not depend on the form of the $\omega(k)$ dispersion relations, a consequence of the cancellation of the group velocity and density of states factors in the formula for the one dimensional heat current. Wires made from different insulating materials and with different cross section geometries will therefore all have the same limiting single channel thermal conductance value for ballistic transport at low temperatures. For this reason, the conductance quantum is often termed 'universal'.

The thermal conductance is in fact universal in a much wider sense. For a single channel connecting two heat reservoirs with (quasi)particles obeying fractional statistics according to Haldane's definition (which generalizes Bose and Fermi statistics) [5], it was recently found that the maximum, limiting thermal conductance quantum is independent of the particle statistics as well [6,7]. For example, in the case of an ideal electron gas, the limiting single channel thermal conductance coincides with the above thermal conductance quantum for phonons. While dimensional analysis would lead us to expect the same factor $k_B^2 T/\hbar$ to occur independently of the statistics, there is no a priori reason to expect the same numerical factor $\pi/6$ as well, given that the latter results from integrating with respect to the energy the expansion to first order in small temperature differences of the thermal reservoir distributions, which have qualitatively different forms for particles obeying different statistics. This remarkable property is unique to the thermal conductance: all other single channel transport coefficients depend on the particle statistics.

In an earlier and unrelated investigation concerning the quantum limits on single channel information and entropy flow [8], Pendry showed that the bound $\dot{S}^2/\dot{E} \leq \pi k_B^2/3\hbar$, involving the averaged single channel entropy and energy currents, is obeyed for both bosons and fermions. The striking resemblance between this bound and that for the single channel thermal conductance suggests the possible existence of an universal and more general, attainable bound relating the entropy and energy currents, from which the thermal conductance bound would follow as a special case. In particular, there is the possibility of a bound which would be independent of the channel materials properties and particle statistics and which would apply even far from equilibrium where the temperatures (and perhaps also the chemical potentials) of the two heat reservoirs connecting the ends of the channel are significantly different.

Given that entropy and information are closely related, the existence of such a universal upper bound on the entropy flow rate would in turn suggest the existence of an optimum capacity for single channel classical information transmission, which is also universal in the wider sense (i.e., independent of channel materials properties and particle statistics). This is in fact the main subject of Ref. [8]. However, there the analysis is restricted to situations in which the channel is noiseless, with the information encoded and decoded in terms of the boson/fermion number eigenstates. A proper determination of the optimum capacity would consider all possible input quantum states for encoding letters and all possible detection schemes at the output. The crucial result which allows one to generalize the analysis of Ref. [8] is Holevo's theorem [9], which bounds the mutual information between channel output and input with a quantity involving the quantum entropies of the input states. Caves and Drummond [10] have carried out the more general analysis for particles obeying Bose statistics only and confirm Pendry's upper bound as the optimum channel capacity.

Thus, there is the possibility of an optimum, universal limiting capacity which bounds all possible methods of encoding and detection, and which is independent of the physical properties of the channel. We emphasize that the existence of such a single channel optimum capacity is suggested by the established existence of the universal thermal conductance quantum.

The fact that the thermal conductance by its very definition requires that the channel be connected at each end to a heat reservoir which can act both as an emitter and absorber of quanta, suggests that in order to attain the above-conjectured optimum capacity, a generalization of the mutual information will be required in which there is a sender/receiver pair at both ends of the channel. The pairs thus share the same channel and information can now flow in opposite directions. In fact, as we shall see, with the exception of bosons the attainability of Pendry's upper bound for particles obeying arbitrary statistics necessarily requires that the chemical potentials of the two reservoirs coincide and be nonzero, so that both reservoirs are sources of particles.

In the following sections, we provide evidence for the validity of the two conjectures outlined above: namely (1) the existence of an universal bound relating the entropy and energy flow rates of a single quantum channel, with the universal thermal conductance bound following as a special case, and (2) the existence of an universal bound on the optimum capacity for single-channel information transmission, subject to certain constraints on the channel and input states.

In Sec. II, we first show that Pendry's bound holds for particles obeying arbitrary statistics under quite general conditions for the reservoir temperatures and chemical potentials. We then introduce a less general but tighter bound, which requires that the chemical potentials of the two reservoirs coincide, and replaces the energy current with the heat current $\dot{E} - \mu\dot{N}$. We show that the thermal conductance bound follows as a special case from this latter bound when the temperature difference between the two reservoirs approaches zero.

In Sec. III, we first repeat the analysis of Caves and Drummond [10] in the conventional case for unidirectional information transmission, but generalizing to particles obeying fractional statistics, and obtain the limiting statistics-dependent optimum capacity. We then generalize the definition of mutual information and Holevo's theorem in a simple way to allow for non-interfering, two-way information flow. Using this generalized definition we show, subject to certain constraints on the channel and input states, that the limiting optimum capacity is now independent of the particle statistics and coincides with that of the bosonic case. In the final part of the section, we introduce a further generalization of the mutual information which allows for the possibility of interference between the 'left-moving' and 'right-moving' information flows.

In Sec. IV, we conclude and also briefly outline various open problems which have a bearing on the two conjectures.

II. ENTROPY BOUNDS

As our generic model structure, we consider some confining 'wire' which supports particles obeying a given statistics and which is connected adiabatically at each end to two particle reservoirs characterized by temperatures T_L and T_R and chemical potentials μ_L and μ_R, where the subscripts L and R denote the left and right reservoirs, respectively. The device of Ref. [1] is one possible realisation of the model structure in the case of phonons. Typically, a wire will provide several available parallel channels for given reservoir chemical potential and temperature values. However, in the case of ballistic transport the channel currents don't interfere with each other and thus can be treated independently. We will restrict ourselves to ballistic transport in the present investigation.

