Structure behind Mechanics II: Deduction

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Abstract

This paper proves that protomechanics, previously introduced in quant-ph/9909025, deduces both quantum mechanics and classical mechanics. It does not only solve the problem of the arbitrariness on the operator ordering for the quantization procedure, but also that of the analyticity at the exact classical-limit of $\hbar = 0$. In addition, proto-mechanics proves valid also for the description of a half-spin.

To be Submitted to Found. Phys.

1 INTRODUCTION

Previous paper [1] proposed a basic theory on physical reality, named as *Structure behind Mechanics* (SbM).[1] It supposed that a field or a particle X on the four-dimensional spacetime has its internal-time $\tilde{o}_{\mathcal{P}(t)}(X)$ relative to a domain $\mathcal{P}(t)$ of the four-dimensional spacetime, whose boundary and interior represent the present and the past at ordinary time $t \in \mathbf{R}$, respectively. The classical action $S_{\mathcal{P}(t)}(X)$ realizes internal-time $\tilde{o}_{\mathcal{P}(t)}(X)$ in the following relation:

$$\tilde{o}_{\mathcal{P}(t)}(X) = e^{iS_{\mathcal{P}(t)}(X)}. \tag{1}$$

It further considered that object X also has the external-time $\tilde{o}^*_{\mathcal{P}(t)}(X)$ relative to $\mathcal{P}(t)$ which is the internal-time of all the rest but X in the universe. Object X gains the actual existence on $\mathcal{P}(t)$ if and only if the internal-time coincides with the external-time:

$$\tilde{o}_{\mathcal{P}(t)}(X) = \tilde{o}^*_{\mathcal{P}(t)}(X). \tag{2}$$

This condition discretizes or quantizes the ordinary time passing from the past to the future, and realizes the mathematical representation of Whitehead's philosophy. It also shows that object X has its actual reality only when it is related with or exposed to the rest of the world. The both sides of relation (2) further obey the variational principle as

$$\delta\tilde{o}_{\mathcal{P}(t)}(X) = 0 \quad , \quad \delta\tilde{o}^*_{\mathcal{P}(t)}(X) = 0. \tag{3}$$

These equations produce the equations of motion in the deduced mechanics.

SbM provided a foundation for quantum mechanics and classical mechanics, named as *protomechanics* [1], originated by the past work [2]. The sapce M of all the objects over present hypersurface $\partial\mathcal{P}(t)$ had an mapping $o_t : TM \to S^1$ for the position $(x_t, \dot{x}_t) \in TM$ in the cotangent space TM corresponding to an object $X \in \tilde{M}$:

$$o_t\,(x_t, \dot{x}_t) = \tilde{o}_{\mathcal{P}(t)}\,(X)\,. \tag{4}$$

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[1]Consult the letter [3] on the overview of the present theory.

For the velocity field $v_t \in X(M)$ such that $v_t(x_t) = \frac{dx_t}{dt}$, we will introduce a section $\eta_t \in \Gamma[E(M)]$ and call it *synchronicity* over M:

$$\eta_t(x) = o_t(x, v_t(x)); \tag{5}$$

thereby, synchronicity η_t has an information-theoretical sense, as defined for the collective set of the objects X that have different initial conditions from one another. On the other hand, the emergence-frequency $f_t(\eta_t)$ represent the frequency that object X satisfies condition (2) on M, and the true probability measure ν_t on TM representing the ignorance of the initial position, defined the *emergence-measure* $\mu_t(\eta_t)$ as follows:

$$d\mu_t(\eta_t)(x) = d\nu_t(x, v_t(x)) \cdot f_t(\eta_t)(x). \tag{6}$$

The induced Hamiltonian $H_t^{T^*M}$ on T^*M, further, redefines the velocity field v_t and the Lagrangian L_t^{TM} as follows:

$$v_t(x) = \frac{\partial H_t^{T^*M}}{\partial p}(x, p(\eta_t)(x)) \tag{7}$$

$$L_t^{TM}(x, v(x)) = v(x) \cdot p(\eta_t)(x) - H_t^{T^*M}(x, p(\eta_t)(x)), \tag{8}$$

where mapping p satisfies the modified Einstein-de Broglie relation:

$$p(\eta_t) = -i\bar{h}\eta_t^{-1} d\eta_t. \tag{9}$$

The equation of motion is the set of the following equations:

$$\left(\frac{\partial}{\partial t} + \mathcal{L}_{v_t}\right)\eta_t(x) = -i\bar{h}^{-1} L_t^{TM}(x, v_t(x))\,\eta_t(x), \tag{10}$$

$$\left(\frac{\partial}{\partial t} + \mathcal{L}_{v_t}\right) d\mu_t(\eta_t) = 0. \tag{11}$$

Protomechanics had the statistical description on an ensemble of all the synchronicities η_t^τ for the labeling-time τ defined in the previous paper such that $\eta_\tau^\tau = \eta$. The next section will be devoted to the review of such statistical description for protomechanics. Sections 3 and 4 will explain how protomechanics deduces classical mechanics and quantum mechanics, respectively. They will consider the space of the synchronicities such that

$$\Gamma_k^A = \left\{ \eta \;\middle|\; \sup_U p_j(\eta)(x) = \hbar^A k_j \in \mathbf{R} \right\} \tag{12}$$

which requires $A = 0$ and $A = 1$ for classical case and quantum case, respectively. Both cases will consider a Lagrange foliation $\bar{p}$ in TM such that it has a synchronicity $\bar{\eta}[k] \in \Gamma_k^A$

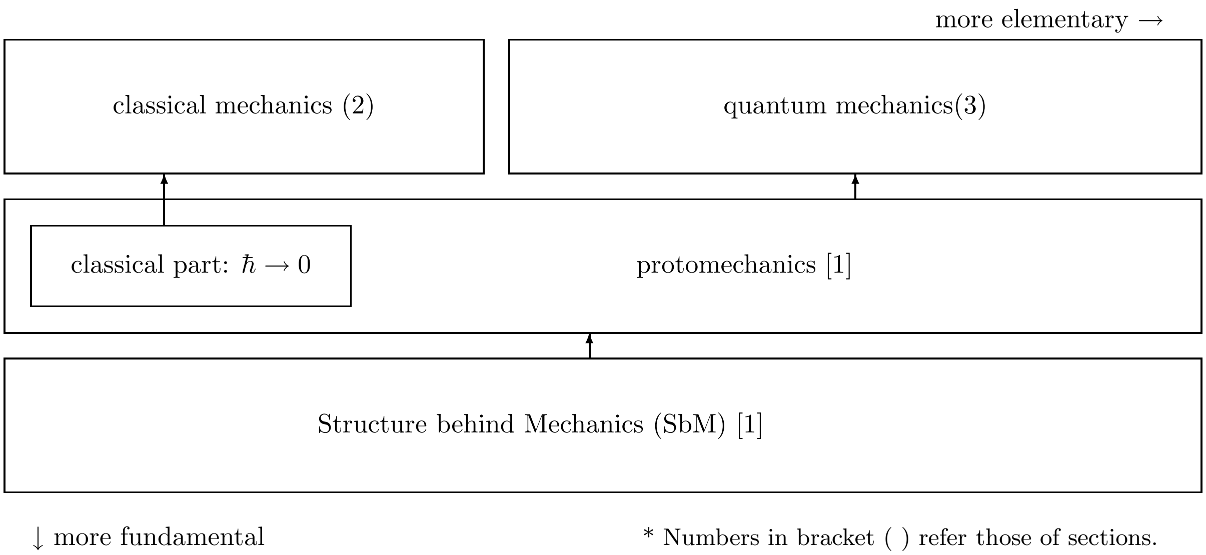
$$\bar{p}[k] = p(\bar{\eta}[k]), \tag{13}$$

and will separate every synchronicity $\eta[k] \in \Gamma_k^A$ into two parts:

$$\eta[k] = \bar{\eta}[k] \cdot \xi. \tag{14}$$

where $\xi \in \Gamma_0^A$. Finally, these sections will compress all the infinite information of back ground ξ to produce classical mechanics and quantum mechanics. Section 3 will additionally discuss a consequent interpretation for the half-spin of a particle; a brief statement of the conclusion will immediately follow.

Let me summarize the construction of the present paper in the following diagram.



In this paper, c and h denote the speed of light and Planck's constant, respectively. I will use Einstein's rule in the tensor calculus for Roman indices' $i, j, k \in \mathbf{N}^N$ and Greek indices' $\nu, \mu \in \mathbf{N}^N$, and not for Greek indices' $\alpha, \beta, \gamma \in \mathbf{N}^N$, and I further denote the trace (or supertrace) operation of a quantum observable $\hat{F}$ as $\langle \hat{F} \rangle$ that is only one difference from the ordinary notations in quantum mechanics, where $i = \sqrt{-1}$.

2 Review on Protomechanics

Let us review the protomechanics in the statistical way for the ensemble of all the synchronicities on M, and construct the dynamical description for the collective motion of the sections of $E(M)$. Such statistical description realizes the description within a long-time interval through the introduced relabeling process so as to change the labeling time, that is the time for the initial condition before analytical problems occur. In addition, it clarifies the relationship between classical mechanics and quantum mechanics under the assumption that the present theory safely induces them, and that will be proved in the following sections.[2] For mathematical simplicity, the discussion below suppose that M is a $N-$dimensional manifold for a finite natural number $N \in \mathbf{N}$.

The derivative operator $D = \hbar dx^j \partial_j : T_0^m(M) \to T_0^{m+1}(M)$ $(m \in \mathbf{N})$ for the space $T_0^n(M)$ of all the $(0, n)$-tensors on M can be described as

$$D^n p(x) = \hbar^n \left(\prod_{k=1}^{n} \partial_{j_k} p_j(x) \right) dx^j \otimes \left(\otimes_{k=1}^n dx^{j_k} \right). \tag{15}$$

By utilizing this derivative operator D, the following norm for every $p \in \Lambda^1(M)$ endows space $\Lambda^1(M)$ with a norm topology:

$$\|p\| = \sup_M \sum_{\kappa \in \mathbf{Z}_{\geq 0}} |D^\kappa p(x)|_x \, , \tag{16}$$

where $| \ \ |_x$ is a norm of covectors at $x \in M$. In terms of this norm topology, we can consider the space $C^\infty\left(\Lambda^1(M), C^\infty(M)\right)$ of all the C^∞-differentiable mapping from $\Lambda^1(M)$ to $C^\infty(M) = C^\infty(M, \mathbf{R})$ and the

[2] In another way, consult quant-ph/9906130.

subspaces of the space $C(\Gamma[E(M)])$ such that

$$C\left(\Gamma\left[E\left(M\right)\right]\right) = \left\{ p^{*}F : \Gamma\left[E(M)\right] \to C^{\infty}(M) \mid F \in C^{\infty}\left(\Lambda^{1}(M), C^{\infty}(M)\right)\right\}. \tag{17}$$

Classical mechanics requires the local dependence on the momentum for functionals, while quantum mechanics needs the wider class of functions that depends on their derivatives. The space of the classical functionals and that of the quantum functionals are defined as

$$C_{cl}\left(\Gamma\left[E\left(M\right)\right]\right) = \left\{p^{*}F \in C\left(\Gamma\left[E\left(M\right)\right]\right) \mid p^{*}F\left(\eta\right)\left(x\right) = \mathbf{F}\left(x, p(\eta)(x)\right) \right\} \tag{18}$$

$$C_{q}\left(\Gamma\left[E\left(M\right)\right]\right) = \{p^{*}F \in C\left(\Gamma\left[E\left(M\right)\right]\right) \mid \tag{19}$$

$$p^{*}F\left(\eta\right)\left(x\right) = \mathbf{F}\left(x, p(\eta)(x), ..., D^{n}p(\eta)(x), ...\right) \}, \tag{20}$$

and related with each other as

$$C_{cl}\left(\Gamma\left[E\left(M\right)\right]\right) \subset C_{q}\left(\Gamma\left[E\left(M\right)\right]\right) \subset C\left(\Gamma\left[E\left(M\right)\right]\right). \tag{21}$$

In other words, the classical-limit indicates the limit of $\hbar \to 0$ with fixing $|p(\eta)(x)|$ finite at every $x \in M$, or what the characteristic length $[x]$ and momentum $[p]$ such that $x/[x] \approx 1$ and $p/[p] \approx 1$ satisfies

$$[p]^{-n-1}D^{n}p(\eta)(x) \ll 1. \tag{22}$$

On the other hand, the emergence-measure $\mu(\eta)$ has the Radon measure $\tilde{\mu}(\eta)$ for section $\eta \in \Gamma[E(M)]$ such that

$$\tilde{\mu}(\eta)\ \ \left(p^{*}F\left(\eta\right)\right) = \int_{M} d\mu(\eta)(x) p^{*}F\left(\eta\right)\left(x\right). \tag{23}$$

Let us assume set $\Gamma\left(E(M)\right)$ is a measure space having the probability measure $\mathcal{M}$ such that

$$\mathcal{M}\left(\Gamma\left(E(M)\right)\right) = 1. \tag{24}$$

For a subset $C_{n}\left(\Gamma\left(E(M)\right)\right) \subset C\left(\Gamma\left(E(M)\right)\right)$, an element $\bar{\mu} \in C_{n}\left(\Gamma\left(E(M)\right)\right)^{*}$ is a linear functional $\bar{\mu}$: $C_{n}\left(\Gamma\left[E(M)\right]\right) \to \mathbf{R}$ such that

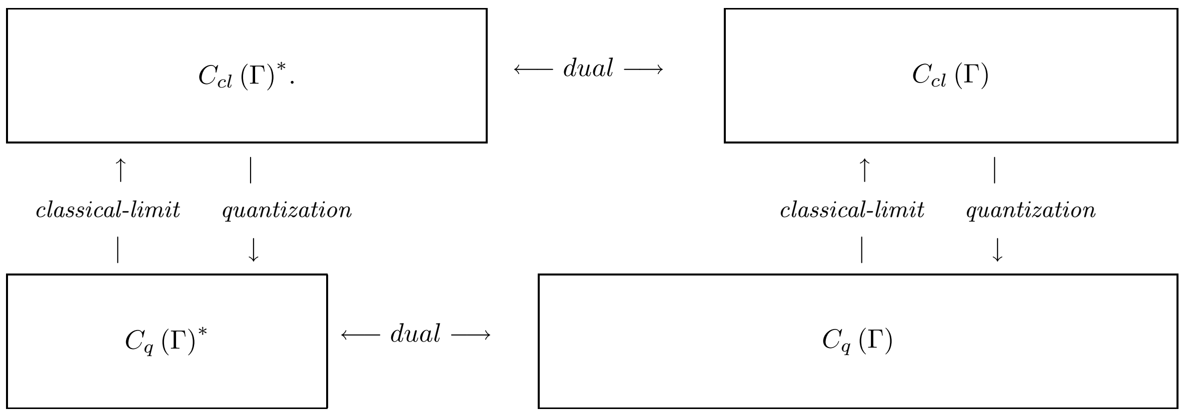
$$\bar{\mu}\left(p^{*}F\right) = \int_{\Gamma[E(M)]} d\mathcal{M}(\eta)\ \tilde{\mu}(\eta)\ \ \left(p^{*}F\left(\eta\right)\right) \tag{25}$$

$$= \int_{\Gamma[E(M)]} d\mathcal{M}(\eta) \int_{M} dv(x)\ \rho\left(\eta\right)(x) F\left(p(\eta)\right)(x), \tag{26}$$

where $d\mu(\eta) = dv\ \rho\left(\eta\right)$. Let us call mapping $\rho : \Gamma[E(M)] \to C^{\infty}(M)$ as the *emergence-density*. The dual spaces make an decreasing series of subsets:

$$C_{cl}\left(\Gamma\left(E(M)\right)\right)^{*} \supset C_{q}\left(\Gamma\left(E(M)\right)\right)^{*} \supset C\left(\Gamma\left(E(M)\right)\right)^{*}. \tag{27}$$

Let us summarize how the relation between quantum mechanics and classical mechanics in the following diagram.



