

Cooling Many Particles to Very Low Temperatures

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In a recent paper [Beige, Knight, and Vitiello, quant-ph/0404160], we showed that a large number N of particles can be cooled very efficiently. The particles should be excited by red-detuned laser fields while coupling to the quantized field mode inside a resonant and leaky optical cavity. When the coupling constants are for all particles the same, a collective behavior can be generated and the cooling rate can be as large as \sqrt{N} times the single-particle coupling constants. Here we study the algebraic structure of the dynamics and the origin of the collective cooling process in detail.

1 Introduction

Collective features in the microscopic dynamics often lead to the emergence of surprising and unexpected effects in the evolution of a physical system at the macroscopic level [1]. The collective cooling of many two-level particles to very low temperatures is discussed here as an example of such a macroscopic manifestation of microscopic collective behavior [2]. It is shown that the collective behavior of a large number of particles can produce much higher cooling rates than could be obtained by means of individual cooling based on the spontaneous decay of the individual particles [3-5].

As in laser sideband cooling techniques for single two-level atoms [3], we consider an experimental setup, where red-detuned laser fields increase the excitation of the particles, thereby continuously reducing the number of phonons. Afterwards, the phonon energy can be removed constantly from the system. This requires energy dissipation and yields an overall decrease of the von Neumann entropy in the setup [6]. One possible decay channel is spontaneous emission from the excited states of the particles. During such a photon emission, a particle returns most likely into its ground state without regaining the phonon energy lost in the excitation process. The net result is a *conversion* of the phonons, originally existing in the setup in the form of thermal energy, into photons escaping the system.

Here we are interested in realizing much higher cooling rates than could be achieved with the help of the above described spontaneous decay of individual particles. This is possible, when the time evolution of the system remains restricted onto a subspace of states presenting high symmetry and strong coupling. Maximum cooling is obtained when the particles exhibit *cooperative* behavior in the excitation step as well as in the de-excitation step. To achieve this we

assume as in Ref. [2], that the outward energy dissipation is conducted by an optical cavity. As shown in Fig. 1, the particles should be placed inside a resonant optical cavity with a relatively large rate for the leakage of photons through the cavity mirrors. The particles can then transfer their excitation collectively into the resonator field, from where the energy leaves the system without affecting the number of phonons in the setup and without changing the symmetry of the involved states.

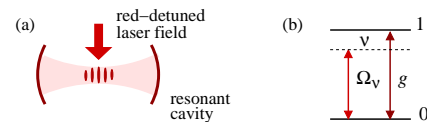


Figure 1. (a) Experimental setup for the collective cooling of many particles. The particles should be trapped inside an optical cavity, where they arrange themselves in the antinodes of the resonator field. (b) Level configuration of a single two-level particle for the cooling of a vibrational mode with frequency ν driven by a red-detuned laser field with Rabi frequency Ω_ν and coupling to a resonant optical cavity with strength g .

Collective behavior of the system requires furthermore that the Rabi frequency Ω_ν of the laser field for the cooling of a vibrational mode with frequency ν and the cavity coupling constant g are for all particles (practically) the same. Initially, the particles should all be prepared in their ground state (in the large N limit fluctuations can be neglected). As shown below, the collective states of the assembly of N particles then experience a very strong coupling to the laser field as well as to the cavity mode [2]. As a consequence, the number of phonons decreases exponentially with a rate as large as \sqrt{N} times the single particles coupling constants, which can be much higher than previously predicted rates in comparable setups [7-15].

The proposed cooling scheme might be used to cool a large number of particles very efficiently. So it should be applicable to the preparation of Bose Einstein condensates. Currently, these experiments mainly use evaporative cooling [16] which systematically removes those atoms with a relatively high temperature from the trap. Consequently, only a small percentage of the initially trapped atoms is finally included in the condensate. If one could instead cool all the atoms efficiently, yet at the same time avoid the loss of particles, it should become easier to experiment with large condensates. Cooling is also crucial for ion trap quantum computing, where the achievable gate operation times can depend primarily on the efficiency of the cooling of a common vibrational mode [17]. First cavity-cooling experiments involving many particles and observing enhanced cooling rates have already been performed [18, 19].

In this paper, we consider some algebraic features of the dynamics ruling the collective cooling of many particles and further clarify some aspects of the underlying mechanisms, which have already been studied in Ref. [2]. A remarkable feature is, for example, the presence of a relatively large and negative coherence k_3 . Instead of solving the time evolution of the system explicitly, we avoid certain approximations made in Ref. [2] by referring to the Heisenberg picture. The system considered in this paper and its collective dynamics may be a paradigmatic example for other applications of physical interest, like the evolution of a system undergoing a continuous phase transition, yet preserving some specific features during such an evolution.