The distribution function for particles obeying fractional statistics is [11]

$$f_g(E) = \left[w\left(\frac{E-\mu}{k_B T} \right) + g \right]^{-1}, \tag{1}$$

where the function $w(x)$ satisfies

$$w(x)^g [1+w(x)]^{1-g} = e^x. \tag{2}$$

The parameter g, assumed to be a rational number, determines the statistics. From these equations, we can see immediately that $g=0$ describes bosons and $g=1$ fermions. The left(right) components of the single channel energy and entropy currents are [6,7]

$$\dot{E}_{L(R)} = \frac{(k_B T_{L(R)})^2}{2\pi\hbar} \int_{x^0_{L(R)}}^{\infty} dx \left(x + \mu_{L(R)}/k_B T_{L(R)} \right) f_g(x) \tag{3}$$

and

$$\begin{aligned} \dot{S}_{L(R)} = & -\frac{k_B^2 T_{L(R)}}{2\pi\hbar} \int_{x^0_{L(R)}}^{\infty} dx \left\{ f_g \ln f_g + (1-gf_g)\ln(1-gf_g) \right. \\ & \left. - [1+(1-g)f_g] \ln[1+(1-g)f_g] \right\}, \end{aligned} \tag{4}$$

where $x^0_{L(R)} = -\mu_{L(R)}/k_B T_{L(R)}$, and where we define the energy origin such that the minimum energy of a channel particle is zero, i.e., the energy is given by the longitudinal kinetic component. The total energy and entropy channel currents are then just $\dot{E} = \dot{E}_L - \dot{E}_R$ and $\dot{S} = \dot{S}_L - \dot{S}_R$, respectively.

The first conjectured bound involving these single channel entropy and energy currents is [8]

$$\dot{S}^2 \leq \frac{\pi k_B^2}{3\hbar}\dot{E}, \tag{5}$$

provided $T_L > T_R$ and $\mu_L \geq \mu_R$. We have numerically tested this bound extensively in μ and T parameter space, for several rational values of the statistical parameter g ranging between between zero and one. Fig. 1 gives an initial idea of the bound by showing the dependence of the ratio $3\hbar\dot{S}^2/\pi k_B^2\dot{E}$ on a selected parameter range.

In the case of bosons with constant $\mu_L = \mu_R = 0$ (e.g., photons or phonons), evaluating Eqs. (3) and (4) gives $\dot{E}_{L(R)} = \pi(k_B T_{L(R)})^2/12\hbar$ and $\dot{S}_{L(R)} = \pi k_B^2 T_{L(R)}/6\hbar$, respectively. Setting $T_R = 0$ and eliminating T_L by solving for $\dot{S}(= \dot{S}_L)$ in terms of $\dot{E}(= \dot{E}_L)$, we obtain equality in bound (5). For all other physically achievable parameter choices, we have strict inequality in (5). The key point, however, is that the bound can be approached arbitrarily closely, no matter the particle statistics. For example, in the case of bosons with non-constant reservoir chemical potentials, the bound is approached asymptotically in the degenerate limit $-x_L^0 = \mu_L/k_B T_L \to 0^-$, with $\mu_R = 0$ and $T_R = 0$. For particles with $g > 0$, the bound is approached asymptotically in the degenerate limit $-x_L^0 = \mu_L/k_B T_L \to +\infty$, with $\mu_L = \mu_R$ and $T_R = 0$. That these are the correct conditions for approaching the upper bound can be seen more clearly after transforming the integrals in Eqs. (3) and (4) for the bosonic and fermionic cases as in, e.g., Sec. 58 of Ref. [12]. For example, in the case of fermions, the single channel energy and entropy currents can be rewritten as follows

$$\begin{aligned}\dot{E} = \frac{\pi(k_B T_L)^2}{12\hbar}\Bigg[1 - \left(\frac{T_R}{T_L}\right)^2 + \frac{3}{(\pi k_B T_L)^2}\left(\mu_L^2 - \mu_R^2\right) + \frac{6}{\pi^2}\int_{-x_L^0}^{\infty} dx\,(\mu_L/k_B T_L - x)\,f(x) \\ - \frac{6}{\pi^2}\left(\frac{T_R}{T_L}\right)^2 \int_{-x_R^0}^{\infty} dx\,(\mu_R/k_B T_R - x)\,f(x)\Bigg]\end{aligned} \tag{6}$$

and

$$\begin{aligned}\dot{S} = \frac{\pi k_B^2 T_L}{6\hbar}\Bigg\{1 - \frac{T_R}{T_L} + \frac{3}{\pi^2}\int_{-x_L^0}^{\infty} dx\,[f\ln f + (1-f)\ln(1-f)] \\ - \frac{3}{\pi^2}\frac{T_R}{T_L}\int_{-x_R^0}^{\infty} dx\,[f\ln f + (1-f)\ln(1-f)]\Bigg\}.\end{aligned} \tag{7}$$

Taking the limit $-x_L^0 \to +\infty$, with the conditions $\mu_L = \mu_R$ and $T_R = 0$, only the first term remains on the right-hand-sides of Eqs. (6) and (7) and the energy and entropy currents coincide with those for bosons with $\mu_L = \mu_R = 0$ and $T_R = 0$.

It is not possible to recover the single channel thermal conductance bound from (5). The best we can do is to derive an upper bound on the rate of heat emission from an isolated reservoir for bosons with zero chemical potential [8] (see also Ref. [13]). Setting $\mu_L = \mu_R = 0$, $T_R = 0$, identifying the heat emission rate with the total energy emission rate $\dot{Q}_L = \dot{E}_L$, and using $\dot{Q}_L/T_L \leq \dot{S}_L$, bound (5) gives

$$\dot{Q}_L \leq \frac{\pi k_B^2 T_L^2}{3\hbar}. \tag{8}$$

Note that for particles with nonzero chemical potential, the heat emission rate is $\dot{Q}_L = \dot{E}_L - \mu_L\dot{N}_L$, where $\dot{N}$ denotes the number current, and in this case (8) does not follow from

(5). If we had equality in (8), then the thermal conductance could be obtained by taking the difference $\dot{Q}_L - \dot{Q}_R = \pi k_B^2(T_L^2 - T_R^2)/3\hbar = 2\pi k_B^2 \bar{T}\delta T/3\hbar$, where $\bar{T} = (T_L + T_R)/2$. But this gives the incorrect coefficient (2/3 instead of 1/6). What is wrong with this argument is the assumption that $\dot{Q}_{L(R)}/T_{L(R)} = \dot{S}_{L(R)}$. In fact, $\dot{Q}_{L(R)}/T_{L(R)} = \dot{S}_{L(R)}/2$, signalling the irreversible nature of the heat emission.