To investigate the time-development of the statistical state discussed so far, we will introduce the related group. The group $\mathcal{D}(M)$ of all the C^∞-diffeomorphisms of M and the abelian group $C^\infty(M)$ of all the C^∞-functions on M construct the semidirect product $S(M) = \mathcal{D}(M) \times_{semi.} C^\infty(M)$ of $\mathcal{D}(M)$ with $C^\infty(M)$, and define the multiplication $\cdot$ between $\Phi_1 = (\varphi_1, s_1)$ and $\Phi_2 = (\varphi_2, s_2) \in S(M)$ as

$$\Phi_1 \cdot \Phi_2 = (\varphi_1 \circ \varphi_2, (\varphi_2^* s_1) \cdot s_2), \tag{28}$$

for the pullback φ^* by $\varphi \in \mathcal{D}(M)$. The Lie algebra $s(M)$ of $S(M)$ has the Lie bracket such that, for $V_1 = (v_1, U_1)$ and $V_2 = (v_2, U_2) \in s(M)$,

$$[V_1, V_2] = ([v_1, v_2], v_1 U_2 - v_2 U_1 + [U_1, U_2]) ; \tag{29}$$

and its dual space $s(M)^*$ is defined by natural pairing $\langle \ , \ \rangle$. Lie group $S(M)$ now acts on every C^∞ section of $E(M)$ (consult *APPENDIX*). We shall further introduce the group $Q(M) = Map\left(\Gamma\left[E(M)\right], S(M)\right)$ of all the mapping from $\Gamma\left[E(M)\right]$ into $S(M)$, that has the Lie algebra $q(M) = Map\left(\Gamma\left[E(M)\right], s(M)\right)$ and its dual space $q(M)^* = Map\left(\Gamma\left[E(M)\right], s(M)^*\right)$.

Let us consider the time-development of the section $\eta_t^\tau(\eta) \in \Gamma[E(M)]$ such that the *labeling time* τ satisfies $\eta_\tau^\tau(\eta) = \eta$. It has the momentum $p_t^\tau(\eta) = -i\bar{h}\eta_t^\tau(\eta)^{-1} d\eta_t^\tau(\eta)$ and the emergence-measure $\mu_t^\tau(\eta)$ such that

$$d\mathcal{M}(\eta)\ \tilde{\mu}_t^\tau(\eta) = d\mathcal{M}\left(\eta_t^\tau(\eta)\right)\ \tilde{\mu}_t\left(\eta_t^\tau(\eta)\right) : \tag{30}$$

$$\begin{aligned}
\bar{\mu}_t\left(p^* F_t\right) &= \int_{\Gamma[E(M)]} d\mathcal{M}(\eta)\ \tilde{\mu}_t(\eta)\ \left(p^* F_t(\eta)\right) && (31)\\
&= \int_{\Gamma[E(M)]} d\mathcal{M}(\eta)\ \tilde{\mu}_t^\tau(\eta)\ \left(p^* F\left(\eta_t^\tau(\eta)\right)\right) && (32)\\
&= \int_{\Gamma[E(M)]} d\mathcal{M}(\eta) \int_M dv(x)\ \rho_t^\tau(\eta)(x) F_t\left(p_t^\tau(\eta)\right)(x). && (33)
\end{aligned}$$

The introduced labeling time τ can always be chosen such that $\eta_t^\tau(\eta)$ does not have any singularity within a short time for every $\eta \in \Gamma\left[E(M)\right]$. The emergence-momentum $\mathcal{J}_t^\tau \in q(M)^*$ such that

$$\begin{aligned}
\mathcal{J}_t^\tau(\eta) &= d\mathcal{M}\left(\eta_t^\tau(\eta)\right)\ \left(\tilde{\mu}_t\left(\eta_t^\tau(\eta)\right) \otimes p_t^\tau(\eta), \tilde{\mu}_t\left(\eta_t^\tau(\eta)\right)\right) && (34)\\
&= d\mathcal{M}(\eta)\ \left(\tilde{\mu}_t^\tau(\eta) \otimes p_t^\tau(\eta), \tilde{\mu}_t^\tau(\eta)\right) && (35)
\end{aligned}$$

satisfies the following relation for the functional $\mathcal{F}_t : q(M)^* \to \mathbf{R}$:

$$\mathcal{F}_t\left(\mathcal{J}_t^\tau\right) = \bar{\mu}_t\left(p^* F_t\right), \tag{36}$$

whose value is independent of labeling time τ. The operator $\hat{F}_t^\tau = \frac{\partial \mathcal{F}_t}{\partial \mathcal{J}}\left(\mathcal{J}_t^\tau\right)$ is defined as

$$\left.\frac{d}{d\epsilon}\right|_{\epsilon=0} \mathcal{F}_t\left(\mathcal{J}_t^\tau + \epsilon\mathcal{K}\right) = \left\langle \mathcal{K}, \hat{F}_t^\tau \right\rangle, \tag{37}$$

i.e.,

$$\hat{F}_t^\tau = \left(\mathcal{D}_{\rho_t^\tau(\eta)} F_t\left(p_t^\tau(\eta)\right), -p_t^\tau(\eta)\cdot\mathcal{D}_{\rho_t^\tau(\eta)} F_t\left(p_t^\tau(\eta)\right) + F_t\left(p_t^\tau(\eta)\right)\right), \tag{38}$$

where the derivative $\mathcal{D}_\rho F(p)$ can be introduced as follows excepting the point where the distribution ρ becomes zero:

$$\mathcal{D}_\rho F(p)(x) = \sum_{(n_1,\ldots,n_N)\in\mathbf{N}^N} \frac{1}{\rho(x)} \left\{ \prod_i^N \left(-\partial_i\right)^{n_i} \left(\rho(x) p(x) \frac{\partial F}{\partial\left\{\left(\prod_i^N \partial_i^{n_i}\right) p_j\right\}} \right) \right\} \partial_j. \tag{39}$$

Thus, the following null-lagrangian relation can be obtained:

$$\mathcal{F}_t\left(\mathcal{J}_t^\tau\right) = \langle \mathcal{J}_t^\tau, \hat{F}_t^\tau \rangle, \tag{40}$$

while the normalization condition has the following expression:

$$\mathcal{I}\left(\mathcal{J}_t^\tau\right) = 1 \qquad for \qquad \mathcal{I}\left(\mathcal{J}_t^\tau\right) = \int_{\Gamma[E(M)]} d\mathcal{M}(\eta)\ \mu_t(\eta)(M). \tag{41}$$

For Hamiltonian operator $\hat{H}_t^\tau = \frac{\partial \mathcal{H}_t}{\partial \mathcal{J}}\left(\mathcal{J}_t^\tau\right) \in q(M)$ corresponding to Hamiltonian $p^* H_t(\eta)(x) = H_t^{T^*M}(x, p(\eta))$, equations (10) of motion becomes Lie-Poisson equation

$$\frac{\partial \mathcal{J}_t^\tau}{\partial t} = ad^*_{\hat{H}_t^\tau} \mathcal{J}_t^\tau, \tag{42}$$

which can be expressed as

$$\frac{\partial}{\partial t}\rho_t^\tau(\eta)(x) = -\sqrt{\ }^{-1}\partial_j \left(\frac{\partial H_t^{T^*M}}{\partial\ p_j}\left(x, p_t^\tau(\eta)(x)\right) \rho_t^\tau(\eta)(x)\sqrt{\ } \right), \tag{43}$$

$$\begin{aligned}
\frac{\partial}{\partial t}\left(\rho_t^\tau(\eta)(x) p_{tk}^\tau(\eta)(x)\right) \quad = \quad & -\sqrt{\ }^{-1}\partial_j \left(\frac{\partial H_t^{T^*M}}{\partial\ p_j}\left(x, p_t^\tau(\eta)(x)\right) \rho_t^\tau(\eta)(x) p_{tk}^\tau(\eta)(x)\sqrt{\ } \right) \\
& -\rho_t^\tau(\eta)(x) p_{tj}^\tau(\eta)(x)\partial_k \frac{\partial H_t^{T^*M}}{\partial\ p_j}\left(x, p_t^\tau(\eta)(x)\right) \\
& +\rho_t^\tau(\eta)(x)\partial_k \left(p_t^\tau(\eta)(x)\cdot \frac{\partial H_t^{T^*M}}{\partial\ p}\left(x, p_t^\tau(\eta)(x)\right) \right. \qquad (44) \\
& \qquad \left. -H_t^{T^*M}\left(x, p_t^\tau(\eta)(x)\right) \right). \qquad (45)
\end{aligned}$$

Equation (42) will prove in the following sections to include the Schrödinger equation in canonical quantum mechanics and the classical Liouville equations in classical mechanics.

For $\mathcal{U}_t^\tau \in Q(M)$ such that $\frac{\partial \mathcal{U}_t^\tau}{\partial t} \circ \left(\mathcal{U}_t^\tau\right)^{-1} = \hat{H}_t^\tau(\eta) \in q(M)$, let us introduce the following operators:

$$\tilde{H}_t^\tau(\eta) = Ad^{-1}_{\mathcal{U}_t^\tau} \hat{H}_t^\tau(\eta) \quad \left(= \hat{H}_t^\tau(\eta)\right), \ and \quad \tilde{F}_t^\tau(\eta) = Ad^{-1}_{\mathcal{U}_t^\tau} \hat{F}_t^\tau(\eta). \tag{46}$$

Lie-Poisson equation (42) is equivalent to the following equation:

$$\frac{\partial}{\partial t}\tilde{F}_t^\tau = \left[\tilde{H}_t^\tau, \tilde{F}_t^\tau\right] + \left(\widetilde{\frac{\partial F_t^\tau}{\partial t}}\right). \tag{47}$$

The general theory for Lie-Poisson systems certificates that, if a group action of Lie group $Q(M)$ keeps the Hamiltonian $\mathcal{H}_t : q(M)^* \to \mathbf{R}$ invariant, there exists an invariant charge functional $Q : \Gamma\left[E(M)\right] \to C(M)$ and the induced function $\mathcal{Q} : q(M)^* \to \mathbf{R}$ such that

$$\left[\hat{H}_t^\tau, \hat{Q}^\tau\right] = 0. \tag{48}$$

3 DEDUCTION OF CLASSICAL MECHANICS

In classical Hamiltonian mechanics, the state of a particle on manifold M can be represented as a position in the cotangent bundle T^*M. In this section, we will reproduce the classical equation of motion from the general theory presented in the previous section. Let us here concentrate ourselves on the case where M is N-dimensional manifold for simplicity, though the discussion below would still be valid if substituting an appropriate Hilbert space when M is infinite-dimensional ILH-manifold[4].

3.1 Description of Statistical State

Now, we must be concentrated on the case where the physical functional $F \in C^\infty\left(\Lambda^1(M), C^\infty(M)\right)$ does *not* depend on the derivatives of the C^∞ 1-form $p\left(\eta\right) \in \Lambda^1(M)$ induced from $\eta \in \Gamma\left[E(M)\right]$, then it has the following expression:

$$p^*F\left(\eta\right)(x) = F^{T^*M}\left(x, p\left(\eta\right)(x)\right). \tag{49}$$

Let us choose a coordinate system $\left(U_\alpha, \mathbf{x}_\alpha\right)_{\alpha\in\Lambda_M}$ for a covering $\left\{U_\alpha\right\}_{\alpha\in\Lambda_M}$ over M, i.e., $M = \bigcup_{\alpha\in\Lambda_M} U_\alpha$. Let us further choose a reference set $U \subset U_\alpha$ such that $v(U) \neq 0$ and consider the set $\Gamma_{Uk}\left[E(M)\right]$ of the C^∞ sections of $E(M)$ having corresponding momentum $p\left(\eta\right)$ the supremum of whose every component $p_j\left(\eta\right)$ in U becomes the value k_j for $k = (k_1, ..., k_N) \in \mathbf{R}^N$:[3]

$$\Gamma_{Uk}\left[E(M)\right] = \left\{\eta \in \Gamma\left[E(M)\right] \;\middle|\; \sup_U p_j\left(\eta\right)(x) = k_j\right\}. \tag{50}$$

Thus, every section $\eta \in \Gamma\left[E(M)\right]$ has some $k \in \mathbf{R}^N$ such that $\eta = \eta[k] \in \Gamma_{Uk}\left[E(M)\right]$. Notice that $\Gamma_{Uk}\left[E(M)\right]$ can be identified with $\Gamma_{U'k}\left[E(M)\right]$ for every two reference sets U and $U' \in M$, since there exists a diffeomorphism φ satisfying $\varphi\left(U\right) = U'$; thereby, we will simply denote $\Gamma_{Uk}\left[E(M)\right]$ as $\Gamma_k\left[E(M)\right]$.