2 The experimental setup

The cavity coupling constant g is for all particles the same, when the particles distribute themselves in the antinodes of the resonator field, as discussed in Ref. [12, 13] (see Fig. 1(a)). Alternatively, a ring resonator could be used, as proposed in Refs. [12, 15, 19]. In the following we consider a collection of N two-level particles (atoms, ions or molecules) with ground states $|0\rangle_i$ and excited states $|1\rangle_i$ (see Fig. 1(b)). The setup should be operated in a parameter regime, where

$$\kappa \sim \sqrt{N}g, \frac{1}{2}\sqrt{N}\eta\Omega \gg \Gamma \text{ with } \Omega \equiv \left(\sum_{\nu} \Omega_{\nu}^2\right)^{1/2}. \quad (1)$$

Here κ is the decay rate of a single photon in the cavity mode, Γ is the spontaneous decay rate of a particle in the excited state and η denotes the Lamb-Dicke parameter characterizing the steepness of the trap [20]. As in Ref. [2], we discuss in this paper the two extreme cases, namely the cooling of common vibrational modes and the cooling of the individual phonon modes in the absence of common vibrational modes.

2.1 Cooling of common modes

In the following, b_{ν} denotes the annihilation operator for a phonon in the common vibrational mode with frequency ν , while c is the annihilation operator for the cavity photons

and $\sigma_i = |0\rangle_i\langle 1|$ is the lowering operator for particle i . Using the rotating wave approximation [21] and going over to the interaction picture with respect to the interaction-free Hamiltonian, the time evolution of the system can be described by the Hamiltonian

$$H_{\text{comm}} = \hbar \sum_{i,\nu} \frac{1}{2}\eta\Omega_{\nu} \sigma_i^{\dagger} b_{\nu} + \sum_i g \sigma_i^{\dagger} c + \text{H.c.} \quad (2)$$

One way to simplify this Hamiltonian is to introduce the effective phonon annihilation operator

$$b \equiv \sum_{\nu} (\Omega_{\nu}/\Omega) b_{\nu} \text{ with } [b, b^{\dagger}] = 1, \quad (3)$$

which allows to write the Hamiltonian (2) as

$$H_{\text{comm}} = \hbar \sum_i \left(\frac{1}{2}\eta\Omega \sigma_i^{\dagger} b + g \sigma_i^{\dagger} c + \text{H.c.} \right). \quad (4)$$

2.2 Cooling of individual particles

We now consider the operator $b_{\nu,i}$ for the annihilation of the phonons of mode ν of particle i . Again we assume that the corresponding laser Rabi frequencies Ω_{ν} are for all particles the same. Then the Hamiltonian of the system equals in the interaction picture and in the rotating wave approximation

$$H_{\text{indi}} = \hbar \sum_{i,\nu} \frac{1}{2}\eta\Omega_{\nu} \sigma_i^{\dagger} b_{\nu,i} + \sum_i g \sigma_i^{\dagger} c + \text{H.c.} \quad (5)$$

Proceeding as above, introducing the effective phonon annihilation operator

$$b_i \equiv \sum_{\nu} (\Omega_{\nu}/\Omega) b_{\nu,i} \text{ with } [b_i, b_i^{\dagger}] = 1 \quad (6)$$

and using Eq. (1), the Hamiltonian (5) becomes

$$H_{\text{ind}} = \hbar \sum_i \left(\frac{1}{2}\eta\Omega \sigma_i^{\dagger} b_i + g \sigma_i^{\dagger} c + \text{H.c.} \right). \quad (7)$$

Although this Hamiltonian has some similarities with the Hamiltonian (4), it describes a physically different situation. Instead of coupling to a set of common vibrational modes, each particle sees his own set of phonons.

2.3 Spontaneous emission

Spontaneous emission is described in the following by the master equation [22]

$$\begin{aligned} \dot{\rho} = & -\frac{i}{\hbar} [H_I, \rho] + \kappa (c\rho c^{\dagger} - \frac{1}{2}c^{\dagger}c\rho - \frac{1}{2}\rho c^{\dagger}c) \\ & + \Gamma \sum_i (\sigma_i \rho \sigma_i^{\dagger} - \frac{1}{2}\sigma_i^{\dagger} \sigma_i \rho - \frac{1}{2}\rho \sigma_i^{\dagger} \sigma_i) \end{aligned} \quad (8)$$

with H_I being H_{comm} or H_{ind} , respectively, κ being the decay rate of a photon in the cavity and Γ being the decay rate of the particle excited state. From this equation one can easily see that the dissipation reduces the energy in the system. This is achieved by lowering the number of photons

inside the resonator but does not affect the state of the particles. Cavity decay therefore does not disturb the collective behavior of the system. However, this does not apply to the emission of photons from the particles, which can negatively interfere with the collective cooling process.

3 Bosonic subsystems

In case of the cooling of common vibrational modes, the setup consists effectively of *two* different subsystems. One is the two-level particle system. As an effect of the emergence of the cooperative behavior, the collection of particles manifests itself as a *bosonic* system. We will see below that a further consequence of this is the transition to a strong coupling regime, which in turn implies a much shorter time scale for the system evolution. The other subsystem is the bosonic system of phonons and photons with a continuous *conversion* of phonons into photons. We will see that it is convenient to consider a boson mode which is a superposition of them. This is analogous to the field-atom polariton of Hopfield [23, 24].