A conjectured, tighter bound suggested by the form of expressions (6) and (7) which does yield the thermal conductance bound as a special case, is the following

$$\dot{S}^2 \leq \frac{\pi k_B^2}{3\hbar}\left(\frac{T_L - T_R}{T_L + T_R}\right)\left(\dot{E} - \mu\dot{N}\right), \tag{9}$$

provided $T_L > T_R$ and $\mu_L = \mu_R = \mu$. Again, we have numerically tested this bound extensively in μ and T parameter space, for several rational values of the statistical parameter g ranging between between zero and one (see Figs. 2 and 3). In the case of bosons with constant $\mu = 0$, we obtain equality for all $T_L > T_R$. For bosons with non-constant $\mu \leq 0$, the bound is approached asymptotically in the degenerate limit $\mu/k_B T_{L(R)} \to 0^-$. For particles with $g > 0$, the bound is approached asymptotically in the degenerate limit $\mu/k_B T_{L(R)} \to +\infty$. Note that the heat current $\dot{Q} = \dot{E} - \mu\dot{N}$ appears instead of the energy current $\dot{E}$ on the right-hand-side of bound (9). This replacement is essential: if the energy current is used, then the bound can be violated for bosons with $\mu < 0$. It is remarkable that the need to recover the thermal conductance and also to satisfy the bound both lead to the replacement of the energy current with the heat current.

III. INFORMATION BOUNDS

Consider a communication channel, characterised by an input alphabet A with letters labeled by an index $a = 1, \ldots, \mathcal{A}$ and a set of probabilities $p_A(a)$ for transmitting letter a, an output alphabet B labeled by $b = 1, \ldots, \mathcal{B}$, and a set of conditional probabilities $p_{B|A}(b|a)$ for receiving letter b, given transmission of letter a. The mutual information gives the measure of the information successfully transmitted from input to output of the communication channel:

$$H(B;A) = \sum_{a,b} p_{B|A}(b|a)p_A(a)\log_2\left(\frac{p_{B|A}(b|a)}{p_B(b)}\right), \tag{10}$$

where $p_B(b) = \sum_a p_{B|A}(b|a)p_A(a)$.

Suppose the quantum channel medium supports particles for some given rational, statistical parameter value g, $0 \leq g \leq 1$. Let the input letter a be encoded in some quantum state $\hat{\rho}_a$, and the output detection scheme be described, in the most general case, by a set of non-negative, bounded Hermitian operators $\hat{F}_b$ satisfying $\sum_b \hat{F}_b = \hat{1}$, with $p_{B|A}(b|a) = \mathrm{tr}(\hat{\rho}_a\hat{F}_b)$. The operators $\hat{\rho}_a$ and $\hat{F}_b$ act on the channel Fock space for statistical parameter value g. More precisely, since information is transmitted in only one direction, these operators act on the subspace describing right-moving states, say.

Holevo's theorem [9] provides an upper bound on the mutual information for all possible detection schemes:

$$\max_{\{\hat{F}_b\}} H(B;A) \leq S(\hat{\rho}) - \sum_a p_A(a) S(\hat{\rho}_a), \tag{11}$$

where $\hat{\rho} = \sum_a p_A(a)\hat{\rho}_a$ and $S(\hat{\rho}) = -\mathrm{tr}(\hat{\rho}\log_2\hat{\rho})$ is the quantum entropy in bits. Note that, while this theorem is usually applied to bosonic communication channels (c.f., Ref. [10]), it is in fact applicable to channels for arbitrary fractional statistics. All that is required is that the channel obeys the usual rules of quantum mechanics.

Maximizing the mutual information with respect to the output detection scheme and the input states and probabilities gives the optimum capacity C of the channel. From (11), we have [10]

$$C = \frac{1}{\mathcal{T}} \max_{\hat{\rho}} \max_{\{\hat{F}_b\}} H(B;A) \leq \frac{1}{\mathcal{T}} \max_{\hat{\rho}} S(\hat{\rho}) = \frac{S_{\max}}{\mathcal{T}}, \tag{12}$$

where $\mathcal{T}$ is the transmission time. As shown in Sec. IV.B of Ref. [10], the upper bound $S_{\max}$ can in fact be attained: Find a complete, orthonormal set of diagonalizing basis states $|a\rangle$ for the $\hat{\rho}$ which maximizes $S(\hat{\rho})$, i.e., $\hat{\rho} = \sum_a q(a)|a\rangle\langle a|$. Choose $\hat{\rho}_a = |a\rangle\langle a| = \hat{F}_a$ and $p_A(a) = q(a)$. Then $H(B;A) = -\sum_a p_A(a)\log_2 p_A(a) = S(\hat{\rho}) = S_{\max}$.

Thus, the optimum capacity is just the maximum quantum entropy in bits divided by the transmission time, subject to the given constraints on the channel. One common constraint is to fix the total energy of the transmitted message; the optimum capacity then gives the maximum information that can be transmitted in a time $\mathcal{T}$ for a given, allowed signal energy. For a single, wideband channel with longitudinal single-particle energies $hf_j = hj/\mathcal{T}$, $j = 1, 2, \ldots$, the total longitudinal energy E_N of a given Fock state is $E_N = \sum_j hf_j n_j = Nh/\mathcal{T}$, where $N = \sum_{j=1}^{\infty} jn_j$, and n_j is the occupation number of, say, the right-propagating mode j. The maximum entropy is then $S_{\max} = \log_2 \mathcal{N}_N$, where $\mathcal{N}_N$ is just the number of different ways the sum N can be partitioned. For bosons ($g = 0$), $\mathcal{N}_N$ is given by the number of unrestricted partitions, while for the fermions ($g = 1$), $\mathcal{N}_N$ is given by the number of partitions into distinct parts. More generally, for particles obeying fractional statistics with $g = 1/n$, $n = 1, 2, \ldots$, no number can appear more than n times in a given partition. Note, however, that for $0 < g < 1$ there are additional constraints on the allowed partitions, as discussed in Ref. [14].