On the other hand, let us consider the space $L\left(T^*M\right)$ of all the Lagrange foliations, i.e., every element $\bar{p} \in L\left(T^*M\right)$ is a mapping $\bar{p} : \mathbf{R}^N \to \Lambda^1(M)$ such that each $q \in T^*M$ has a unique $k \in \mathbf{R}^N$ as

$$q = \bar{p}[k]\left(\pi(q)\right). \tag{51}$$

For every $\bar{p} = p \circ \bar{\eta} \in L\left(T^*M\right)$ such that $\bar{\eta}[k] \in \Gamma_k\left[E(M)\right]$, it is possible to separate an element $\eta[k] \in \Gamma_k\left[E(M)\right]$ for a $\xi \in \Gamma_0\left[E(M)\right]$ as

$$\eta[k] = \bar{\eta}[k] \cdot \xi, \tag{52}$$

or to separate momentum $p\left(\eta[k]\right)$ as

$$p\left(\eta[k]\right) = \bar{p}[k] + p\left(\xi\right); \tag{53}$$

[3] To substitute $\Gamma_{Uk}\left[E(M)\right] = \left\{\eta \in \Gamma\left[E(M)\right] \;\middle|\; \int_U dv(x)\; p_j\left(\eta\right)(x) = k_j v\left(U\right)\right\}$ for definition (50) also induces the similar discussion below, while there exist a variety of the classification methods that produce the same result.

thereby, we can express the emergence-density $\rho : \Gamma[E(M)] \to C^{\infty}(M)$ in the following form for the function $\varrho(\xi) \in C^{\infty}(T^*M, \mathbf{R})$ on T^*M:

$$\rho(\eta[k])(x)\surd = \varrho(\xi)(x, p(\eta[k])(x)) . \tag{54}$$

We call the set $B[E(M)] = \Gamma_0[E(M)]$ the **back ground** of $L(T^*M)$. For the Jacobian-determinant $\sigma[k] = det\left(\frac{\partial \bar{p}^{\tau}_{ti}[k]}{\partial k_j}\right)$, we will define the measure $\mathcal{N}$ on $B[E(M)]$ for the σ-algebra induced from that of $\Gamma[E(M)]$:

$$d\mathcal{M}(\eta[k]) \ dv(x) = d^N k d\mathcal{N}(\xi) dv(x) \ \sigma[k](x) \ . \tag{55}$$

For separation (53), the Radon measure $\tilde{\mu}(\eta)$ induces the measure ω^N on T^*M in the following lemma such that $\omega^N = \phi_{U_\alpha *} d^N x \wedge d^N k$ for $d^N x = dx^1 \wedge ... dx^N$ and $d^N k = dk^1 \wedge ... dk^N$.

Lemma 1 *The following relation holds:*

$$\bar{\mu}(p^* F) = \int_{T^*M} \omega^N(q) \ \rho^{T^*M}(q) F^{T^*M}(q) , \tag{56}$$

where

$$\rho^{T^*M}(q) = \int_{B[E(M)]} d\mathcal{N}(\xi) \ \varrho(\xi)(q) . \tag{57}$$

Proof. *The direct calculation based on separation (53) shows*

$$\begin{aligned}
\bar{\mu}(p^* F) &= \int_{\Gamma[E(M)]} d\mathcal{M}(\eta) \ \tilde{\mu}(\eta)(p^* F(\eta)) \\
&= \int_{\Gamma[E(M)]} d\mathcal{M}(\eta[k]) \int_M dv(x) \ \varrho(\xi)(x, p(\eta[k])(x)) F^{T^*M}(x, p(\eta[k])(x)) \\
&= \int_{\mathbf{R}^N} d^N k \int_{B[E(M)]} d\mathcal{N}(\xi) \int_M dv(x) \ \sigma[k](x) \\
&\qquad \times \ \varrho(\xi)(x, p(\eta[k])(x)) F^{T^*M}(x, p(\eta[k])(x)) \\
&= \int_{\mathbf{R}^N} d^N k \int_{B[E(M)]} d\mathcal{N}(\xi) \int_M dv(x) \ \sigma[k](x) \\
&\qquad \times \ \varrho(\xi)(x, \bar{p}[k](x) + p(\xi)(x)) F^{T^*M}(x, \bar{p}[k](x) + p(\xi)(x)) \qquad (58) \\
&= \int_{B[E(M)]} d\mathcal{N}(\xi) \sum_{\alpha \in \Lambda_M} \int_{\phi_{U_\alpha}(A_\alpha)} d^N k \wedge d^N x \ \phi^*_{U_\alpha} \varrho(\xi)(x, k) \phi^*_{U_\alpha} F^{T^*M}(x, k) \\
&= \int_{B[E(M)]} d\mathcal{N}(\xi) \int_{T^*M} \omega^N(q) \ \varrho(\xi)(q) F^{T^*M}(q) , \qquad (59)
\end{aligned}$$

where $T^*M = \bigcup_{\alpha \in \Lambda_M} A_\alpha$ is the disjoint union of $A_\alpha \in \mathcal{B}(\mathcal{O}_{T^*M})$ such that (1) $\pi(A_\alpha) \subset U_\alpha$ and that (2) $A_\alpha \cap A_\beta = \emptyset$ for $\alpha \neq \beta \in \Lambda_M$ (consult APPENDIX).

If defining the probability function $\rho^{T^*M} : T^*M \to \mathbf{R}$ such that

$$\rho^{T^*M}(q) = \int_{B[E(M)]} d\mathcal{N}(\xi) \ \varrho(\xi)(q) , \tag{60}$$

we can obtain this lemma. □

3.2 Description of Time-Development

Let us consider the time-development of the functional $\bar{\mu}_t : C^1(\Gamma(M), C(M)) \to \mathbf{R}$ for $p_t^\tau(\eta[k]) = \bar{p}_t^\tau[k] + p(\xi)$. For the Jacobian-determinant $\sigma_t^\tau[k] = det\left(\frac{\partial \bar{p}_{ti}^\tau[k]}{\partial k_j}\right)$, the following relation holds:

$$\bar{\mu}_t(p^* F_t) = \int_{T^*M} \omega^N(q)\ \rho_t^{T^*M}(q)\, F^{T^*M}(q) \tag{61}$$

$$= \int_{\mathbf{R}^N} d^N k \int_M dv(x)\ \bar{\rho}_t^\tau[k](x) F^{T^*M}\left(x, \bar{p}_t^\tau[k](x)\right), \tag{62}$$

where

$$\bar{\rho}_t^\tau[k](x)\surd = \sigma_t^\tau[k](x)\rho^{T^*M}\left(x, \bar{p}_t^\tau[k](x)\right). \tag{63}$$

The Jacobian-determinant $\sigma_t^\tau[k]$ satisfies the following relation:

$$\frac{d\mathcal{M}(\eta_t^\tau(\eta))}{d\mathcal{M}(\eta)} = \frac{\sigma_t^\tau[k]}{\sigma[k]}. \tag{64}$$

Thus, we can define the reduced emergence-momentum $\bar{\mathcal{J}}_t \in \bar{q}(M)^* = q(M)^*/B[E(M)]$ as follows:

$$\bar{\mathcal{J}}_t(\bar{\eta}[k]) = \left(d^N k\ \wedge dv\ \bar{\rho}_t^\tau[k] \otimes \bar{p}_t^\tau[k], d^N k\ \wedge dv\ \bar{\rho}_t^\tau[k]\right); \tag{65}$$

and we can define the functional $\bar{\mathcal{F}}_t \in C^\infty\left(\bar{q}(M)^*, \mathbf{R}\right)$ as

$$\bar{\mathcal{F}}_t\left(\bar{\mathcal{J}}_t\right) = \bar{\mu}_t(p^* F_t) \tag{66}$$

$$= \int_{\mathbf{R}^N} d^N k \int_M dv(x)\ \bar{\rho}_t^\tau[k](x) F_t^{T^*M}\left(x, \bar{p}_t^\tau[k](x)\right), \tag{67}$$

which is independent of labeling time τ.

Then, the operator $\hat{F}_t^{cl} = \frac{\partial \bar{\mathcal{F}}_t}{\partial \mathcal{J}}\left(\bar{\mathcal{J}}_t\right)$ satisfies

$$\hat{F}_t^{cl} = \left(\frac{\partial F_t^{T^*M}}{\partial p}\left(x, \bar{p}_t^\tau[k](x)\right), -L^{F_t^{T^*M}}\left(x, \frac{\partial F_t^{T^*M}}{\partial p}\left(x, \bar{p}_t^\tau[k](x)\right)\right)\right), \tag{68}$$

where

$$L^{F_t^{T^*M}}\left(x, \frac{\partial F_t^{T^*M}}{\partial p}(x,p)\right) = p \cdot \frac{\partial F_t^{T^*M}}{\partial p}(x,p) - F_t^{T^*M}(x,p) \tag{69}$$

is the Lagrangian if function F_t is Hamiltonian H_t. Thus, the following null-lagrangian relation can be obtained:[4]

$$\bar{\mathcal{F}}_t\left(\bar{\mathcal{J}}_t\right) = \langle \bar{\mathcal{J}}_t, \hat{F}_t^{cl} \rangle. \tag{70}$$

Besides, the normalization condition becomes

$$\bar{\mathcal{I}}\left(\bar{\mathcal{J}}_t\right) = 1 \qquad for \qquad \bar{\mathcal{I}}\left(\bar{\mathcal{J}}_t\right) = \int_{\mathbf{R}^N} d^N k \int_M dv(x)\ \bar{\rho}_t^\tau[k](x). \tag{71}$$

Theorem 1 *For Hamiltonian operator $\hat{H}_t = \frac{\partial \mathcal{H}_t}{\partial \mathcal{J}}\left(\bar{\mathcal{J}}_t\right) \in \bar{q}(M)$, the equation of motion becomes Lie-Poisson equation:*

$$\frac{\partial \bar{\mathcal{J}}_t}{\partial t} = ad^*_{\hat{H}_t^{cl}} \bar{\mathcal{J}}_t, \tag{72}$$

[4] The Lagrangian corresponding to this Lie-Poisson system is $\langle \bar{\mathcal{J}}_t, \hat{H}_t^{cl} \rangle - \mathcal{H}_t\left(\bar{\mathcal{J}}_t\right)$, while the usual Lagrangian is $L^{H_t^{T^*M}}$.

that is calculated as follows:

$$\frac{\partial}{\partial t}\bar{\rho}_t^\tau[k](x) = -\sqrt{}^{-1}\partial_j\left(\frac{\partial H_t^{T^*M}}{\partial p_j}\left(x, \bar{p}_t^\tau[k](x)\right)\bar{\rho}_t^\tau[k](x)\sqrt{}\right), \tag{73}$$

$$\begin{aligned}\frac{\partial}{\partial t}\left(\bar{\rho}_t^\tau[k](x)\bar{p}_{tk}^\tau[k](x)\right) &= -\sqrt{}^{-1}\partial_j\left(\frac{\partial H_t^{T^*M}}{\partial p_j}\left(x, \bar{p}_t^\tau[k](x)\right)\bar{\rho}_t^\tau[k](x)\bar{p}_{tk}^\tau[k](x)\sqrt{}\right)\\ &\quad -\bar{\rho}_t^\tau[k](x)\bar{p}_{tj}^\tau[k](x)\partial_k\left(\frac{\partial H_t^{T^*M}}{\partial p_j}\left(x, \bar{p}_t^\tau[k](x)\right)\right)\\ &\quad +\bar{\rho}_t^\tau[k](x)\partial_k L^{H_t^{T^*M}}\left(x, \bar{p}_t^\tau[k](x)\right).\end{aligned} \tag{74}$$

Proof. *The above equation can be obtained from the integration of general equations (**??**) and (**??**) on the space Γ_{U0}; thereby, it proves the reduced equation from original Lie-Poisson equation (42).* □

As a most important result, the following theorem shows that Lie-Poisson equation (72), or the set of equations (73) and (74), actually represents the classical Liouville equation.