3.1 Bosonic behavior of the particles

Suppose the time evolution of the particles is solely governed by the operators $\sigma^+ = \sum_i \sigma_i^+$, $\sigma^- = \sum_i \sigma_i^-$ and $\sigma_3 = \frac{1}{2} \sum_i (|1\rangle_{ii}\langle 1| - |0\rangle_{ii}\langle 0|)$. The σ_i are the Pauli matrices. These operators obey the $SU(2)$ commutation relations of a *fermion-like* N -body system

$$[\sigma_3, \sigma^\pm] = \pm \sigma^\pm \text{ and } [\sigma^-, \sigma^+] = -2\sigma_3. \quad (9)$$

Under this condition, the time evolution of the system with all particles initially prepared in $|0\rangle_i$ remains restricted to a subspace of highly symmetric particle Dicke states (for more details see [2,25-27]). Since we assume a large number N of particles, small fluctuations in the initial state of the system can be neglected. Let us denote by l the number of particles excited by the laser action into the upper state $|1\rangle$. Then it can be shown that for $N \gg l$ the particle system can be described by the collective operators [2]

$$S^+ \equiv \frac{1}{\sqrt{N}} \sum_i \sigma_i^+, \quad S^- \equiv \frac{1}{\sqrt{N}} \sum_i \sigma_i^-, \quad S_3 \equiv \sigma_3, \quad (10)$$

with $\sigma_3 = S^+ S^- - \frac{1}{2} N$. In the large N limit ($N \gg l$), the operators in Eq. (10) obey the relations

$$[S_3, S^\pm] = \pm S^\pm, \quad [S^-, S^+] = 1. \quad (11)$$

This means, the $su(2)$ algebra (9) written in terms of S^\pm and S_3 contracts in the large- N limit to the (projective) $e(2)$ (or Heisenberg-Weyl) algebra (11) [28-30].

The physical implication of the familiar commutator relations (11) is that the particles behave no longer like *individual fermions*. Instead, the time evolution generates the excitation of bosonic modes, namely collective dipole waves, with S^\pm denoting the creation and annihilation operators of the associated quanta obeying the usual commutation relation (11). As a consequence, the collection of

the single two-level particles manifests itself as a *bosonic* system. In other words, the ladder of equally-spaced Dicke states approximates to a weakly-excited harmonic oscillator.

Using the operators (10), the Hamiltonian (4) for the cooling of *common* vibrational modes can simply be written as

$$H_{\text{comm}} = \hbar(x S^+ b + y S^+ c + \text{H.c.}), \quad (12)$$

where we have introduced the notation

$$x \equiv \frac{1}{2} \sqrt{N} \eta \Omega \text{ and } y \equiv \sqrt{N} g. \quad (13)$$

From this one sees that the time evolution of the system is mainly governed by the parameters x , y and κ , which scale as \sqrt{N} . We thus have, as a consequence of the emergence of the particle collective behavior, the transition to the strong coupling regime, $\frac{1}{2} \eta \Omega \rightarrow \frac{1}{2} \sqrt{N} \eta \Omega$ and $g \rightarrow \sqrt{N} g$. The evolution of the system happens no longer on the time scale given by the parameters $\frac{1}{2} \eta \Omega$ and g but on a much shorter time scale defined by $\frac{1}{2} \sqrt{N} \eta \Omega$ and $\sqrt{N} g$.

3.2 Photon-phonon exchange

Another factor that contributes significantly to the described cooling mechanism is the *interface* or *ambivalent* role played by the particles with respect to the phonons and the photons in the setup. The only difference between the particle-phonon and particle-photon interaction is the difference of the coupling constants, given by $\sqrt{N} \frac{1}{2} \eta \Omega$ and $\sqrt{N} g$, respectively. In some sense, the particles act as an engine transforming phonon energy into photon energy. This last one is then dissipated outward through the coupling with the cavity which allows energy leakage. However, in the absence of the leakage of photons through the cavity, the inverse transformation, photons to phonons, is also possible in principle. In such a situation, interesting interference and coherence effects arise, which we analyze below in detail.