For long transmission times $\mathcal{T}$, or equivalently large N, one obtains the following asymptotic approximation to the optimum capacity of a wideband, bosonic channel for fixed energy [10]:

$$C_{\text{boson}} = \frac{\pi}{\ln 2}\sqrt{\frac{2P}{3h}} - \frac{1}{\mathcal{T}}\log_2\left(\frac{4\sqrt{3}P\mathcal{T}^2}{h}\right), \tag{13}$$

where $P = E_N/\mathcal{T}$ is the time-averaged power. Caves *et al.* [10] also derive the bosonic optimum capacity subject to the alternative constraints that the maximum-energy or the message-ensemble-averaged-energy of the channel be fixed. All give the same leading order term as on the right-hand-side of Eq. (13), with P appropriately defined in each case.

In order to write down the long transmission-time optimum capacity of a wideband, fermionic channel for fixed energy, we require the asymptotic approximation to the number of distinct partitions of N (see, e.g., Sec. 24.2.2 of Ref. [15]):

$$\mathcal{N}_N \sim \frac{1}{4 \cdot 3^{1/4} \cdot N^{3/4}} e^{\pi\sqrt{1/3}\sqrt{N}}. \tag{14}$$

This gives

$$C_{\text{fermion}} = \frac{\pi}{\ln 2}\sqrt{\frac{P}{3h}} - \frac{1}{\mathcal{T}}\log_2\left[4 \cdot 3^{1/4}\left(\frac{P\mathcal{T}^2}{h}\right)^{3/4}\right]. \tag{15}$$

Note that, in the limit $\mathcal{T} \to \infty$, the fermionic optimum capacity is smaller than the bosonic optimum capacity by a factor $\sqrt{2}$ for given power P. Note also that these optimum capacities satisfy the information-theoretic counterpart to bound (5) [8]:

$$C < \frac{\pi}{\ln 2}\sqrt{\frac{2P}{3h}} \tag{16}$$

for finite $\mathcal{T}$.

Asymptotic approximations to C, analogous to Eqs. (13) and (15), can no doubt also be written down for certain other rational g values. However, rather than attempting to derive C through the nontrivial route which involves first obtaining the asymptotic approximation to the number of partitions, we can appeal to the fact that different ensemble derivations of the entropy give the same result in the thermodynamic limit when the ensemble energies coincide. In particular, we can instead use expressions (3) and (4) for $\mu = 0$ (only the energy is constrained and not the particle number) to derive the leading order term in the asymptotic approximation to C. For example, in the case of 'semions' ($g = 1/2$), carrying out the integrals in (3) and (4) gives $C_{\text{semion}}(\mathcal{T} \to \infty) = (\pi/\ln 2)\sqrt{2P/5h}$, which falls between the Fermi and Bose capacities, again satisfying bound (16). Solving numerically Eqs. (3) and (4) for a range of g-values, we find that $C_g(\mathcal{T} \to \infty)$ decreases monotonically as g increases from 0 to 1 (Fig. 4). Thus bound (16) holds for all $0 \leq g \leq 1$.

However, unlike the analogous upper bound (5) on the single-channel physical entropy current, the information-theoretic bound (16) cannot be approached arbitrarily closely independently of the particle statistics: only for bosons is the upper bound approached in the limit $\mathcal{T} \to \infty$. But recall, from the form of the conditions for approaching the upper bound (5), and also from the form of Eqs. (6) and (7) for fermions, that it is crucial for both ends of the channel to be connected to reservoirs providing two-way energy and entropy flows in the channel. This suggests that, with a suitable generalization of the communication channel allowing for two-way information flow, the channel capacity will approach the upper bound (16) arbitrarily closely independently of the particle statistics.

Consider, therefore, two sender-receiver 'stations', with one station at each end of the single channel, thus sharing the channel. Station L at the left end encodes information in right-moving states and decodes information from left-moving states, while station R at the right end encodes information in left-moving states and decodes information from right-moving states. A single 'use' of the channel involves L and R each sending and subsequently detecting a message, the whole operation taking place during an interval $\mathcal{T}$. Station L uses an input alphabet A_L with letters labeled by an index $a_L = 1, \ldots, \mathcal{A}_L$ and a set of probabilities $p_{A_L}(a_L)$ for transmitting letter a_L, and an output alphabet B_L labeled by $b_L = 1, \ldots, \mathcal{B}_L$. Station R similarly uses an input alphabet A_R with transmission probabilities $p_{A_R}(a_R)$, and an output alphabet B_R. The probability that R receives letter b_R, given that L sends letter a_L is denoted as $p_{B_R|A_L}(b_R|a_L)$, and analogously for the conditional probability $p_{B_L|A_R}(b_L|a_R)$. We assume throughout that the joint probabilities for L and R to send a message are uncorrelated, i.e., $p_{A_L,A_R}(a_L, a_R) = p_{A_L}(a_L)p_{A_R}(a_R)$. We also assume to begin with that the left- and right-moving information flows do not interfere with each other.

Using the formula for the single channel entropy current as a guide [see, e.g., Eqs. (11) and (12) of Ref. [16]], we define the *net* information transmitted from the L and R inputs to the R output during a single use of the channel to be

$$H(B_R; A_L, A_R) = H(B_R; A_L) - H(A_R), \tag{17}$$

where $H(B_R; A_L)$ is defined as in Eq. (10) and $H(A_R) = -\sum_{a_R} p_{A_R}(a_R)\log_2 p_{A_R}(a_R)$. Similarly, the net information transmitted to the L output is $H(B_L; A_R, A_L) = H(B_L; A_R) - H(A_L)$. Note the asymmetry of the two terms on the right-hand-side of definition (17), reflecting an analogous asymmetry of the left- and right-moving components making up the net entropy current [16]. With the information defined with respect to the receiver at the right end of the channel, it makes more sense to use the information $H(A_R)$ rather than the mutual information $H(B_L; A_R)$ which takes into account the channel noise and receiver properties at the other end of the channel. As we shall soon see when we generalize (17) to include interfering left- and right-moving information, one reason why it might be a good thing to subtract, rather than to add, the information $H(A_R)$ is that it gives reasonable answers in familiar examples such as that of a returned or 'bounced' message.