Theorem 2 *Lie-Poisson equation (72) is equivalent to the classical Liouville equation for the probability density function (PDF) $\rho_t^{T^*M} \in C^\infty(T^*M, \mathbf{R})$ of a particle on cotangent space T^*M:*

$$\frac{\partial}{\partial t}\rho_t^{T^*M} = \{\rho_t^{T^*M}, H^{T^*M}\}, \tag{75}$$

where the Poisson bracket $\{\ ,\ \}$ is defined for every A, $B \in C^\infty(M)$ as

$$\{A, B\} = \frac{\partial A}{\partial p_j}\frac{\partial B}{\partial x^j} - \frac{\partial B}{\partial p_j}\frac{\partial A}{\partial x^j}. \tag{76}$$

Proof. *Classical equation (75) is equivalent to the canonical equations of motion through the local expression such that $\phi_{U_\alpha}(q_t) = (x_t, p_t)$ for the bundle mapping $\phi_{U_\alpha} : \pi^{-1}(U_\alpha) \to U_\alpha \times \mathbf{R}^N$:*

$$\frac{dp_{jt}}{dt} = -\frac{\partial H^{T^*M}}{\partial x^j}(x_t, p_t) \qquad\qquad \frac{dx_t^j}{dt} = \frac{\partial H^{T^*M}}{\partial p_j}(x_t, p_t). \tag{77}$$

If $q_t = (x_t, \bar{p}_t^\tau[k](x_t))$ satisfies canonical equations of motion (77), the above equation of motion induces

$$\frac{\partial \bar{p}_{tk}^\tau}{\partial t}[k](x) = -\frac{\partial H^{T^*M}}{\partial x^k}\left(x, \bar{p}_t^\tau[k](x)\right) - \frac{\partial H^{T^*M}}{\partial p_j}\left(x, \bar{p}_t^\tau[k](x)\right)\partial_j \bar{p}_{tk}^\tau[k](x), \tag{78}$$

then relation (63) satisfies the following equation:

$$\begin{aligned}\frac{\partial}{\partial t}\bar{\rho}_t^\tau[k](x) &= \sqrt{}^{-1}\partial_j\left(\sigma_t^\tau[k](x)\frac{\partial H^{T^*M}}{\partial p_j}\left(x, \bar{p}_t^\tau[k](x)\right)\right)\rho_t^{T^*M}(x, \bar{p}_t^\tau[k](x))\\ &\quad +\sqrt{}^{-1}\sigma_t^\tau[k](x)\frac{\partial \rho_t^{T^*M}}{\partial t}(x, \bar{p}_t^\tau[k](x))\\ &\quad -\sqrt{}^{-1}\sigma_t^\tau[k](x)\frac{\partial H^{T^*M}}{\partial x^j}\left(x, \bar{p}_t^\tau[k](x)\right)\frac{\partial \rho_t^{T^*M}}{\partial p_j}(x, \bar{p}_t^\tau[k](x))\\ &\quad -\sqrt{}^{-1}\sigma_t^\tau[k](x)\frac{\partial H^{T^*M}}{\partial p_j}\left(x, \bar{p}_t^\tau[k](x)\right)\\ &\quad \times\partial_j\bar{p}_{tk}[k](x)\frac{\partial \rho_t^{T^*M}}{\partial p_k}(x, \bar{p}_t^\tau[k](x))\\ &= -\sqrt{}^{-1}\partial_j\left(\frac{\partial H^{T^*M}}{\partial p_j}\left(x, \bar{p}_t^\tau[k](x)\right)\rho_t^\tau[k](x)\sqrt{}\right).\end{aligned} \tag{79}$$

Equations (78) and (79) lead to the following equation:

$$\begin{aligned}\frac{\partial}{\partial t}\{\bar{\rho}^{\tau}_t[k](x)\bar{p}^{\tau}_{tk}[k](x)\} &= -\bar{p}_{tk}[k](x)\surd^{-1}\partial_j\left(\frac{\partial H^{T^*M}}{\partial p_j}\left(x,\bar{p}^{\tau}_t[k](x)\right)\bar{\rho}^{\tau}_t[k](x)\surd\right)\\ &\quad -\bar{\rho}^{\tau}_t[k](x)\frac{\partial H^{T^*M}}{\partial x^k}\left(x,\bar{p}^{\tau}_t[k](x)\right)\\ &\quad -\bar{\rho}^{\tau}_t[k](x)\frac{\partial H^{T^*M}}{\partial p_j}\left(x,\bar{p}^{\tau}_t[k](x)\right)\partial_j\bar{p}^{\tau}_{tk}[k](x)\\ &= -\surd^{-1}\partial_j\left(\frac{\partial H^{T^*M}}{\partial p_j}\left(x,\bar{p}^{\tau}_t[k](x)\right)\bar{\rho}^{\tau}_t[k](x)\bar{p}^{\tau}_{tk}[k](x)\surd\right)\\ &\quad -\bar{\rho}^{\tau}_t[k](x)\left\{\bar{p}^{\tau}_{tj}[k](x)\partial_k\left(\frac{\partial H^{T^*M}}{\partial p_j}\left(x,\bar{p}^{\tau}_t[k](x)\right)\right)\right.\\ &\quad \left.+\partial_k L^H\left(x,\bar{p}^{\tau}_t[k](x)\right)\right\}. \end{aligned} \tag{80}$$

Equations (79) and (80) are equivalent to equations (73) and (74); thereby, canonical equation (75) is equivalent to Lie-Poisson equation (72). □

The above discussion has a special example of the following Hamiltonian:

$$H^{T^*M}_t(x,p) = g^{ij}(x)\left(p_i + A_i\right)\left(p_j + A_j\right) + U(x), \tag{81}$$

where corresponding Hamiltonian operator $\hat{H}_t$ is calculated as

$$\hat{H}_t[k] = \left(g^{ji}\left(\bar{p}_{ti}[k] + A_i\right)\partial_j, -g^{ji}\bar{p}_{tj}[k]\bar{p}_{ti}[k] + g^{ji}A_jA_i + U\right); \tag{82}$$

thereby, equation (72) is described for special Hamiltonian (81) as

$$\begin{aligned}\frac{\partial}{\partial t}\left(\bar{\rho}^{\tau}_t[k](x)\bar{p}^{\tau}_{tj}[k](x)\right) &= -\surd^{-1}\partial_i\left\{g^{ik}(x)\left(\bar{p}^{\tau}_{tk}[k](x) + A_k(x)\right)\bar{\rho}^{\tau}_t[k](x)\bar{p}^{\tau}_{tj}[k](x)\surd\right\}\\ &\quad -\bar{\rho}^{\tau}_t[k](x)\left(\partial_j g^{ik}(x)\right)\bar{p}^{\tau}_{ti}[k](x)\bar{p}^{\tau}_{tk}[k](x)\\ &\quad -\left(\partial_j g^{ik}(x)A_k(x)\right)\bar{\rho}^{\tau}_t[k](x)\bar{p}^{\tau}_{ti}[k](x)\\ &\quad -\bar{\rho}^{\tau}_t[k](x)\partial_j\left\{U(x) + g^{ik}(x)A_i(x)A_k(x)\right\}, \end{aligned} \tag{83}$$

$$\frac{\partial}{\partial t}\bar{\rho}^{\tau}_t[k](x) = -\surd^{-1}\partial_i\left\{g^{ik}(x)\left(\bar{p}_{tk}[k](x) + A_k(x)\right)\bar{\rho}^{\tau}_t[k](x)\surd\right\}. \tag{84}$$

For $\bar{\mathcal{U}}_t \in \bar{Q}(M)$ such that $\frac{\partial\bar{\mathcal{U}}_t}{\partial t}\circ\bar{\mathcal{U}}_t^{-1} = \hat{H}^{cl}_t \in \bar{q}(M)$, let us introduce operators

$$\tilde{H}^{cl}_t = Ad^{-1}_{\bar{\mathcal{U}}_t}\hat{H}^{cl}_t, \tag{85}$$

$$\tilde{F}^{cl}_t = Ad^{-1}_{\bar{\mathcal{U}}_t}\hat{F}^{cl}_t, \tag{86}$$

which induces the following equation equivalent to equation (72):

$$\frac{\partial}{\partial t}\tilde{F}^{cl}_t = \left[\tilde{H}^{cl}_t, \tilde{F}^{cl}_t\right] + \left(\widetilde{\frac{\partial F_t}{\partial t}}\right)^{cl}. \tag{87}$$

This expression of the equations of motion coincides with the following Poisson equation because of Theorem 2:

$$\frac{d}{dt}F^{T^*M}_t = \left\{H^{T^*M}_t, F^{T^*M}_t\right\} + \frac{\partial F^{T^*M}_t}{\partial t}. \tag{88}$$

As discussed in Section 3, if a group action of Lie group $Q(M)$ keeps the Hamiltonian $\bar{\mathcal{H}}_t : \bar{q}(M)^* \to \mathbf{R}$ invariant, there exists an invariant charge function $Q^{T^*M} \in C^\infty(T^*M)$ and the induced function $\bar{\mathcal{Q}} : \bar{q}(M)^* \to \mathbf{R}$ such that

$$\left[\hat{H}^{cl}_t, \hat{Q}^{cl}\right] = 0, \tag{89}$$

where $\hat{Q}^{cl}$ is expressed as

$$\hat{Q}^{cl} = \left(\frac{\partial Q^{T^*M}}{\partial p} \left(x, \bar{p}^\tau_t[k](x)\right), -p(\eta) \cdot \frac{\partial Q^{T^*M}}{\partial p} \left(x, \bar{p}^\tau_t[k](x)\right) + Q^{T^*M} \left(x, \bar{p}^\tau_t[k](x)\right) \right). \tag{90}$$

Relation (89) is equivalent to the following convolution relation:

$$\left\{ H^{T^*M}_t, Q^{T^*M} \right\} = 0. \tag{91}$$

In the argument so far on the dynamical construction of classical mechanics, the introduced infinite-dimensional freedom of the background $B\left[E(M)\right]$ seems to be redundant, while they appear as a natural consequence of the general theory on protomechanics discussed in the previous section. In fact, it is really true that one can directly induce classical mechanics as the dynamics of the Lagrange foliations of T^*M in $L\left(T^*M\right)$. In the next section, however, it is observed that we will encounter difficulties without those freedom if moving onto the dynamical construction of quantum mechanics.

4 DEDUCTION OF QUANTUM MECHANICS

In canonical quantum mechanics, the state of a particle on manifold M can be represented as a position in the Hilbert space $\mathcal{H}(M)$ of all the L_2-functions over M. In this section, we will reproduce the quantum equation of motion from the general theory presented in Section 4. Let us here concentrate ourselves on the case where M is N-dimensional manifold for simplicity, though the discussion below is still valid if substituting an appropriate Hilbert space when M is infinite-dimensional ILH-manifold[4].

4.1 Description of Statistical-State

Now, we must be concentrated on the case where the physical functional $F \in C^\infty\left(\Lambda^1(M), C^\infty(M)\right)$ depends on the derivatives of the 1-form $p\left(\eta\right) \in \Lambda^1(M)$ induced from $\eta \in \Gamma\left[E(M)\right]$, then it has the following expression:

$$p^*F\left(\eta\right)(x) = F^Q\left(x, p\left(\eta\right)(x), Dp\left(\eta\right)(x), ..., D^n p\left(\eta\right)(x), ...\right). \tag{92}$$

Let us assume that M has a finite covering $M = \bigcup_{\alpha\in\Lambda_M} U_\alpha$ for the mathematical simplicity such that $\Lambda_M = \{1, 2, ..., \Lambda\}$ for some $\Lambda \in \mathbf{R}$, and choose a coordinate system $\left(U_\alpha, \mathbf{x}_\alpha\right)_{\alpha\in\Lambda_M}$. Let us further choose a reference set $U \subset U_\alpha$ such that $v(U) \neq 0$ and consider the set $\Gamma^\hbar_{Uk}\left[E(M)\right]$ of the C^∞ sections of $E(M)$ for $k = (k_1, ..., k_N) \in \mathbf{R}^N$ such that[5]

$$\Gamma^\hbar_{Uk}\left[E(M)\right] = \left\{ \eta \in \Gamma\left[E(M)\right] \;\middle|\; \sup_U p_j\left(\eta\right)(x) = \hbar k_j \right\}. \tag{93}$$

As in classical mechanics, we will simply denote $\Gamma^\hbar_{Uk}\left[E(M)\right]$ as $\Gamma^\hbar_k\left[E(M)\right]$, since $\Gamma^\hbar_{Uk}\left[E(M)\right]$ can be identified with $\Gamma^\hbar_{U'k}\left[E(M)\right]$ for every two reference sets U and $U' \subset M$.

For every $\bar{p} = p \circ \bar{\eta} \in L\left(T^*M\right)$ such that $\bar{\eta}[k] \in \Gamma^\hbar_k\left[E(M)\right]$, it is further possible to separate an element $\eta[k] \in \Gamma^\hbar_k\left[E(M)\right]$ for a $\xi \in \Gamma^\hbar_0\left[E(M)\right]$ as

$$\eta[k] = \bar{\eta}[k] \cdot \xi, \tag{94}$$

[5] As in classical mechanics, to substitute $\Gamma^\hbar_{Uk}\left[E(M)\right] = \left\{ \eta \in \Gamma\left[E(M)\right] \;\middle|\; \int_U dv(x)\ p_j\left(x\right) = \hbar k_j v\left(U\right) \right\}$ for definition (93) also induces the similar discussion below, while there exist a variety of the classification methods that produce the same result.

or to separate momentum $p\left(\eta[k]\right)$ as

$$p\left(\eta[k]\right)=\bar{p}[k]+p\left(\xi\right). \tag{95}$$

The emergence density $\rho\left(\eta[k]\right)$ can have the same expression as the classical one (54) for the function $\varrho\left(\xi\right)\in C^{\infty}\left(T^{*}M,\mathbf{R}\right)$ on $T^{*}M$ since $C_{q}\left(\Gamma\right)^{*}\subset C_{cl}\left(\Gamma\right)^{*}$:

$$\rho\left(\eta[k]\right)(x)\surd=\varrho\left(\xi\right)\left(x,p\left(\eta[k]\right)(x)\right), \tag{96}$$

which has only the restricted values if compared with the classical emergence density; it sometimes causes the discrete spectra of the wave-function in canonical quantum mechanics. We call the set $B^{\hbar}\left[E(M)\right]=\Gamma_{0}^{\hbar}\left[E(M)\right]$ as the **back ground** of $L\left(T^{*}M\right)$ for quantum mechanics. For the measure $\mathcal{N}$ on $B^{\hbar}\left[E(M)\right]$ for the σ-algebra induced from that of $\Gamma\left[E(M)\right]$:

$$d\mathcal{M}\left(\eta[k]\right)\ dv(x)=d^{N}kd\mathcal{N}\left(\xi\right)dv(x)\ \sigma[k](x)\ . \tag{97}$$

Let us next consider the disjoint union $M=\bigcup_{\alpha\in\Lambda_{M}}A_{\alpha}$ for $A_{\alpha}\in\mathcal{B}\left(\mathcal{O}_{E(M)}\right)$ such that (1) $\pi\left(A_{\alpha}\right)\subset U_{\alpha}$ and that (2) $A_{\alpha}\cap A_{\beta}=\emptyset$ for $\alpha\neq\beta\in\Lambda_{M}$ (consult *APPENDIX*). Thus, every section $\eta\in\Gamma\left[E(M)\right]$ has some $k\in\mathbf{R}^{N}$ such that $\eta=\eta[k]\in\Gamma_{k}^{\hbar}\left[E(M)\right]$; and, it will be separated into the product of a $\xi\in B^{\hbar}\left[E(M)\right]$ and the fixed $\bar{\eta}[k]=e^{2i\{k_{j}x^{j}+\zeta\}}\in\Gamma_{k}\left[E(M)\right]$ that induces one of the Lagrange foliation $\bar{p}=p\circ\bar{\eta}\in L\left(T^{*}M\right)$:

$$\begin{aligned}\eta\left[k\right] &= \sum_{\alpha\in A_{\alpha}}\chi_{A_{\alpha}}\cdot e^{2i\{k_{j}x^{j}+\zeta\}}\cdot\xi & (98)\\ &= \prod_{\alpha\in A_{\alpha}}\left(e^{2i\{k_{j}x^{j}+\zeta\}}\cdot\xi\right)^{\chi_{A_{\alpha}}}, & (99)\end{aligned}$$

where the test function $\chi_{A_{\alpha}}:M\rightarrow\mathbf{R}$ satisfies

$$\chi_{A_{\alpha}}(x)=\begin{cases}1 & at\ x\in A_{\alpha}\\ 0 & at\ x\notin A_{\alpha}\end{cases} \tag{100}$$

and has the projection property $\chi_{A_{\alpha}}^{2}=\chi_{A_{\alpha}}$.