Indeed, a closer look at Eq. (12) reveals that the Hamiltonian for the cooling of common vibrational modes can alternatively be written as

$$H_{\text{comm}} = \hbar z S^+ a + \text{H.c.} \quad (14)$$

with

$$z \equiv \sqrt{x^2 + y^2}, \quad a \equiv \frac{1}{z} (x b + y c) \text{ and } [a, a^\dagger] = 1. \quad (15)$$

Instead of interacting with the phonons and photons separately, the particles see the boson mode with annihilation operator a and number operator

$$a^\dagger a = \frac{1}{z^2} [x^2 b^\dagger b + y^2 c^\dagger c + y x k_3] \quad (16)$$

with

$$k_3 \equiv b^\dagger c + b c^\dagger. \quad (17)$$

Physically, the creation of bosons corresponding to a^\dagger does not only affect the number of phonons and the number of

photons in the system. Inevitably, it also creates a coherence between the b and the c subsystem. This coherence k_3 provides a ‘‘symmetric’’ channel for the phonon-photon energy transformation. However, the leakage of energy outside the cavity perturbs such a symmetric phonon-photon balancing due to the k_3 action. The system reacts by subsequent adjustments, trying to recover its lost balance. Crucial to such a re-adjustment mechanism is the difference in the time scales. The time scale for reaching the quasi-stationary state is of the order $1/N$, while the time scale for spontaneous decay is, as we see in the next Section, of order $1/\sqrt{N}$.

4 Collective cooling of common vibrational modes

The time evolution of the system turns out to be highly non-linear since the second and higher order derivatives of the physical observables are much larger than their first order derivatives. The system reaches a quasi-stationary state on a time scale of the order $1/N$. The formation of this local equilibrium is solely governed by the effect of the Hamiltonian H_{comm} . Let us therefore first consider the situation, where we can neglect spontaneous emission and $\kappa \approx 0$ and $\Gamma \approx 0$.

4.1 Conservation laws in the absence of dissipation

In this case, the time evolution of the system, governed by the Hamiltonian (14), results in a redistribution of population between the bosonic subsystem described by $a^\dagger a$ and the particles described by $S^+ S^-$. To analyse this process we introduce the operators

$$\begin{aligned} L_1 &\equiv \frac{1}{2}(S^+ a + S^- a^\dagger), \\ L_2 &\equiv -\frac{i}{2}(S^+ a - S^- a^\dagger), \\ L_3 &\equiv \frac{1}{2}(S^+ S^- - a^\dagger a) \end{aligned} \quad (18)$$

with the familiar $SU(2)$ commutators

$$[L_i, L_j] = i\epsilon_{ijk} L_k. \quad (19)$$

With this notation, the Hamiltonian (14) becomes

$$H_{\text{comm}} = 2\hbar z L_1 \quad (20)$$

and this, formally, simply generates a rotation around the 1-axis. In such an algebraic picture, we can immediately conclude that the angular momentum L_1 and the total angular momentum

$$\begin{aligned} \underline{L}^2 &= L_1^2 + L_2^2 + L_3^2 \\ &= \frac{1}{2}(S^+ S^- + a^\dagger a) \left[1 + \frac{1}{2}(S^+ S^- + a^\dagger a)\right] \end{aligned} \quad (21)$$

are conserved during the time evolution of the system under the considered conditions. Especially, the conservation of \underline{L}^2 implies the conservation of the total number of bosons, as accounted for by the number operator $S^+ S^- + a^\dagger a$.

When the number of particles in the excited state $|1\rangle_i$ remains small compared to N , the commutator relations (11) hold. In such a case, we can assume that the expectation value of $S^+ S^- = \frac{1}{N} \sum_{i,j} \sigma_i^+ \sigma_j^-$ remains small. Neglecting $S^+ S^-$ in Eq. (21) implies

$$\frac{d}{dt}(a^\dagger a) = 0. \quad (22)$$

Using the definition (15), this conservation law translates into

$$\frac{d}{dt}(y^2 b^\dagger b + x^2 c^\dagger c - xy k_3) = 0, \quad (23)$$

which coincides with Eq. (16) in Ref. [2] and describes the continual balance between the rate of change of the expectation value of k_3 and the rate of change of the total number of phonons and photons in the system.

To show that Eq. (23) does not violate the conservation of the number of particles in the setup, we remark that the system obeys a second conservation law. Considering again the Heisenberg picture, we indeed find

$$\frac{d}{dt}(b^\dagger b + c^\dagger c) = -\frac{d}{dt}(S^+ S^-) = -2z L_2. \quad (24)$$

In the presence of many particles, with most of them remaining in their ground state, the expectation value of $S^+ S^-$ remains negligible. This implies $\frac{d}{dt}(S^+ S^-) = 0$ and, consequently, also $L_2 = 0$ [31]. Inserting this into Eq. (24), we obtain the particle number conservation law

$$\frac{d}{dt}(b^\dagger b + c^\dagger c) = 0. \quad (25)$$

Despite being a coherence, the expectation value of k_3 acts like a population. This allows the system to obey conservation of the particle number $a^\dagger a$ as well as conservation of $b^\dagger b + c^\dagger c$ by balancing the coherence k_3 accordingly. In the following, we study the conversion of phonons into cavity photons and creation of a non-zero coherence k_3 in more detail.