But perhaps the most appealing property of $H(B_R; A_L, A_R)$ as defined is that it satisfies a generalized Holevo theorem:

$$\max_{\{\hat{F}_{b_R}\}} H(B_R; A_L, A_R) \leq S(\hat{\rho}_L) - S(\hat{\rho}_R) - \sum_{a_L} p_{A_L}(a_L) S(\hat{\rho}_{a_L}) + \sum_{a_R} p_{A_R}(a_R) S(\hat{\rho}_{a_R}), \tag{18}$$

where equality holds if and only if the left input states $\hat{\rho}_{a_L}$ commute and the right input states $\hat{\rho}_{a_R}$ are orthogonal (see, e.g., Sec. IV.B of Ref. [10]). Inequality (18) is a consequence both of the Holevo theorem (11), which bounds $H(B_R; A_L)$, and also of the inequality $H(A_R) \geq S(\hat{\rho}_R) - \sum_{a_R} p_{A_R}(a_R) S(\hat{\rho}_{a_R})$. Note that the latter inequality goes in the opposite direction to that of (11), so that one must subtract $H(A_R)$ in order that $H(B_R; A_L, A_R)$ be bounded.

Maximizing the information $H(B_R; A_L, A_R)$ with respect to the R output detection scheme and the L and R input states and probabilities gives the optimum capacity of the channel. From (18), we have

$$C = \frac{1}{\mathcal{T}} \max_{\hat{\rho}_L, \hat{\rho}_R} \max_{\{\hat{F}_{b_R}\}} H(B_R; A_L, A_R) \leq \frac{1}{\mathcal{T}} \max_{\hat{\rho}_L} S(\hat{\rho}_L) = \frac{S_{\max}}{\mathcal{T}}. \tag{19}$$

The upper bound $S_{\max}$ can in fact be attained: find a complete, orthonormal set of diagonalizing basis states $|a_L\rangle$ for the $\hat{\rho}_L$ which maximizes $S(\hat{\rho}_L)$, i.e., $\hat{\rho}_L = \sum_{a_L} q(a_L)|a_L\rangle\langle a_L|$. Choose $\hat{\rho}_{a_L} = |a_L\rangle\langle a_L| = \hat{F}_{a_L}$ and $p_{A_L}(a_L) = q(a_L)$. Choose any set $\{\hat{\rho}_{a_R}\}$ and probabilities $p_{A_R}(a_R) = \delta_{a_R, a'_R}$ for some fixed a'_R. Then $H(B_R; A_L, A_R) = H(B_R; A_L) = -\sum_{a_L} p_{A_L}(a_L)\log_2 p_{A_L}(a_L) = S(\hat{\rho}_L) = S_{\max}$. The choice for $\hat{\rho}_{a_R}$ and $p_{A_R}(a_R)$ reflects the obvious fact that, for the definition (17), maximizing $H(B_R; A_L, A_R)$ requires that $H(A_R)$

be minimized, so that any left-moving message component can be sent, provided it is with probability one so that its information content is zero.

Thus, the optimum capacity is just the maximum quantum entropy in bits for right-moving states divided by the transmission time, subject to the given constraints on the channel. Of particular interest are the constraints for which the optimum capacity in the limit $\mathcal{T} \to \infty$ is independent of the statistical parameter g. Recalling the conditions for approaching asymptotically the entropy bound (5) for $g > 0$, namely $\mu_L/k_B T_L \to +\infty$ with $\mu_L = \mu_R$ and $T_L > T_R = 0$, a little thought establishes that two constraints are: fixed power (i.e., fixed energy current) $P > 0$ and fixed number current $\dot{N} = 0$. Again, as for the unidirectional optimum capacity, the choice of ensemble for the definition of P and $\dot{N}$—microcanonical, grand canonical etc.—is immaterial in the limit $\mathcal{T} \to \infty$. Given that the unidirectional optimum capacity for $0 < g \leq 1$ is strictly less than the bosonic optimum capacity $(\pi/\ln 2)\sqrt{2P/3h}$ in the limit $\mathcal{T} \to \infty$, it may seem paradoxical that additional constraints have to be imposed (namely, $\dot{N} = 0$) in order to attain the latter, larger capacity. The resolution lies in the fact that the dimension of the channel Hilbert space accessible for information and energy transmission has been doubled through the accomodation of left-moving states.

The two above constraints, while necessary, are not sufficient. The problem lies in the fact that optimization step (19) places no conditions on the left-moving states $\hat{\rho}_{a_R}$, with the result that it is rather easy to find examples where the power P can be made arbitrarily small for given $S_{\max}$, while at the same time satisfying the constraint $\dot{N} = 0$. One possible way to overcome this problem is to introduce the further constraint on the left-moving states $\hat{\rho}_{a_R}$ that they be completely degenerate. Then it is possible to show that $S_{\max} = (\pi/\ln 2)\sqrt{2P/3h}$ in the limit $\mathcal{T} \to \infty$, i.e., $S_{\max}$ coincides with the limiting, unidirectional bosonic optimum capacity independently of $0 < g \leq 1$. Furthermore, in the case of bosons adding a left-moving degenerate state does not change the energy current, so that the above constraints can also be applied to bosons with the unidirectional bosonic optimum capacity again being obtained in the limit.