If defining the *window mapping* $\chi_{A_{\alpha}}^{*}:C^{\infty}(M)\rightarrow L^{1}\left(\mathbf{R}^{N}\right)$ for any $f\in C^{\infty}(M)$ such that

$$\chi_{A_{\alpha}}^{*}f\left(\mathbf{x}\right)=\begin{cases}\varphi_{\alpha}^{*}f\left(\mathbf{x}\right) & at\ \mathbf{x}\in\varphi_{\alpha}\left(A_{\alpha}\right)\\ 0 & at\ \mathbf{x}\notin\varphi_{\alpha}\left(A_{\alpha}\right)\end{cases}, \tag{101}$$

we can *locally* transform the function $\rho[k]\left(\xi\right)=\sigma[k]\rho\left(\eta\left[k\right]\right)\surd$ into Fourier coefficients as follows:

$$\chi_{A_{\alpha}}^{*}\rho[k]\left(\xi\right)\ \left(\mathbf{x}\right)=\int_{\mathbf{R}^{N}}d^{N}k'\ \tilde{\varrho}_{\alpha}\left(\xi\right)\left(\frac{2k+k'}{2},\frac{2k-k'}{2}\right)e^{ik'\mathbf{x}^{j}}, \tag{102}$$

where introduced function $\tilde{\varrho}_{\alpha}$ should satisfies

$$\tilde{\varrho}_{\alpha}\left(\xi\right)(k,k')^{*}=\tilde{\varrho}_{\alpha}\left(\xi\right)(k',k), \tag{103}$$

for the value $\rho[k]\left(\xi\right)(x)$ is real at every $x\in M$; thereby, the collective expression gives

$$\begin{aligned}\rho\left[k\right]\left(\xi\right) &= \sum_{\alpha\in A_{\alpha}}\chi_{A_{\alpha}}\cdot\int_{\mathbf{R}^{N}}d^{N}k'\ \tilde{\varrho}_{\alpha}\left(\xi\right)\left(\frac{2k+k'}{2},\frac{2k-k'}{2}\right)e^{ik'x^{j}} & (104)\\ &= \int_{\mathbf{R}^{N}}d^{N}k'\ \tilde{\varrho}\left(\xi\right)\left(\frac{2k+k'}{2},\frac{2k-k'}{2}\right)\cdot\eta\left[k-\frac{k'}{2}\right]^{-\frac{1}{2}}\eta\left[k+\frac{k'}{2}\right]^{\frac{1}{2}}, & (105)\end{aligned}$$

where

$$\tilde{\varrho}\left(\xi\right)\left(\frac{2k+k'}{2},\frac{2k-k'}{2}\right)=\prod_{\alpha\in A_\alpha}\left(\tilde{\varrho}_\alpha\left(\xi\right)\left(\frac{2k+k'}{2},\frac{2k-k'}{2}\right)\right)^{\chi_{A_\alpha}}. \tag{106}$$

Let us introduce the ketvector $|k\rangle$ and bravector $\langle k|$ such that

$$|k\rangle=\prod_{\alpha\in\Lambda_M}|k,\alpha\rangle \quad , \quad \langle k|=\prod_{\alpha\in\Lambda_M}\langle k,\alpha|\,, \tag{107}$$

where the local vectors $|k,\alpha\rangle$ and $\langle k,\alpha|$ satisfy

$$\langle x\,|k,\alpha\rangle=e^{2i\{k_jx^j+\zeta\}\chi_{A_\alpha}}\sqrt{}^{-\frac{1}{2}} \quad , \quad \langle k,\alpha\,|x\rangle=e^{2i\{-k_jx^j+\zeta\}\chi_{A_\alpha}}\sqrt{}^{-\frac{1}{2}}. \tag{108}$$

We can define the Hilbert space $\mathcal{H}\left(M\right)$ of all the vectors that can be expressed as a linear combination of vectors $\{|k\rangle\}_{k\in\mathbf{R}}$. Now, let us construct the *density matrix* in the following definition.

Definition 1 *The* **density matrix** $\hat{\rho}$ *is an operator such that*

$$\begin{aligned}\hat{\rho} &= \int_{B^\hbar[E(M)]}d\mathcal{N}(\xi)\int_{\mathbf{R}^N}d^Nn\int_{\mathbf{R}^N}d^Nn'\ \tilde{\varrho}(\xi)\left(n,n'\right)\ \xi^{\frac{1}{2}}\left|n\right\rangle\left\langle n'\right|\xi^{-\frac{1}{2}} &(109)\\ &= \int_{B^\hbar[E(M)]}d\mathcal{N}(\xi)\int_{\mathbf{R}^N}d^Nk\ \hat{\rho}[k]\left(\xi\right), &(110)\end{aligned}$$

where

$$\hat{\rho}\left[k\right]\left(\xi\right)=\int_{\mathbf{R}^N}d^Nk'\ \tilde{\varrho}(\xi)\left(k+\frac{k'}{2},k-\frac{k'}{2}\right)\ \xi^{\frac{1}{2}}\left|k+\frac{k'}{2}\right\rangle\left\langle k-\frac{k'}{2}\right|\xi^{-\frac{1}{2}}. \tag{111}$$

Let $\mathcal{O}\left(M\right)$ be the set of all the hermite operators acting on Hilbert space $\mathcal{H}\left(M\right)$, which has the bracket $\langle\ \ \rangle:\mathcal{O}\left(M\right)\to\mathbf{R}$ for every hermite operator $\hat{\mathbf{F}}$ such that

$$\left\langle\hat{\mathbf{F}}\right\rangle=\int_{\mathbf{R}^N}d^Nk\int_M dv(x)\ \left\langle x\left|\hat{\mathbf{F}}\right|x\right\rangle. \tag{112}$$

Set $\mathcal{O}\left(M\right)$ becomes the algebra with the product, scalar product and addition; thereby, we can consider the commutation and the anticommutaion between operators $\hat{\mathbf{A}}$, $\hat{\mathbf{B}}\in\mathcal{O}\left(M\right)$:

$$\left[\hat{\mathbf{A}},\hat{\mathbf{B}}\right]_\pm=\hat{\mathbf{A}}\hat{\mathbf{B}}\pm\hat{\mathbf{B}}\hat{\mathbf{A}}. \tag{113}$$

Consider the momentum operator $\hat{\mathbf{p}}$ that satisfies the following relation for any $|\psi\rangle\in\mathcal{H}\left(M\right)$:

$$\langle x\,|\hat{\mathbf{p}}|\,\psi\rangle=-iD\,\langle x\,|\psi\,\rangle\,, \tag{114}$$

where $D=\hbar dx^j\partial_j$ is the derivative operator (15). Further, the function operator $\hat{\mathbf{f}}$ induced from the function $f\in C^\infty(M)$ is an operator that satisfies the following relation for any $|\psi\rangle\in\mathcal{H}\left(M\right)$:

$$\left\langle x\left|\hat{\mathbf{f}}\right|\psi\right\rangle=f(x)\,\langle x\,|\psi\,\rangle\,. \tag{115}$$

The following commutation relation holds:

$$\left[\hat{\mathbf{p}}_j,\hat{\mathbf{f}}\right]_-=\frac{\hbar}{i}\widehat{\partial_j\mathbf{f}}. \tag{116}$$

Those operators $\hat{\mathbf{f}}$ and $\hat{\mathbf{p}}$ induces a variety of operators in the form of their polynomials.

Definition 2 *The hermite operator* $\hat{\mathbf{F}}$ *is called an* **observable**, *if it can be represented as the polynomial of the momentum operators* $\hat{p}$ *weighted with function operators* $\hat{\mathbf{f}}_n^j$ *independent of* k *such that*

$$\hat{\mathbf{F}} = \sum_{n=0}^{\infty} \left[\hat{\mathbf{f}}_n^j, \hat{\mathbf{p}}_j^n \right]_+ . \tag{117}$$

The following lemma shows that every *observable* has its own physical functional.

Lemma 2 *Every observable* $\hat{\mathbf{F}}$ *has a corresponding functional* $F : \Gamma[E] \to C^\infty(M)$*:*

$$\bar{\mu}\left(p^* F\right) = \left\langle \hat{\rho} \ \hat{\mathbf{F}} \right\rangle . \tag{118}$$

Proof. *There are corresponding functionals* $g_{nl}^j : \Lambda^1(M) \to C(M)$ $(l \in \{1, 2, ..., n\})$ *such that*

$$\begin{aligned}
\left\langle \hat{\rho} \ \left[\hat{\mathbf{f}}_n^j, \hat{\mathbf{p}}_j^n \right]_+ \right\rangle &= \int_{B^\hbar[E(M)]} d\mathcal{N}(\xi) \int_{\mathbf{R}^N} d^N n \int_{\mathbf{R}^N} d^N n' \ \tilde{\varrho}(\xi)\left(n, n'\right) \left\langle n' \middle| \ \xi^{-\frac{1}{2}} \ \left[\hat{\mathbf{f}}_n^j, \hat{\mathbf{p}}_j^n \right]_+ \ \xi^{\frac{1}{2}} \middle| n \right\rangle \\
&= \int_{B^\hbar[E(M)]} d\mathcal{N}(\xi) \int_{\mathbf{R}^N} d^N k \int_{\mathbf{R}^N} d^N k' \\
&\quad \times \sum_{\alpha \in \Lambda_M} \int_{U_\alpha} d^N x \ \tilde{\varrho}(\xi) \left(k - \frac{k'}{2}, k + \frac{k'}{2} \right) e^{i k'_j x^j} \left\{ \sum_{l=0}^{n} g_{nl}^j \left(p\left(\eta[k]\right)\right)(x) k_j'^l \right\} \\
&= \int_{B^\hbar[E(M)]} d\mathcal{N}(\xi) \int_{\mathbf{R}^N} d^N k \int_{\mathbf{R}^N} d^N k' \\
&\quad \times \sum_{\alpha \in \Lambda_M} \int_{U_\alpha} d^N x \ \tilde{\varrho}(\xi) \left(k - \frac{k'}{2}, k + \frac{k'}{2} \right) e^{i k'_j x^j} \left\{ \sum_{l=0}^{n} \left(-\hbar \frac{\partial}{\partial x^j} \right)^l g_{nl}^j \left(p\left(\eta[k]\right)\right)(x) \right\} \\
&= \int_{B^\hbar[E(M)]} d\mathcal{N}(\xi) \int_{\mathbf{R}^N} d^N k \int_M dv(x) \ \rho\left(\eta[k]\right)(x) \ p^* F_j^n \left(\eta[k]\right)(x) \\
&= \int_{\Gamma[E(M)]} d\mathcal{M}(\eta) \int_M dv(x) \ \rho(\eta)(x) \ p^* F_j^n(\eta)(x) \\
&= \bar{\mu}\left(p^* F_j^n\right) .
\end{aligned} \tag{119}$$

where

$$p^* F_j^n \left(\eta[k]\right)(x) = \sum_{l=0}^{n} \left\{ \left(-\hbar \frac{\partial}{\partial x^j} \right)^l g_{nl}^j \left(p\left(\eta[k]\right)\right)(x) \right\} . \tag{120}$$

□

4.2 Description of Time-Development

Now, we can describe a $\eta_t^\tau\left(\eta[k]\right) \in \Gamma_{Uk}\left[E(M)\right]$ as

$$\begin{aligned}
\eta_t^\tau\left(\eta[k]\right) &= \sum_{\alpha \in A_\alpha} \chi_{A_\alpha} \cdot e^{2i\{k_{\alpha j} x^j + \zeta_t^\tau[k]\}} \cdot \xi & (121) \\
&= \prod_{\alpha \in A_\alpha} \left(e^{2i\{k_{\alpha j} x^j + \zeta_t^\tau[k]\}} \cdot \xi \right)^{\chi_{A_\alpha}} , & (122)
\end{aligned}$$

where the function $\zeta_t^\tau[k] \in C^\infty(M)$ labeled by *labeling time* $\tau \le t \in \mathbf{R}$ satisfies

$$\zeta_\tau^\tau[k] = \zeta \qquad : independent\ of\ k; \tag{123}$$

thereby, the momentum $p_t^\tau(\eta[k]) = \bar{p}_t^\tau[k] + p(\xi) \in \Lambda^1(M)$ for $\bar{p}_t^\tau = p_t^\tau \circ \bar{\eta} \in L(T^*M)$ satisfies the Einstein-de Broglie relation:[6]

$$\bar{p}_t^\tau[k] = -i\frac{\hbar}{2}\bar{\eta}_t^\tau[k]^{-1} d\bar{\eta}_t^\tau[k]. \tag{124}$$

The density operator $\hat{\rho}_t^\tau[k](\xi)$ is introduced as

$$\hat{\rho}_t^\tau[k](\xi) = \int_{\mathbf{R}^N} d^N k' \; \tilde{\varrho}_t^\tau(\xi)\left(k+\frac{k'}{2}, k-\frac{k'}{2}\right)\; \xi^{\frac{1}{2}} \left| k+\frac{k'}{2}\right\rangle \left\langle k-\frac{k'}{2}\right| \xi^{-\frac{1}{2}}, \tag{125}$$

which satisfies the following lemma.