4.2 Quasi-stationary states

In the absence of spontaneous emission, the system reaches after a very short time a stationary state with constant expectation values for $b^\dagger b$, $c^\dagger c$ and k_3 . To calculate the corresponding values of the cavity photon number and coherence k_3 as a function of the phonon number, we use again the Heisenberg picture, and obtain the second order differential equations

$$\begin{aligned} \frac{d^2}{dt^2}(b^\dagger b) &= -2x^2 b^\dagger b - xy k_3, \\ \frac{d^2}{dt^2}(c^\dagger c) &= -2y^2 c^\dagger c - xy k_3, \\ \frac{d^2}{dt^2} k_3 &= -2xy (b^\dagger b + c^\dagger c) - z^2 k_3. \end{aligned} \quad (26)$$

These equations imply that the first order derivatives of the operators $b^\dagger b$, $c^\dagger c$ and k_3 change on a time scale of the order $1/N$, which is, for large N , much faster than the time scale on which the time evolution of $b^\dagger b$, $c^\dagger c$ and k_3 takes place. Since we are only interested in the time dependence of the phonon number $b^\dagger b$, we can safely assume that the first order

derivatives $\frac{d}{dt}(b^\dagger b)$, $\frac{d}{dt}(c^\dagger c)$ and $\frac{d}{dt}k_3$ adapt adiabatically to the state of the system. Setting the right hand sides of the differential equations (26) equal to zero, we find

$$c^\dagger c = \frac{x^2}{y^2} b^\dagger b \quad \text{and} \quad k_3 = -\frac{2x}{y} b^\dagger b. \quad (27)$$

Moreover, Eq. (27) implies, due to the conservation laws (23) and (25), that

$$\frac{d}{dt}(b^\dagger b) = \frac{d}{dt}(c^\dagger c) = \frac{d}{dt}k_3 = 0. \quad (28)$$

This means that, after a short time of order $1/N$, the system reaches a quasi-stationary state with a constant ratio between the expectation value of the coherence k_3 and the number of photons inside the cavity, respectively, compared to the total number of phonons in the system. To a very good approximation, these ratios remain constant throughout the whole cooling process.

4.3 Cooling equations

In the absence of spontaneous emission, at most a redistribution of phonon energy can occur in the system [6]. Efficient cooling and the irreversible removal of energy from the system requires dissipation. We therefore now consider the effect of dissipation in more detail. Using the master equation (8) we find

$$\begin{aligned} \frac{d}{dt}(b^\dagger b) &= ix(S^+ b - S^- b^\dagger), \\ \frac{d}{dt}(c^\dagger c) &= iy(S^+ c - S^- c^\dagger) - \kappa c^\dagger c, \\ \frac{d}{dt}k_3 &= iy(S^+ b - S^- b^\dagger) + ix(S^+ c - S^- c^\dagger) \\ &\quad - \frac{1}{2}\kappa k_3. \end{aligned} \quad (29)$$

The leakage of photons through the cavity mirrors with decay rate κ not only decreases the number of photons in the cavity mode but also affects the size of k_3 .

In the previous subsection, we have seen that the system reaches a stationary state on a time scale of the order $1/N$ with $c^\dagger c$ and k_3 being a multiple of $b^\dagger b$. Using Eq. (29), we find

$$\frac{d}{dt}(b^\dagger b) = -\frac{x^2}{y^2} \left[\kappa c^\dagger c + \frac{d}{dt}(c^\dagger c) \right] + \frac{x}{y} \left[\frac{1}{2}\kappa k_3 + \frac{d}{dt}k_3 \right]. \quad (30)$$

This shows that on the time scale we are interested in, namely a time scale of the order $1/\sqrt{N}$, both, the operator $c^\dagger c$ and the presence of a negative valued operator k_3 , provide effective cooling channels in the system. Inserting the results (27) and (28) into Eq. (30) we obtain indeed

$$\frac{d}{dt}(b^\dagger b) = -\frac{x^2 z^2}{y^4} \kappa b^\dagger b. \quad (31)$$

If m denotes the phonon number expectation value $\langle b^\dagger b \rangle_\rho$ and $m_0 = m(0)$, we finally obtain

$$m = m_0 \exp \left[-\frac{x^2 z^2}{y^4} \kappa t \right]. \quad (32)$$

This describes exponential cooling of the atomic sample with a rate that can be as large as $\sqrt{N}g$ and $\frac{1}{2}\sqrt{N}\eta\Omega$ (cf. Eq. (1)). The result (32) coincides with Eq. (17) in Ref. [2] and is in good agreement with an exact numerical solution of the time evolution of the system (see Fig. 2(a)). Spontaneous emission from excited atomic levels with decay rate Γ is too slow to contribute to the cooling of the system, since most of the particles remain in their ground states and $\kappa \gg \Gamma$.