What we have essentially done both here and in the previous section is cancel part of the right-moving energy current component with a left-moving, degenerate component, leaving the information and entropy currents unchanged, thus increasing the optimum capacity and entropy current bound for given energy current [Eqs. (6) and (7) show this more explicitly]. What is remarkable is that the optimum capacity (16) and entropy current bound (5) are attained asymptotically for a common set of constraints independent of the statistics $0 \leq g \leq 1$.

In the final part of this section, we generalize our two-way information definition (17) so as to allow for the possibility of interference between the left- and right-moving information flows. Our definition is motivated by the formula for the single-channel entropy current in the presence of elastic scattering in the channel [16]. We define the *net* information transmitted from the L and R inputs to the R output during a single use of the channel to be

$$H(B_R; A_L, A_R) = \sum_{a_L, a_R, b_R} p_{B_R|A_L,A_R}(b_R|a_L, a_R)\; p_{A_L}(a_L)\; p_{A_R}(a_R)$$

$$\times \log_2 \left(p_{B_R|A_L,A_R}(b_R|a_L,a_R)/p_{B_R}(b_R) \right) + \sum_{a_R} p_{A_R}(a_R) \log_2 p_{A_R}(a_R), \quad (20)$$

with an analogous definition for $H(B_L; A_R, A_L)$ and where we again assume that the joint probabilities for L and R to send a message are uncorrelated. The channel interference is conveniently implemented by a unitary 'scattering' operator $\hat{\mathcal{S}}$, acting on the states as $\hat{\mathcal{S}}(\hat{\rho}_L \otimes \hat{\rho}_R)\hat{\mathcal{S}}^\dagger \rightarrow \hat{\rho}'_L \otimes \hat{\rho}'_R$, where we restrict ourselves to non-correlating interfering processes. The conditional probabilities are constructed as follows:

$$p_{B_R|A_L,A_R}(b_R|a_L,a_R) = \mathrm{tr}\left[(\hat{F}_{b_R} \otimes \hat{1})\hat{\mathcal{S}}(\hat{\rho}_{a_L} \otimes \hat{\rho}_{a_R})\hat{\mathcal{S}}^\dagger \right] \quad (21a)$$

and

$$p_{B_L|A_L,A_R}(b_L|a_L,a_R) = \mathrm{tr}\left[(\hat{1} \otimes \hat{F}_{b_L})\hat{\mathcal{S}}(\hat{\rho}_{a_L} \otimes \hat{\rho}_{a_R})\hat{\mathcal{S}}^\dagger \right], \quad (21b)$$

where the the right and left detector operators are written as $\hat{F}_{b_R} \otimes \hat{1}$ and $\hat{1} \otimes \hat{F}_{b_L}$, respectively, reflecting the fact that, in the absence of interference, i.e., when $\hat{\mathcal{S}}$ is the identity operator, the right(left) detector can only receive left(right) input states. Note that definition (20) reduces to the two-way information definition (17) when there is no interference.

The more general, two-way information definition (20) can be applied to certain situations which are beyond the scope of the unidirectional mutual information (10). As a simple example, consider the situation of a 'bounced' message, an all too common occurence with electronic mail. This example can be modeled as follows: let the right letters be encoded in the orthonormal states $|a_R\rangle$ and sent with probabilities $p_{A_R}(a_R)$. Let the states $|a_L\rangle$ encoding the left letters, and sent with probabilities $p_{A_L}(a_L)$, be in one-to-one correspondence with the right states $|a_R\rangle$, with the mapping achieved simply by reversing the propagation direction. Let the right detector be characterized by projection operators $\hat{F}_{a_R} = |a_L\rangle\langle a_L|$. Finally, suppose the scattering operator reverses the direction of the propagating states, i.e., $\hat{\mathcal{S}}|a_{R(L)}\rangle = |a_{L(R)}\rangle$. Then evaluating the two-way information (20), we find that the first term on the right-hand-side reduces to the information $H(A_R)$, thus cancelling the second term and giving the value $H(B_R; A_L, A_R) = 0$. This coincides with our common-sense measure: if a message gets bounced back, then no information was sent.

From the Holevo theorem (11) for unidirectional information flow and also the inequality $H(A_R) \geq S(\hat{\rho}_R) - \sum_{a_R} p_{A_R}(a_R) S(\hat{\rho}_{a_R})$, it is straightforward to show that the information $H(B_R; A_L, A_R)$ as defined in (20) also satisfies a generalized Holevo theorem:

$$\max_{\{\hat{F}_{b_R}\},\hat{\mathcal{S}}} H(B_R; A_L, A_R) \leq S(\hat{\rho}_L) - \sum_{a_L} p_{A_L}(a_L) S(\hat{\rho}_{a_L}). \quad (22)$$

Such a bound enables us to determine the optimum capacity allowing also for interfering left- and right-moving information flows. We shall leave this to a future investigation.

IV. CONCLUSION

We have provided evidence for the validity of two related conjectures which state that the entropy current and optimum capacity for information transmission of a single channel are universally bounded for given energy current/power, independently of the channel materials properties and particle statistics according to Haldane's definition. What is most notable, is that these bounds can be approached arbitrarily closely no matter the particle statistics. A less general, tighter bound on the entropy current was also conjectured, from which the recently discovered statistics-independent thermal conductance bound follows as a special case. The statistics-independent, limiting bound on the optimum capacity required a generalisation of the definition for the transmitted information, allowing for two-way information flow. The bound then followed from a generalized Holevo theorem, with certain constraints placed on the channel and input states.

The results presented here can be extended in several ways. It would be more satisfying to have an analytic proof of the conjectured bounds, rather than an exhaustive numerical check. The entropy current bound should be tested under more general conditions, for example in the presence of channel scattering. Similarly, the optimum capacity bound should be tested also allowing for interference between two-way information flows.

Finally, we point out the recent demonstration that Holevo's theorem follows from Landauer's principle of information erasure [17]. In the light of this, it would be interesting to try to rederive the universal upper bound on the optimum capacity starting from Landauer's erasure principle.