Lemma 3

$$\hat{\rho}_t = \int_{\Gamma_U} d\mathcal{N}(\xi) \int_{\mathbf{R}^N} d^N k \; U_t^\tau[k]\hat{\rho}_t^\tau[k](\xi)\, U_t^\tau[k]^{-1}, \tag{126}$$

where

$$U_t^\tau[k] = e^{i\{\zeta_t^\tau[k]-\zeta\}}. \tag{127}$$

Proof. *The direct calculation shows for the observable $\hat{\mathbf{F}}_t$ corresponding to every functional F*

$$\begin{aligned}
\left\langle \hat{\rho}_t \; \hat{\mathbf{F}}_t \right\rangle &= \bar{\mu}_t\left(p^* F_t\right) \\
&= \int_{\Gamma[E(M)]} d\mathcal{M}(\eta) \int_M dv \; \rho_t^\tau(\eta)(x) \; p^* F_t\left(\eta_t^\tau(\eta)\right) \\
&= \int_{B[E(M)]} d\mathcal{N}(\xi) \int_{\mathbf{R}^N} d^N k \int_M dv(x) \; \rho_t^\tau[k](\xi)(x) \; p^* F\left(\eta_t^\tau[k]\right)(x) \\
&= \int_{B[E(M)]} d\mathcal{N}(\xi) \int_{\mathbf{R}^N} d^N k \int_M dv(x) \; \rho_t^\tau[k](\xi)(x) \; p^* F\left(\eta[k]\cdot e^{i\{\zeta_t^\tau[k]-\zeta\}}.\right)(x) \\
&= \int_{B[E(M)]} d\mathcal{N}(\xi) \int_{\mathbf{R}^N} d^N k \int_M dv(x) \; \left\langle x \left| \frac{1}{2}\left[U_t^\tau[k]\hat{\rho}_t^\tau[k](\xi)\, U_t^\tau[k]^{-1} \; , \; \hat{\mathbf{F}}_t \right]_+ \right| x \right\rangle \\
&= \left\langle \left\{ \int_{B[E(M)]} d\mathcal{N}(\xi) \int_{\mathbf{R}^N} d^N k \; U_t^\tau[k]\hat{\rho}_t^\tau[k](\xi)\, U_t^\tau[k]^{-1} \right\} \; \hat{\mathbf{F}}_t \right\rangle .
\end{aligned} \tag{128}$$

□

Relation (126) represents relation (30):

$$\tilde{\mu}_t(\eta) = \frac{d\mathcal{M}(\eta)}{d\mathcal{M}\left(\eta_t^{\tau\;-1}(\eta)\right)} \cdot \tilde{\mu}_t^\tau\left(\eta_t^{\tau\;-1}(\eta)\right). \tag{129}$$

Emergence-momentum $\mathcal{J}_t^\tau = \mathcal{J}(\eta_t^\tau) \in q(M)^*$ has the following expression:

$$\begin{aligned}
\mathcal{J}_t^\tau &= d^N k d\mathcal{N}(\xi)\, dv \;\left(\rho_t^\tau[k](\xi)\, p_t^\tau(\eta[k]), \rho_t^\tau[k](\xi)\right) && (130)\\
&= d\mathcal{N}(\xi)\, d^N k \wedge dv \;\left(\frac{1}{2}\left\langle x \left| \left[\hat{\rho}_t^\tau[k](\xi), \hat{\mathbf{p}}_t^\tau[k]\right]_+ \right| x \right\rangle \; , \; \left\langle x \left| \hat{\rho}_t^\tau[k](\xi) \right| x\right\rangle \right), && (131)
\end{aligned}$$

where the momentum operator $\hat{\mathbf{p}}_t^\tau[k]$ satisfies

$$\hat{\mathbf{p}}_t^\tau[k] = U_t^\tau[k]^{-1} \; \hat{\mathbf{p}} \; U_t^\tau[k]. \tag{132}$$

[6] Relation (124) is the most crucial improvement from the corresponding relation in previous letter [2].

The following calculus of the fourier basis for $2k_j = n_j + m_j$ justifies expression (131):

$$e^{-i\{n_j\mathbf{x}^j+\zeta_t^\tau[k]\}}de^{+i\{m_j\mathbf{x}^j+\zeta_t^\tau[k]\}} - e^{+i\{m_j\mathbf{x}^j+\zeta_t^\tau[k]\}}de^{-i\{n_j\mathbf{x}^j+\zeta_t^\tau[k]\}} =$$
$$e^{-i\{n_j\mathbf{x}^j+\zeta_t^\tau[k]\}}d\left\{e^{+i\{(m_j+n_j)\mathbf{x}^j+2\zeta_t^\tau[k]\}}\cdot e^{-i\{n_j\mathbf{x}^j+\zeta_t^\tau[k]\}}\right\} - e^{+i\{m_j\mathbf{x}^j+\zeta_t^\tau[k]\}}de^{-i\{n_j\mathbf{x}^j+\zeta_t^\tau[k]\}} =$$
$$e^{-i(n_j-m_j)\mathbf{x}^j}\cdot e^{-i\{(n_j+m_j)\mathbf{x}^j+2\zeta_t^\tau[k]\}}de^{+i\{(n_j+m_j)\mathbf{x}^j+2\zeta_t^\tau[k]\}}. \tag{133}$$

For Hamiltonian operator $\hat{H}_t^\tau = \frac{\partial\mathcal{H}_t}{\partial\bar{\mathcal{J}}}\left(\bar{\mathcal{J}}_t^\tau\right) \in q(M)$, the equation of motion is the Lie-Poisson equation

$$\frac{\partial\mathcal{J}_t^\tau}{\partial t} = ad^*_{\hat{H}_t}\mathcal{J}_t^\tau, \tag{134}$$

that is calculated as follows:

$$\frac{\partial}{\partial t}\rho_t^\tau[k](\xi)(x) = -\surd^{-1}\partial_j\left(\frac{\partial H_t^{T^*M}}{\partial p_j}\left(x, p_t^\tau(\eta[k])(x)\right)\rho_t^\tau[k](\xi)(x)\surd\right), \tag{135}$$

$$\begin{aligned}\frac{\partial}{\partial t}\left(\rho_t^\tau[k](\xi)(x)p_{tk}^\tau(\eta[k])(x)\right) \quad = \quad & -\surd^{-1}\partial_j\left(\frac{\partial H_t^{T^*M}}{\partial p_j}\left(x, p_t^\tau(\eta[k])(x)\right)\rho_t^\tau[k](\xi)(x)p_{tk}^\tau(\eta[k])(x)\surd\right)\\ & -\rho_t^\tau[k](\xi)(x)p_{tj}^\tau(\eta[k])(x)\partial_k\left(\frac{\partial H_t^{T^*M}}{\partial p_j}\left(x, p_t^\tau(\eta[k])(x)\right)\right)\\ & +\rho_t^\tau[k](\xi)(x)\partial_k L^{H_t^{T^*M}}\left(x, p_t^\tau(\eta[k])(x)\right).\end{aligned} \tag{136}$$

Notice that the above expression is still valid even if Hamiltonian $H_t^{T^*M}$ has the ambiguity of the operator ordering such as that for the Einstein gravity.

To elucidate the relationship between the present theory and canonical quantum mechanics, we will concentrate on the case of the canonical Hamiltonian having the following form:

$$H_t^{T^*M}(x,p) = \frac{1}{2}h^{ij}\left(p_i + A_{ti}\right)\left(p_j + A_{tj}\right) + U_t(x), \tag{137}$$

where $dh^{ij} = 0$. Notice that almost all the canonical quantum theory including the standard model of the quantum field theory, that have empirically been well-established, really belong to this class of Hamiltonian systems. For Hamiltonian (137), we will define the Hamiltonian operator $\hat{\mathbf{H}}_t$ as

$$\hat{\mathbf{H}}_t = \frac{1}{2}\left(\hat{p}_i + A_{ti}\right)h^{ij}\left(\hat{p}_j + A_{tj}\right) + U_t, \tag{138}$$

or $\langle x|\hat{\mathbf{H}}_t|\psi\rangle = \mathcal{H}_t\langle x|\psi\rangle$ where

$$\mathcal{H}_t = \frac{1}{2}\left(-i\hbar\partial_i + A_{ti}(x)\right)h^{ij}\left(-i\hbar\partial_j + A_{tj}(x)\right) + U_t(x). \tag{139}$$

Lemma 4 *Lie-Poisson equation (134) for Hamiltonian (137) induces the following equation:*

$$i\hbar\frac{\partial}{\partial t}\left\langle x\left|\hat{\rho}_t^\tau[k](\xi)\right|x\right\rangle = -\left\langle x\left|\left[\hat{\rho}_t^\tau[k](\xi), \hat{\mathbf{H}}_t^\tau[k]\right]_-\right|x\right\rangle \tag{140}$$

$$i\hbar\frac{\partial}{\partial t}\left\langle x\left|\frac{1}{2}\left[\hat{\rho}_t^\tau[k](\xi), \hat{\mathbf{p}}_t^\tau[k]\right]_+\right|x\right\rangle = -\left\langle x\left|\left[\frac{1}{2}\left[\hat{\rho}_t^\tau[k](\xi)\ ,\ \hat{\mathbf{H}}_t^\tau[k]\right]_-\ ,\ \hat{\mathbf{p}}_t^\tau[k]\right]_+\right|x\right\rangle. \tag{141}$$

Proof. *If we define the operators:*

$$\hat{\mathbf{H}}_{(0)} = \frac{1}{2} h^{ij} \hat{\mathbf{p}}^{\tau}_{ti}[k] \hat{\mathbf{p}}^{\tau}_{tj}[k] \Big/ i\hbar \tag{142}$$

$$\hat{\mathbf{H}}_{(1)} = \frac{1}{2} \{ \hat{\mathbf{A}}_i h^{ij} \hat{\mathbf{p}}^{\tau}_{tj}[k] + \hat{\mathbf{p}}^{\tau}_{ti}[k] h^{ij} \hat{\mathbf{A}}_j \} \Big/ i\hbar \tag{143}$$

$$\hat{\mathbf{H}}_{(2)} = \left(\hat{U} + \frac{1}{2} h^{ij} \hat{\mathbf{A}}_i \hat{\mathbf{A}}_j \right) \Big/ i\hbar, \tag{144}$$

then Hamiltonian operator $\hat{\mathbf{H}}_t$ *can be represented as*

$$\hat{\mathbf{H}}_t \Big/ i\hbar = \hat{\mathbf{H}}_{(0)} + \hat{\mathbf{H}}_{(1)} + \hat{\mathbf{H}}_{(2)}. \tag{145}$$

Thus, for density operator $\hat{\rho}^{\tau}_t[k](\xi)$ *defined as equation (125),*

$$\frac{-1}{2i\hbar} \left\langle x \left| \left[\left[\hat{\rho}^{\tau}_t[k](\xi) \ , \ \hat{\mathbf{H}}^{\tau}_t[k] \right]_- \ , \ \hat{\mathbf{p}}^{\tau}_t[k] \right]_+ \right| x \right\rangle = term_{(1)} \left(\hat{\mathbf{H}}_{(0)} \right) + term_{(1)} \left(\hat{\mathbf{H}}_{(1)} \right) + term_{(1)} \left(\hat{\mathbf{H}}_{(2)} \right), \tag{146}$$

where

$$term_{(1)} \left(\hat{\mathbf{H}}_{(0)} \right) = \frac{-1}{2i\hbar} \left\langle x \left| \left[\frac{1}{2} \left[\hat{\rho}^{\tau}_t[k](\xi) \ , \ \hat{\mathbf{H}}_{(0)} \right]_- \ , \ \hat{\mathbf{p}}^{\tau}_t[k] \right]_+ \right| x \right\rangle$$

$$term_{(1)} \left(\hat{\mathbf{H}}_{(1)} \right) = \frac{-1}{2i\hbar} \left\langle x \left| \left[\frac{1}{2} \left[\hat{\rho}^{\tau}_t[k](\xi) \ , \ \hat{\mathbf{H}}_{(1)} \right]_- \ , \ \hat{\mathbf{p}}^{\tau}_t[k] \right]_+ \right| x \right\rangle$$

$$term_{(1)} \left(\hat{\mathbf{H}}_{(2)} \right) = \frac{-1}{2i\hbar} \left\langle x \left| \left[\frac{1}{2} \left[\hat{\rho}^{\tau}_t[k](\xi) \ , \ \hat{\mathbf{H}}_{(2)} \right]_- \ , \ \hat{\mathbf{p}}^{\tau}_t[k] \right]_+ \right| x \right\rangle .$$