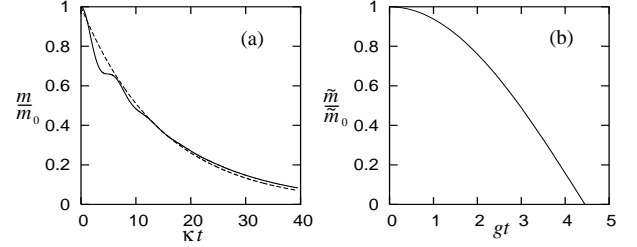


Figure 2. Cooling of common vibrational modes obtained from a numerical solution of the master equation (8) (solid line) in comparison to Eq. (32) (dashed line) for $g = 10^{-3} \kappa$, $\eta\Omega = 5 \cdot 10^{-4} \kappa$, $N = 10^6$ and $m_0 = 10^3$ (a). Cooling of individual phonon modes, obtained from a numerical solution of Eq. (21) in Ref. [2] under the assumption of negligible cavity photon numbers, for $g = 10^{-3} \kappa$, $\eta\Omega = 5 \cdot 10^{-4} \kappa$, $N = 10^6$ and $\tilde{m}_0 = 10^9$ (b).

4.4 Similarities between collective cooling and phase transitions

Finally, we remark that the cooling process we describe here is not simply a redistribution of phonons into cavity photons and k_3 , which then decay into the environment with the spontaneous decay rates κ and $\frac{1}{2}\kappa$, respectively. Instead, the leakage of photons through the cavity mirrors disturbs the equilibrium expressed by Eqs. (23) and (25) perturbing the otherwise conserved quantities

$$Q \equiv \langle a^\dagger a \rangle \quad \text{and} \quad Q' \equiv \langle b^\dagger b \rangle + \langle c^\dagger c \rangle. \quad (33)$$

The coupling to the environment causes a *dynamical response* in the system, which generates a transition from a state with a fixed ratio of phonons to cavity photons (cf. Eq. (27)) into a state with no phonons and photons in the setup.

In atom-cavity systems, the time evolution of the system is, on a very short time scale, usually dominated by spontaneous emission. However, in the presence of many particles, the evolution is primarily governed by the Hamiltonian H_{comm} , which drives the system into a quasi-stationary state characterised by the preserved quantities Q and Q' within a time of the order $1/N$. The presence of the spontaneous decay rate κ , which scales as \sqrt{N} (cf. Eq. (1)), disturbs this stationary state and causes the system to continuously assume new values for Q and Q' . Using Eqs. (27) and (31), we find

$$\frac{d}{dt}Q = -\frac{x^2 z^2}{y^4} \kappa Q \quad \text{and} \quad \frac{d}{dt}Q' = -\frac{x^2 z^2}{y^4} \kappa Q'. \quad (34)$$

While the relative size of the photon number $\langle c^\dagger c \rangle$ and the coherence $\langle k_3 \rangle$ with respect to the number of phonons m remain constant, their sum, as accounted for by Q and Q' decrease exponentially in time due to the presence of dissipation, namely leakage of photons through the cavity mirrors. This is typical for a system undergoing a phase transition.

5 Collective cooling of individual phonon modes

There are many similarities but also many differences between the cooling of the common and the cooling of the individual phonon modes of particles. For example, one cannot simplify the Hamiltonian H_{ind} using the collective lowering operator S^- , as we did in Eq. (12). However, there is still a high symmetry in the system and all particles are treated in exactly the same way. To take this into account we introduce the vector operators

$$\underline{S}^\pm \equiv \frac{1}{\sqrt{N}} \{ \sigma_i^\pm \}, \quad \underline{b} \equiv \{ b_i \} \quad \text{and} \quad \underline{c} \equiv c \{ 1_i \} \quad (35)$$

with the usual scalar product, such that, for example,

$$\underline{S}^\pm \cdot \underline{b} = \frac{1}{\sqrt{N}} \sum_i \sigma_i^\pm b_i. \quad (36)$$

This notation allows us to write the Hamiltonian (7) as

$$H_{\text{ind}} = \hbar (x \underline{S}^+ \cdot \underline{b} + y \underline{S}^+ \cdot \underline{c} + \text{H.c.}), \quad (37)$$

which is of a similar form as the Hamiltonian H_{comm} in Eq. (12).

In analogy to Section 3.2, we proceed by introducing an effective annihilation operator a_i for each particle i with

$$a_i \equiv \frac{1}{z} (x b_i + y c), \quad \underline{a} \equiv \{ a_i \} \quad \text{with} \quad [a_i, a_i^\dagger] = 1. \quad (38)$$

Then the Hamiltonian (37) can be written as

$$H_{\text{ind}} = \hbar z \underline{S}^+ \cdot \underline{a} + \text{H.c.} \quad (39)$$

Each particle couples individually to a new type of bosons, which are a superposition of a single phonon and a cavity photon. The number operator accounting for all bosons a_i equals

$$\underline{a}^\dagger \cdot \underline{a} = \frac{1}{z^2} [x^2 \underline{b}^\dagger \cdot \underline{b} + y^2 \underline{c}^\dagger \cdot \underline{c} + yx k_3], \quad (40)$$

where we defined the coherence

$$k_3 \equiv \underline{b}^\dagger \cdot \underline{c} + \underline{b} \cdot \underline{c}^\dagger. \quad (41)$$

While $\underline{b}^\dagger \cdot \underline{b}$ counts the total number of phonons in the system, the expectation value of $\underline{c}^\dagger \cdot \underline{c} = N c^\dagger c$. Again, the coherence k_3 describes a certain ‘‘symmetry’’ between phonons and photons in the setup and accounts for a continual conversion of the two types of bosons into each other.