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REFERENCES

* Electronic address: miles.p.blencowe@dartmouth.edu
† Address from September 1, 2000: Dept. of Physics, Harvard University, Cambridge MA 02138; electronic address: vincenzo.vitelli@ic.ac.uk

[1] K. Schwab, E.A. Henriksen, J.M. Worlock, and M.L. Roukes, Nature **404**, 974 (2000).
[2] L.G.C. Rego and G. Kirczenow, Phys. Rev. Lett. **81**, 232 (1998); D.E. Angelescu, M.C. Cross, and M.L. Roukes, Superlattices Microstruct. **23**, 673 (1998); M.P. Blencowe, Phys. Rev. B **59**, 4992 (1999).
[3] N. Nishiguchi, Y. Ando, and M.N. Wybourne, J. Phys.: Condens. Matter **9**, 5751 (1997).
[4] B.J. van Wees, H. van Houten, C.W.J. Beenakker, J.G. Williamson, L.P. Kouwenhoven, D. van der Marel, and C.T. Foxon, Phys. Rev. Lett. **60**, 848 (1988); D.A. Wharam, T.J. Thornton, R. Newbury, M. Pepper, H. Ahmed, J.E.F. Frost, D.G. Hasko, D.C. Peacock, D.A. Ritchie, and G.A.C. Jones, J. Phys. C **21**, L209 (1988).
[5] F.D.M. Haldane, Phys. Rev. Lett. **67**, 937 (1991).
[6] L.G.C. Rego and G. Kirczenow, Phys. Rev. B **59**, 13080 (1999).
[7] I.V. Krive and E.R. Mucciolo, Phys. Rev. B **60**, 1429 (1999).
[8] J.B. Pendry, J. Phys. A: Math. Gen. **16**, 2161 (1983).
[9] A.S. Holevo (Kholevo), Probl. Pereda. Inf. **9**(3), 3 (1973) [Probl. Inf. Trans. **9**, 177 (1973)].
[10] C.M. Caves and P.D. Drummond, Rev. Mod. Phys. **66**, 481 (1994).
[11] Y.-S. Wu, Phys. Rev. Lett. **73**, 922 (1994).
[12] L.D. Landau and E.M. Lifshitz, *Statistical Physics*, 3rd ed. Part 1 (Pergamon Press, Oxford, 1980).
[13] J.B. Pendry, J. Phys.: Condens. Matter **11**, 6621 (1999).
[14] M.V.N. Murthy and R. Shankar, Phys. Rev. B **60**, 6517 (1999).
[15] M. Abramowitz and I.A. Stegun, Eds., *Handbook of Mathematical Functions* (U.S. G.P.O., Washington, DC, 1964).
[16] U. Sivan and Y. Imry, Phys. Rev. B **33**, 551 (1986).
[17] M.B. Plenio, Phys. Lett. A **263**, 281 (1999).

FIGURES

FIG. 1. Dependence of the ratio $3\hbar\dot{S}^2/\pi k_B^2\dot{E}$ on ${-x_L^0}^{-1} = k_BT_L/\mu_L$ for $g = 1$ with $\mu_L = \mu_R$ (solid line), $\mu_L = 1.01\mu_R$ (dashed line), and $\mu_L = 1.1\mu_R$ (dotted line), and also for $g = 1/2$ with $\mu_L = \mu_R$ (dot-dashed line). The parameter x_R^0 is chosen to be $x_R^0 = 100x_L^0$.

FIG. 2. Dependence of the ratio $3\hbar(T_L + T_R)\dot{S}^2/[\pi k_B^2(T_L - T_R)\dot{Q}]$ on $x_L^0 = -\mu/k_BT_L$ for $g = 0$ with $T_R = 0.9T_L$ (solid line), $T_R = 0.5T_L$ (dashed line), and $T_R = 0.1T_L$ (dotted line).

FIG. 3. Dependence of the ratio $3\hbar(T_L + T_R)\dot{S}^2/[\pi k_B^2(T_L - T_R)\dot{Q}]$ on ${-x_L^0}^{-1} = k_BT_L/\mu$ for $g = 1$ (solid line), $g = 1/2$ (dashed line), and $g = 1/4$ (dotted line). The temperature T_R is chosen to be $T_R = T_L/2$.

FIG. 4. Dependence of the optimum capacity ratio C_g/C_0 on the statistical parameter g. Note that only rational values of g are physical.