First term results

$$term_{(1)} \left(\hat{\mathbf{H}}_{(0)} \right) = -\partial_j \left\{ h^{ij} p_{ti}(\eta[k]) \, \rho^{\tau}_t[k](\xi) \, p_{tk}(\eta[k]) \right\} dx^k \tag{147}$$

from the following computations:

$$\left\langle x \left| \ \hat{\mathbf{p}}^{\tau}_{tk}[k] \ \hat{\rho}^{\tau}_t[k](\xi) \ \hat{\mathbf{H}}_{(0)} \ \right| x \right\rangle = \frac{1}{2} \int_{\mathbf{R}^N} d^N k' \ \tilde{\rho}^{\tau}_t(\xi) \left(k + \frac{k'}{2}, k - \frac{k'}{2} \right) e^{ik' \cdot x} \ \Bigg\{ \tag{148}$$

$$\left(p_{tk}(\eta[k]) + \hbar \frac{k'_k}{2} \right) h^{ij} \left(p_{ti}(\eta[k]) - \hbar \frac{k'_i}{2} \right) \left(p_{tj}(\eta[k]) - \hbar \frac{k'_j}{2} \right) \tag{149}$$

$$+ i\hbar \left(p_{tk}(\eta[k]) + \hbar \frac{k'_k}{2} \right) h^{ij} \partial_j \left(p_{ti}(\eta[k]) - \hbar \frac{k'_i}{2} \right) \Bigg\} ; \tag{150}$$

$$\left\langle x \left| \ \hat{\mathbf{H}}_{(0)} \ \hat{\rho}^{\tau}_t[k](\xi) \ \hat{\mathbf{p}}^{\tau}_{tk}[k] \ \right| x \right\rangle = \frac{1}{2} \int_{\mathbf{R}^N} d^N k' \ \tilde{\rho}^{\tau}_t(\xi) \left(k + \frac{k'}{2}, k - \frac{k'}{2} \right) e^{ik' \cdot x} \ \Bigg\{ \tag{151}$$

$$\left(p_{tk}(\eta[k]) - \hbar \frac{k'_k}{2} \right) h^{ij} \left(p_{ti}(\eta[k]) + \hbar \frac{k'_i}{2} \right) \left(p_{tj}(\eta[k]) + \hbar \frac{k'_j}{2} \right) \tag{152}$$

$$- i\hbar \left(p_{tk}(\eta[k]) - \hbar \frac{k'_k}{2} \right) h^{ij} \partial_j \left(p_{ti}(\eta[k]) + \hbar \frac{k'_i}{2} \right) \Bigg\} ; \tag{153}$$

$$\left\langle x \right| \hat{\rho}_t^\tau[k](\xi)\ \hat{\mathbf{H}}_{(0)}\ \hat{\mathbf{p}}_{tk}^\tau[k] \left| x \right\rangle = \frac{1}{2}\int_{\mathbf{R}^N} d^N k'\ \tilde{\rho}_t^\tau(\xi)\left(k+\frac{k'}{2}, k-\frac{k'}{2}\right) e^{ik'\cdot x} \Big\{ \tag{154}$$

$$\left(p_{tk}(\eta[k]) - \hbar\frac{k'_k}{2}\right) h^{ij} \left(p_{ti}(\eta[k]) - \hbar\frac{k'_i}{2}\right)\left(p_{tj}(\eta[k]) - \hbar\frac{k'_j}{2}\right) \tag{155}$$

$$+i\hbar\left(p_{tk}(\eta[k]) - \hbar\frac{k'_k}{2}\right) h^{ij}\partial_i \left(p_{tj}(\eta[k]) - \hbar\frac{k'_j}{2}\right) \tag{156}$$

$$-\hbar^2 h^{ij}\partial_k\partial_i\left(p_{tj}(\eta[k]) - \hbar\frac{k'_j}{2}\right) \tag{157}$$

$$+i\hbar\ h^{ij}\partial_k\left\{\left(p_{ti}(\eta[k]) - \hbar\frac{k'_i}{2}\right)\left(p_{tj}(\eta[k]) - \hbar\frac{k'_j}{2}\right)\right\}\Big\}; \tag{158}$$

$$\left\langle x \right| \hat{\mathbf{p}}_{tk}^\tau[k]\ \hat{\mathbf{H}}_{(0)}\ \hat{\rho}_t^\tau[k](\xi) \left| x \right\rangle = \frac{1}{2}\int_{\mathbf{R}^N} d^N k'\ \tilde{\rho}_t^\tau(\xi)\left(k+\frac{k'}{2}, k-\frac{k'}{2}\right) e^{ik'\cdot x} \Big\{ \tag{159}$$

$$+\left(p_{tk}(\eta[k]) + \hbar\frac{k'_k}{2}\right) h^{ij} \left(p_{ti}(\eta[k]) + \hbar\frac{k'_i}{2}\right)\left(p_{tj}(\eta[k]) + \hbar\frac{k'_j}{2}\right) \tag{160}$$

$$-i\hbar\left(p_{tk}(\eta[k]) + \hbar\frac{k'_k}{2}\right) h^{ij}\partial_i \left(p_{tj}(\eta[k]) + \hbar\frac{k'_j}{2}\right) \tag{161}$$

$$-\hbar^2 h^{ij}\partial_k\partial_i\left(p_{tj}(\eta[k]) + \hbar\frac{k'_j}{2}\right) \tag{162}$$

$$-i\hbar\ h^{ij}\partial_k\left\{\left(p_{ti}(\eta[k]) + \hbar\frac{k'_i}{2}\right)\left(p_{tj}(\eta[k]) + \hbar\frac{k'_j}{2}\right)\right\}\Big\}. \tag{163}$$

Further,

$$\begin{aligned} term_{(1)}\left(\hat{\mathbf{H}}_{(1)}\right) &= -\left\{\partial_i\left(h^{ij}A_j\rho_t^\tau[k](\xi)\,p_{tk}(\eta[k])\right)\right. \\ &\quad \left. +\rho_t^\tau[k](\xi)\left(\partial_k h^{ij}A_j\right)p_{ti}(\eta[k])\right\}dx^k; \end{aligned} \tag{164}$$

$$term_{(1)}\left(\hat{\mathbf{H}}_{(2)}\right) = -\rho_t^\tau[k](\xi)\,\partial_k\left(U + \frac{1}{2}h^{ij}A_iA_j\right)dx^k. \tag{165}$$

Thus, second equation (141) in this lemma becomes

$$\frac{\partial}{\partial t}\left\{\rho_t^\tau[k](\xi)\,p_{tk}(\eta[k])\right\} = -\partial_j\left\{h^{ij}\left(p_{ti}(\eta[k]) + A_j\right)\rho_t^\tau[k](\xi)\,p_{tk}(\eta[k])\right\} \tag{166}$$

$$+\rho_t^\tau[k](\xi)\,p_{tj}(\eta[k])\left(\partial_k h^{ij}A_i\right) \tag{167}$$

$$-\rho_t^\tau[k](\xi)\,\partial_k\left(U + \frac{1}{2}h^{ij}A_iA_j\right), \tag{168}$$

which is equivalent to equation (136) for Hamiltonian (137).

On the other hand,

$$\frac{-1}{i\hbar}\left\langle x\right|\left[\hat{\rho}_t^\tau[k](\xi)\ ,\ \hat{\mathbf{H}}_t^\tau[k]\right]_- \left| x\right\rangle = term_{(2)}\left(\hat{\mathbf{H}}_{(0)}\right) + term_{(2)}\left(\hat{\mathbf{H}}_{(1)}\right) + term_{(2)}\left(\hat{\mathbf{H}}_{(2)}\right), \tag{169}$$

where

$$term_{(2)}\left(\hat{\mathbf{H}}_{(0)}\right) = \frac{-1}{i\hbar}\left\langle x\right|\left[\hat{\rho}_t^\tau[k](\xi)\ ,\ \hat{\mathbf{H}}_{(0)}\right]_- \left| x\right\rangle$$

$$term_{(2)}\left(\hat{\mathbf{H}}_{(1)}\right) = \frac{-1}{i\hbar}\left\langle x\right|\left[\hat{\rho}^{\tau}_{t}[k]\left(\xi\right)\ ,\ \hat{\mathbf{H}}_{(1)}\right]_{-}\left|x\right\rangle$$

$$term_{(2)}\left(\hat{\mathbf{H}}_{(2)}\right) = \frac{-1}{i\hbar}\left\langle x\right|\left[\hat{\rho}^{\tau}_{t}[k]\left(\xi\right)\ ,\ \hat{\mathbf{H}}_{(2)}\right]_{-}\left|x\right\rangle .$$

Each term can be calculated as follows:

$$term_{(2)}\left(\hat{\mathbf{H}}_{(0)}\right) = \frac{-1}{2i\hbar}\int_{\mathbf{R}^N} d^N k'\ \tilde{\rho}^{\tau}_{t}\left(\xi\right)\left(k+\frac{k'}{2},k-\frac{k'}{2}\right)e^{ik'\cdot x}\ \Bigg\{ \tag{170}$$

$$h^{ij}\left(p_{ti}\left(\eta[k]\right)-\hbar\frac{k'_i}{2}\right)\left(p_{tj}\left(\eta[k]\right)-\hbar\frac{k'_j}{2}\right) \tag{171}$$

$$+i\hbar h^{ij}\partial_j\left(p_{ti}\left(\eta[k]\right)-\hbar\frac{k'_i}{2}\right) \tag{172}$$

$$-h^{ij}\left(p_{ti}\left(\eta[k]\right)+\hbar\frac{k'_i}{2}\right)\left(p_{tj}\left(\eta[k]\right)+\hbar\frac{k'_j}{2}\right) \tag{173}$$

$$-i\hbar h^{ij}\partial_j\left(p_{ti}\left(\eta[k]\right)-\hbar\frac{k'_i}{2}\right)\Bigg\} \tag{174}$$

$$= -\partial_j\left(\rho^{\tau}_{t}[k]\left(\xi\right)h^{ij}p_{ti}\left(\eta[k]\right)\right) \tag{175}$$

$$term_{(2)}\left(\hat{\mathbf{H}}_{(1)}\right) = -\partial_i h^{ij}\left(A_j\rho^{\tau}_{t}[k]\left(\xi\right)\right);$$

$$term_{(2)}\left(\hat{\mathbf{H}}_{(2)}\right) = 0. \tag{176}$$

Thus, first equation (141) in this lemma becomes

$$\frac{\partial}{\partial t}\rho^{\tau}_{t}[k]\left(\xi\right) = -\partial_j\left\{h^{ij}\left(p_{ti}\left(\eta[k]\right)+A_j\right)\rho^{\tau}_{t}[k]\left(\xi\right)\right\}, \tag{177}$$

which is equivalent to equation (135) for Hamiltonian (137).

Therefore, Lie-Poisson equation (134) proved to be equivalent to the equation set (140) and (141) in this lemma.

□

The above lemma leads us to one of the main theorem in the present paper, declaring that Lie-Poisson equation (134) for Hamiltonian (137) is equivalent to the quantum Liouville equation.

Theorem 3 *Lie-Poisson equation (134) for Hamiltonian (137) is equivalent to the following quantum Liouville equation:*

$$\frac{\partial}{\partial t}\hat{\rho}_t = \left[\hat{\rho}_t,\hat{\mathbf{H}}\right]_{-}/(-i\hbar). \tag{178}$$

Proof. *The following computation proves this theorem based on the previous lemma:*

$$\frac{\partial}{\partial t}\left\langle\hat{\rho}_t\ \hat{\mathbf{F}}_t\right\rangle = \frac{\partial}{\partial t}\left\langle\hat{\rho}^{\tau}_t\ \hat{\mathbf{F}}_t\right\rangle$$

$$= \int_{\Gamma_U} d\mathcal{N}\left(\xi\right)\int_{\mathbf{R}^N} d^N k\int_M dv(x)\times$$

$$\Big\{\left\langle x\left|\hat{\mathbf{F}}^{\tau}_t[k]\ \hat{\rho}^{\tau}_t[k]\left(\xi\right)\ \hat{\mathbf{H}}^{\tau}_t[k]\right|x\right\rangle - \left\langle x\left|\hat{\mathbf{H}}^{\tau}_t[k]\ \hat{\rho}^{\tau}_t[k]\left(\xi\right)\ \hat{\mathbf{F}}^{\tau}_t[k]\right|x\right\rangle$$

$$+\left\langle x\right|\ \hat{\rho}^{\tau}_t[k]\left(\xi\right)\ \left|x\right\rangle\ \frac{\partial p^{\tau}_t[k](x)}{\partial t}\cdot\mathcal{D}F_t\left(\eta^{\tau}_t[k]\right)(x)$$

$$+\langle x|\ \hat{\rho}^{\tau}_t[k]\ |x\rangle\, p^*\frac{\partial F_t}{\partial t}\left(\eta^{\tau}_t[k]\right)(x)\Big\}$$

$$\begin{aligned}
= & \int_{\Gamma_U} d\mathcal{N}(\xi)\int_{\mathbf{R}^N} d^N k\int_M dv(x)\times \\
& \Big\{\left\langle x\left|\left[\hat{\rho}^{\tau}_t[k](\xi),\hat{\mathbf{H}}^{\tau}_t[k]\right]_-\right|x\right\rangle\ p^*F_t\left(\eta^{\tau}_t[k]\right)(x) \\
& +\left(\frac{\partial}{\partial t}\left\langle x\left|\ \frac{1}{2}\left[\ \hat{\rho}^{\tau}_t[k](\xi)\ \ ,\hat{\mathbf{p}}^{\tau}_t[k]\right]_+\ \right|x\right\rangle\right)\cdot\mathcal{D}F_t\left(\eta^{\tau}_t[k]\right)(x) \\
& -\left\langle x\left|\ \frac{\partial\hat{\rho}^{\tau}_t[k](\xi)}{\partial t}\ \right|x\right\rangle\ p^{\tau}_t[k](x)\cdot\mathcal{D}F_t\left(\eta^{\tau}_t[k]\right)(x) \\
& +\langle x|\ \hat{\rho}^{\tau}_t[k]\ |x\rangle\, p^*\frac{\partial F_t}{\partial t}\left(\eta^{\tau}_t[k]\right)(x)\Big\} \\
= & \int_{\Gamma_U} d\mathcal{N}(\xi)\int_{\mathbf{R}^N} d^N k\int_M dv(x)\times \\
& \Big\{\left\langle x\left|\left[\ \frac{1}{2}\left[\ \hat{\rho}^{\tau}_t[k](\xi)\ \ ,\hat{\mathbf{p}}^{\tau}_t[k]\right]_+\ ,\ \hat{\mathbf{H}}^{\tau}_t[k]\right]_-\right|x\right\rangle\cdot\mathcal{D}F_t\left(\eta^{\tau}_t[k]\right)(x) \\
& \left\langle x\left|\left[\hat{\rho}^{\tau}_t[k](\xi)\ ,\ \hat{\mathbf{H}}^{\tau}_t[k]\right]_-\right|x\right\rangle\ \left\{p^*F_t\left(\eta^{\tau}_t[k]\right)(x)-p^{\tau}_t[k](x)\cdot\mathcal{D}F_t\left(\eta^{\tau}_t[k]\right)(x)\right\} \\
& +\langle x|\ \hat{\rho}^{\tau}_t[k]\ |x\rangle\, p^*\frac{\partial F_t}{\partial t}\left(\eta^{\tau}_t[k]\right)(x)\Big\} \\
= & \left\langle ad^*_{\hat{H}^{\tau}_t}\mathcal{J}^{\tau}_t,\hat{F}^{\tau}_t\right\rangle+\left\langle\mathcal{J}_t,\frac{\partial\hat{F}_t}{\partial t}\right\rangle\ .
\end{aligned}\tag{179}$$