5.1 Conservation laws in the absence of dissipation

Neglecting spontaneous emission, i.e. assuming $\kappa \approx 0$ and $\Gamma \approx 0$, and using the Hamiltonian (37), we find

$$\begin{aligned} \frac{d}{dt} (\underline{b}^\dagger \cdot \underline{b}) &= ix (\underline{S}^+ \cdot \underline{b} - \underline{S}^- \cdot \underline{b}^\dagger), \\ \frac{d}{dt} (\underline{c}^\dagger \cdot \underline{c}) &= iNy (\underline{S}^+ \cdot \underline{c} - \underline{S}^- \cdot \underline{c}^\dagger), \\ \frac{d}{dt} k_3 &= iy (\underline{S}^+ \cdot \underline{b} - \underline{S}^- \cdot \underline{b}^\dagger) + ix (\underline{S}^+ \cdot \underline{c} - \underline{S}^- \cdot \underline{c}^\dagger). \end{aligned} \quad (42)$$

In the derivation of these equations, we neglected the operators $\sigma_i^\dagger b_j$ and $\sigma_i b_j^\dagger$ with $i \neq j$ since there are no interactions between particles and the phonons of other particles. From Eq. (42) we see that there are, as before, two conserved quantities in the system, namely

$$\begin{aligned} Q &\equiv y^2 \langle \underline{b}^\dagger \cdot \underline{b} \rangle + \frac{x^2}{N} \langle \underline{c}^\dagger \cdot \underline{c} \rangle - xy \langle k_3 \rangle, \\ Q' &\equiv \langle \underline{b}^\dagger \cdot \underline{b} \rangle + \frac{1}{N} \langle \underline{c}^\dagger \cdot \underline{c} \rangle. \end{aligned} \quad (43)$$

One is associated with the total number of bosonic particles with annihilation operators a_i , the other one counts the total number of phonons and photons in the setup and

$$\frac{d}{dt} Q = \frac{d}{dt} Q' = 0. \quad (44)$$

From the formal equivalence of the Hamiltonians (14) and (39) pointed out in the previous subsection, one might have had expected that the expectation value $\langle \underline{a}^\dagger \cdot \underline{a} \rangle$ is preserved in the time evolution of the system. However, this would only be the case if a_i and a_j^\dagger commute with each other for $i \neq j$ and does not apply here.

5.2 Quasi-stationary states

To calculate the distribution of phonons, cavity photons and coherence k_3 in the system, which builds up in the absence of spontaneous emission, we now consider the second derivatives of the operators $\underline{b}^\dagger \cdot \underline{b}$, $\underline{c}^\dagger \cdot \underline{c}$ and k_3 in the Heisenberg picture. Neglecting again the population in the excited states, as accounted for by $\underline{S}^+ \cdot \underline{S}^-$, we obtain

$$\begin{aligned} \frac{d^2}{dt^2} (\underline{b}^\dagger \cdot \underline{b}) &= -\frac{2x^2}{N} \underline{b}^\dagger \cdot \underline{b} - \frac{xy}{N} k_3, \\ \frac{d^2}{dt^2} (\underline{c}^\dagger \cdot \underline{c}) &= -2y^2 \underline{c}^\dagger \cdot \underline{c} - xy k_3, \\ \frac{d^2}{dt^2} k_3 &= -\frac{2xy}{N} (\underline{b}^\dagger \cdot \underline{b} + \underline{c}^\dagger \cdot \underline{c}) - \frac{z^2}{N} k_3. \end{aligned} \quad (45)$$

Setting the right hand side of these equations equal to zero and taking the conservation laws (44) in the absence of dissipation into account, we find that the system possesses a stationary state with

$$\begin{aligned} \underline{c}^\dagger \cdot \underline{c} &= \frac{x^2}{y^2} \underline{b}^\dagger \cdot \underline{b}, \quad k_3 = -\frac{2x}{y} \underline{b}^\dagger \cdot \underline{b}, \\ \frac{d}{dt} (\underline{b}^\dagger \cdot \underline{b}) &= \frac{d}{dt} (\underline{c}^\dagger \cdot \underline{c}) = \frac{d}{dt} k_3 = 0. \end{aligned} \quad (46)$$

In contrast to the results in Section 4.2, this equation describes a state with only a relatively small number of photons in the cavity mode compared to the total number of phonons in the setup. However, as we see in the next subsection, the presence of a negative coherence k_3 of the same order of magnitude as $\underline{b}^\dagger \cdot \underline{b}$ provides a cooling channel, which is sufficient to obtain a cooling rate of the same order of magnitude as the cavity decay rate κ .