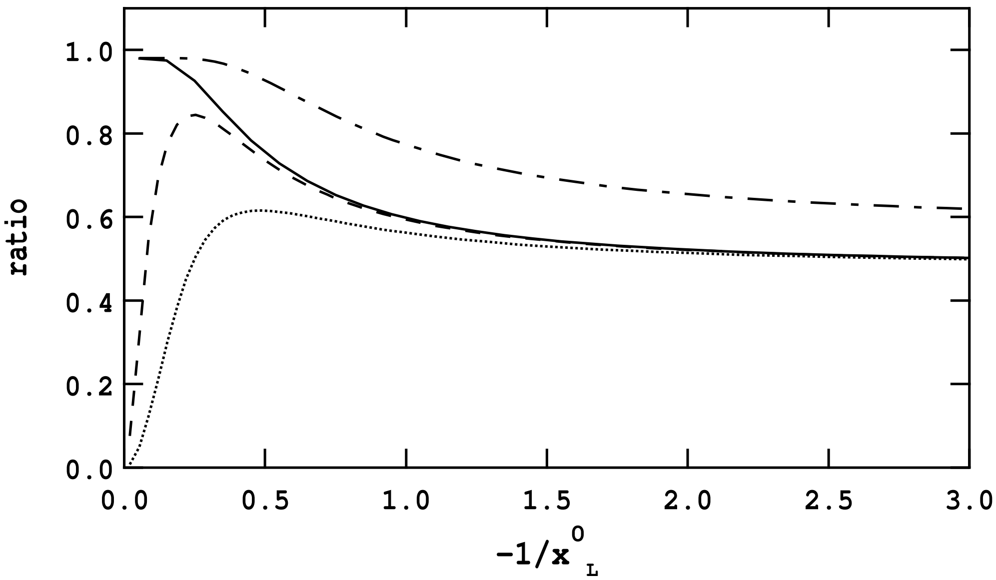

Fig 1 Blencowe

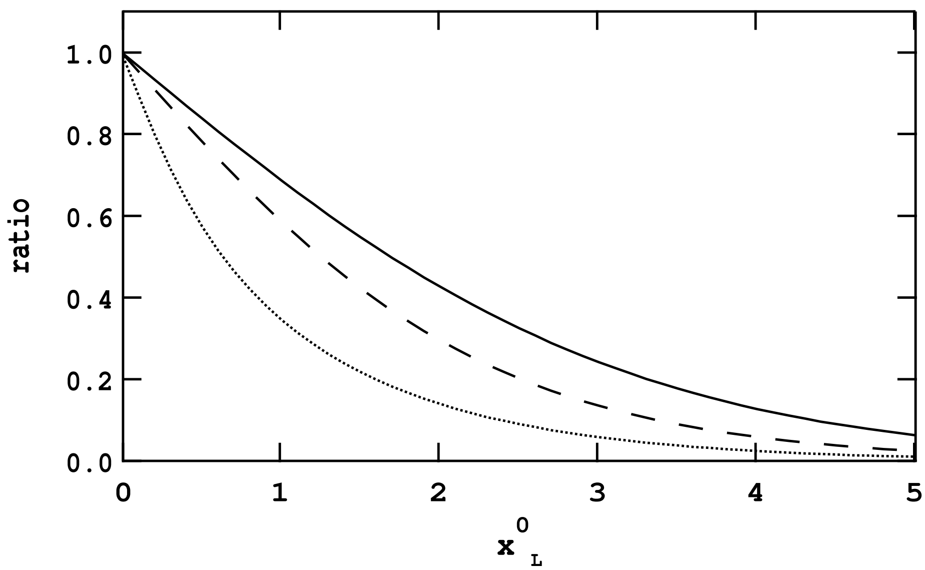

Fig 2 Blencowe

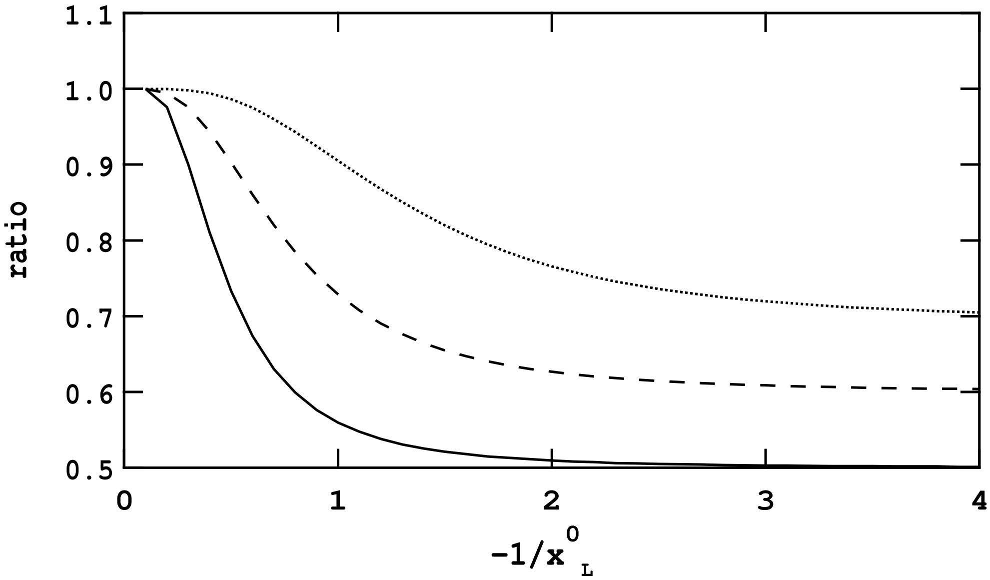

Fig 3 Blencowe

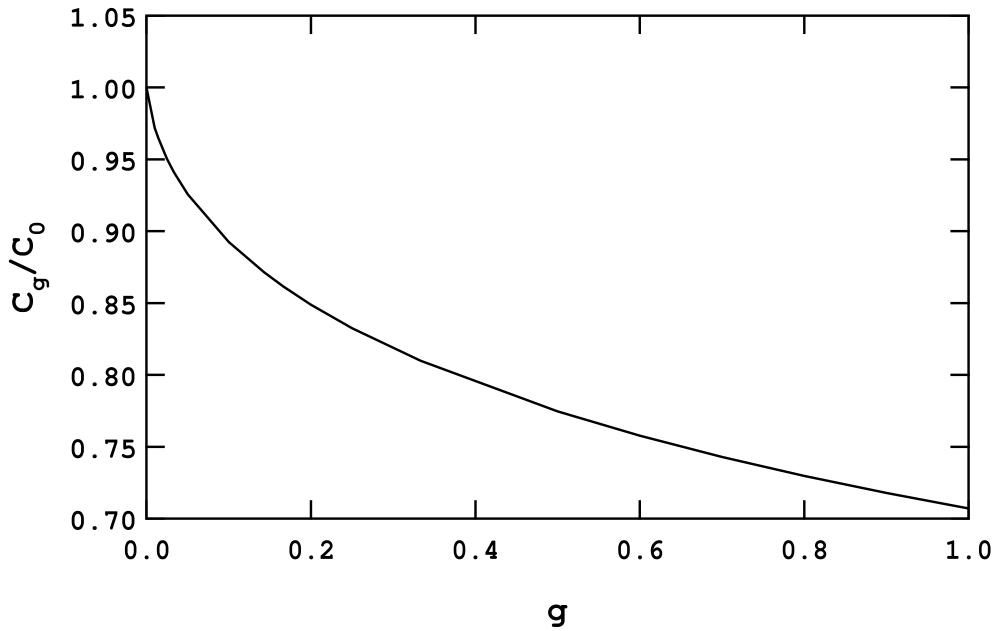

Fig 4 Blencowe