5.3 Cooling equations

As before, the effective removal of phonons from the setup requires leakage of photons through the cavity mirrors. To take spontaneous emission into account we consider again the master equation (8) and find

$$\begin{aligned} \frac{d}{dt}(\underline{b}^\dagger \cdot \underline{b}) &= ix(\underline{S}^+ \cdot \underline{b} - \underline{S}^- \cdot \underline{b}^\dagger), \\ \frac{d}{dt}(\underline{c}^\dagger \cdot \underline{c}) &= iNy(\underline{S}^+ \cdot \underline{c} - \underline{S}^- \cdot \underline{c}^\dagger) - \kappa \underline{c}^\dagger \cdot \underline{c}, \\ \frac{d}{dt}k_3 &= iy(\underline{S}^+ \cdot \underline{b} - \underline{S}^- \cdot \underline{b}^\dagger) + ix(\underline{S}^+ \cdot \underline{c} - \underline{S}^- \cdot \underline{c}^\dagger) \\ &\quad - \frac{1}{2}\kappa k_3, \end{aligned} \quad (47)$$

which implies

$$\begin{aligned} \frac{d}{dt}(\underline{b}^\dagger \cdot \underline{b}) &= -\frac{x^2}{Ny^2} [\kappa \underline{c}^\dagger \cdot \underline{c} + \frac{d}{dt}(\underline{c}^\dagger \cdot \underline{c})] \\ &\quad + \frac{x}{y} \left[\frac{1}{2}\kappa k_3 + \frac{d}{dt}k_3 \right]. \end{aligned} \quad (48)$$

As already mentioned above, the number of photons in the cavity remains relatively small and does not contribute considerably to the cooling process. Neglecting $\underline{c}^\dagger \cdot \underline{c}$ and using the results from the previous subsection, we obtain

$$\frac{d}{dt}(\underline{b}^\dagger \cdot \underline{b}) = -\frac{x^2}{y^2} \kappa \underline{b}^\dagger \cdot \underline{b} \quad (49)$$

and

$$\tilde{m} = \tilde{m}_0 \exp \left[-\frac{x^2}{y^2} \kappa t \right], \quad (50)$$

where \tilde{m} and \tilde{m}_0 now describe the total number of phonons in the system, as accounted for by $\underline{b}^\dagger \cdot \underline{b}$ at time t and $t = 0$, respectively. As in the case of the cooling of common vibrational modes, cooling rates of the same order of magnitude as the cavity decay rate κ can be obtained, which can be as large as \sqrt{N} times the single particle coupling constants.

However, there are some differences with respect to the case considered in Section 4. In the present case, achieving such a high cooling rate and cooling the system to very low temperatures first requires the build up of a reasonably large coherence k_3 , which does not exist in the absence of any laser driving. Suppose, there are no initial correlations between the particles and their motional degrees of freedom. Then $\frac{d}{dt}k_3 = 0$ (cf. Eq. (47)). Moreover, the second order derivative of k_3 increases only on a time scale of order one (cf. Eq. (45)). Initially, the system is far away from the quasi-stationary state described in the previous subsection. However, once k_3 reaches its equilibrium, the collective cooling process can begin. The coherence k_3 needs no

longer to be established; the system only has to adapt to the small changes of Q and Q' caused by the leakage of photons through the cavity mirrors. For a numerical solution of the time evolution of the system see Fig. 2(b).

6 Conclusions

The emergence of collective dynamics in a system of a large number of particles manifests itself in some macroscopic features in the system behavior. As an example, we considered the problem of fast and efficient cooling of an assembly of N two-level particles trapped inside a leaky optical cavity. Results obtained here confirm those derived in Ref. [2]. The particles are excited by red-detuned laser fields. When the coupling constants are for all particles the same, a collective behavior emerges and the cooling rate can be as large as \sqrt{N} times the single-particle coupling constants. The generation of cooperative behavior of the N particles is crucial in the excitation step as well as in the de-excitation step. The collective states of the assembly of N particles then experience a very strong coupling to the laser field as well as to the cavity mode.

We have considered the case of particles sharing a common phonon mode and the case of individual phonon modes for each particle. The two cases are similar and in both cases the collective cooling rate is very fast as compared to the single-particle cooling rate. The two cases are, however, different for the behavior of the coherence k_3 : in the individual mode case, high cooling rates are achieved by first building up a reasonably large coherence k_3 , which does not exist in the absence of any laser driving. Initially, the system is far away from the quasi-stationary state, where the phonon and the photon populations are balanced through the k_3 action. Only when k_3 reaches its equilibrium, the collective cooling process can begin and the system can adapt to the small changes of Q and Q' caused by the leakage of photons through the cavity mirrors. In the common mode case, there is no need for the ‘‘pre-cooling’’ phase for a build up of the coherence k_3 .

The system considered in this paper may be a paradigmatic example for other applications of physical interest, such as systems undergoing continuous phase transitions, yet preserving some specific features during their evolution. Systems presenting such a behavior might be of interest as well in biology.

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