□

Now, the density matrix $\hat{\rho}_t$ becomes the summation of the pure sates $\left|\psi^{(l;\pm)}_t\right\rangle\left\langle\psi^{(l;\pm)}_t\right|$ for the set $\left\{\left|\psi^{(l;\pm)}_t\right\rangle\right\}_{l\in\mathbf{R}^N}$ of the orthonormal wave vectors such that $\left\langle\psi^{(l';s')}_t\right|\left.\psi^{(l;s)}_t\right\rangle=\delta(l'-l)\delta_{s,s'}$:

$$\hat{\rho}_t=\int_\Lambda dP_+(l)\left|\psi^{(l;+)}_t\right\rangle\left\langle\psi^{(l;+)}_t\right|-\int_\Lambda dP_-(l)\left|\psi^{(l;-)}_t\right\rangle\left\langle\psi^{(l;-)}_t\right|,\tag{180}$$

where $P_\pm$ is a corresponding probability measure on the space Λ of a spectrum and the employed integral is the Stieltjes integral [5]. If the system is open and has the continuous spectrum, then it admits Λ be the continuous superselection rules (CSRs). The induced wave function has the following expression for a L^2-function $\psi^{(l;\pm)}_t=\left\langle x\left|\psi^{(l;-)}_t\right.\right\rangle\in L^2(M)$:

$$\chi^*_\alpha\psi^{(l;\pm)}_t(x)=\int_{\mathbf{R}^N}d^N k\ \tilde{\psi}^{(l;\pm)}_{\alpha\ t}(k)e^{i\{k_j\mathbf{x}^j+\zeta_t(x)\}}.\tag{181}$$

The existence of the probability measure P_- would be corresponding to the existence of the antiparticle for the elementary quantum mechanics.

For example, the motion of the particle on a N-dimensional rectangle box $[0,\pi]^N$ needs the following boundary condition on the verge of the box:

if $x_j=0$ or π for some $j\in\{1,...,N\}$, then $\langle x|\hat{\rho}_t|x\rangle=0$, Density matrix $\hat{\rho}_t$ is the summation of integer-labeled pure states:

$$\hat{\rho}_t = \sum_{(n,n')\in\mathbf{Z}^{2N}} \tilde{\rho}^t_t(n',n) \left|n;t\right\rangle \left\langle n';t\right| . \tag{182}$$

Let us now concentrate on the case where $\hat{\rho}_t$ is a pure state in the following form:

$$\hat{\rho}_t = \left|\psi_t\right\rangle \left\langle \psi_t\right| ; \tag{183}$$

there exists a wave function $\psi_t \in L^2(M)$

$$\psi_t(x) = \int_{\mathbf{R}^N} d^N k \ \tilde{\psi}_t(k) e^{i\{k_j \mathbf{x}^j + \zeta_t(x)\}}, \tag{184}$$

where

$$\tilde{\rho}^t_t(k,k') = \tilde{\psi}_t(k)^* \tilde{\psi}_t(k'). \tag{185}$$

Theorem 3 introduces the Schrödinger equation as the following collorary.

Collorary 1 *Lie-Poisson equation (134) for Hamiltonian (137) becomes the following Schrödinger equation:*

$$i\hbar\partial_t\psi_t = \mathcal{H}\psi_t, \tag{186}$$

where

$$\mathcal{H} = \frac{1}{2m}\sqrt{}^{-1}\left(-i\hbar\partial_i + A_{ti}(x)\right) g^{ij}(x) \sqrt{}\left(-i\hbar\partial_j + A_{tj}(x)\right) + U_t(x). \tag{187}$$

Therefore, the presented theory induces not only canonical, nonrelativistic quantum mechanics but also the canonical, relativistic or nonrelativistic quantum field theory if proliferated for the grassmanian field variables. In addition, Section 7 will discuss how the present theory also justifies the regularization procedure in the appropriate renormalization.

On the other hand, if introducing the unitary transformation $\hat{U}_t = e^{it\hat{\mathbf{H}}_t}$, Theorem 3 obtains the Heisenberg equation for Heisenberg's representations $\tilde{\mathbf{H}}_t = \hat{U}_t \hat{\mathbf{H}}_t \hat{U}_t^{-1}$ and $\tilde{\mathbf{F}}_t = \hat{U}_t \hat{\mathbf{F}}_t \hat{U}_t^{-1}$:

$$\frac{\partial}{\partial t}\tilde{\mathbf{F}}_t = \left[\tilde{\mathbf{H}}_t, \tilde{\mathbf{F}}_t\right]_- / (-i\hbar) + \left(\widetilde{\frac{\partial \mathbf{F}_t}{\partial t}}\right), \tag{188}$$

since $\hat{\rho}_t = \hat{U}_t^{-1}\hat{\rho}_0\hat{U}_t$.

As discussed in Section 3, if a group action of Lie group $Q(M)$ keeps the Hamiltonian $\mathcal{H}_t : q(M)^* \to \mathbf{R}$ invariant, there exists an invariant charge functional $Q : \Gamma\left[E(M)\right] \to C(M)$ and the induced function $\mathcal{Q} : q(M)^* \to \mathbf{R}$ such that

$$\left[\hat{H}_t, \hat{Q}\right] = 0, \tag{189}$$

where $\hat{Q}$ is expressed as

$$\hat{Q} = \left(\mathcal{D}_{\rho(\eta)} Q\left(p(\eta)\right), -p(\eta)\cdot\mathcal{D}_{\rho(\eta)} Q\left(p(\eta)\right) + Q\left(p(\eta)\right)\right). \tag{190}$$

Suppose that functional $p^*Q : \Gamma\left[E(M)\right] \to C(M)$ has the canonical form such that

$$Q^{T^*M}(x,p) = A^{ij}p_ip_j + B(x)_ip_j + C(x), \tag{191}$$

then the corresponding generator is equivalent to the observable:

$$\hat{\mathbf{Q}} = A^{ij}\hat{\mathbf{p}}_i\hat{\mathbf{p}}_j + \hat{\mathbf{B}}_i\hat{\mathbf{p}}_j + \hat{\mathbf{p}}_j\hat{\mathbf{B}}_i + \hat{\mathbf{C}}. \tag{192}$$

In this case, relation (189) has the canonical expression:

$$\left[\hat{\mathbf{H}}_t, \hat{\mathbf{Q}}\right] = 0. \tag{193}$$

Those operators can have the eigen values at the same time.

As shown so far, protomechanics successfully deduced quantum mechanics for the canonical Hamiltonians that have no problem in the operator ordering, and proves still valid for the noncanonical Hamiltonian that have the ambiguity of the operator ordering in the ordinary quantum mechanics. In the latter case, the infinitesimal generator $\hat{\mathbf{F}}_t^{tr}$ corresponding to $\hat{F} \in q(M)$ is not always equal to observable $\hat{\mathbf{F}}_t$:

$$\hat{\mathbf{F}}_t \neq \hat{\mathbf{F}}_t^{tr}. \tag{194}$$

If one tries to quantize the Einstein gravity, he or she can proliferate the present theory in a direct way by utilizing Lie-Poisson equation (134). But, some calculation method should be developed for this purpose elsewhere.

4.3 Interpretation of Spin

It has been known that a half-spin in quantum mechanics *does* have a classical analogy as a rigid rotor in classical mechanics [6].[7] Such a model represents the motion of a particle on the three-dimensional orthogonal group $SO(3)$. A spinor is corresponding to an element of the Lie-algebra $so(3)$ of $SO(3)$, which is equivalent to a right-(or left-)invariant vector field over $SO(3)$. This section reviews such an interpretation of a spin in terms of the Euler angles or the coordinates over a three-dimensional special orthogonal group $SO(3)$; and thus, it proves that the present theory is applicable for the description of a half-spin, too.

Now, let us consider the particle motion in a three-dimensional Euclidean space $\mathbf{R}^3$ with the polar coordinates $\mathbf{x} = (r, \theta, \phi) \in [0, +\infty) \times [0, 2\pi) \times (0, \pi)$. Lie group $SO(3)$ acts on $\mathcal{J}_t = (\rho_t^\tau p_t^\tau, \rho_t^\tau)$ by the coadjoint action, where an infinitesimal generator $M = M^j \hat{L}_j \in so(3) \subset q(M)$ ($M_j \in \mathbf{R}$, $j \in \{1, 2, 3\}$) has an corresponding operator $\hat{\mathbf{M}} = M^j \hat{\mathbf{L}}_j \in su(2, \mathbf{C})$ that satisfies

$$\left\langle ad^*_{\hat{M}} \mathcal{J}_t, \hat{F} \right\rangle = -i\hbar^{-1} \left\langle \left[\hat{\rho}_t, \hat{\mathbf{M}}\right]_- \hat{\mathbf{F}} \right\rangle. \tag{195}$$

Infinitesimal generator $\hat{L}_j$ has the following expression:

$$\hat{L}_1 = -sin\phi \frac{\partial}{\partial \theta} - \cot\theta \cos\phi \frac{\partial}{\partial \phi}\ , \tag{196}$$

$$\hat{L}_2 = \cos\phi \frac{\partial}{\partial \theta} - \cot\theta \sin\phi \frac{\partial}{\partial \phi}\ , \tag{197}$$

$$\hat{L}_3 = \frac{\partial}{\partial \phi}\ ; \tag{198}$$

It has an corresponding operator $\hat{\mathbf{M}} = M^j \hat{\mathbf{L}}_j \in su(2, \mathbf{C})$ acting on the Hilbert space $\mathcal{H}(S^2)$ of all the single- or double-valued L_2 functions over S^2:

$$\left\langle \theta, \phi \left| \hat{\mathbf{L}}_j \right| \psi \right\rangle = \frac{\hbar}{i} \hat{L}_j\ \ \langle \theta, \phi \left| \psi \right\rangle, \tag{199}$$

where $|\psi\rangle \in \mathcal{H}(S^2)$. Notice that these operators are hermite or self-conjugate, $\hat{\mathbf{L}}_j^\dagger = \hat{\mathbf{L}}_j$, and induces the angular momentum or the integer spin of the particle:

$$|\psi_t\rangle = \sum_{m=-l}^{l} c^l_m(t) \left| l, m \right\rangle \tag{200}$$

$$for \qquad \langle \theta, \phi \left| l; m \right\rangle = Y_l^m(\theta, \phi), \tag{201}$$

[7] The ignorance on this fact may have prevented quantum mechanics from the realistic interpretation in general.

where

$$\hat{\mathbf{L}}\cdot\hat{\mathbf{L}}\left|l,m\right\rangle = \hbar^2 l(l+1)\left|l;m\right\rangle \quad , \qquad \hat{\mathbf{L}}_3\left|l,m\right\rangle = \hbar m\left|l;m\right\rangle . \tag{202}$$

If the Hamiltonian for the motion in the three-dimensional Euclid space has the following form in a central field of force, it is invariant under the rotation about z-axis:

$$H\left(x,p\right) = p^2 + x\cdot(p\times B) + U(r), \tag{203}$$

where $r = \sqrt{x^2+y^2+z^2} \neq 0$. Since this Hamiltonian has the canonical form, the corresponding infinitesimal generator is equivalent to the following quantum observable [7]:

$$\hat{\mathbf{H}} = \hat{\mathbf{P_r}}^2 + \frac{\hat{\mathbf{L}}\cdot\hat{\mathbf{L}}}{r^2} + \frac{1}{2}\left\{\hat{\mathbf{L}}\cdot B + B\cdot\hat{\mathbf{L}}\right\} + U(r), \tag{204}$$

where

$$\left\langle \theta,\phi,r\left|\hat{\mathbf{P_r}}\right|\psi\right\rangle = -\frac{\hbar}{ir}\frac{\partial}{\partial r}r\left\langle \theta,\phi,r\left|\psi\right.\right\rangle . \tag{205}$$

To realize the representation for a half-spin, let us consider the Hilbert spaces $\mathcal{H}(SO(3))$ of all the single- or double-valued L_2 functions over S^2 which can be reduced to $\mathcal{H}(S^2)$. On the classical level, an infinitesimal generator $N = N^j S_j$ of $SO(3)$ is equivalent to a left-(or right-)invariant vector field:

$$\begin{aligned}
\hat{S}_1 &= \hat{L}_1 + \frac{\hbar}{i}\cdot\frac{\cos\phi}{\sin\theta}\frac{\partial}{\partial\chi}, && (206)\\
\hat{S}_2 &= \hat{L}_2 + \frac{\hbar}{i}\cdot\frac{\sin\phi}{\sin\theta}\frac{\partial}{\partial\chi}\ , && (207)\\
\hat{S}_3 &= \hat{L}_3\ . && (208)
\end{aligned}$$

Notice that infinitesimal generator $N = N^j S_j$ is also an element of the semidirect product $SO(3)\times C^\infty(M)$ of $SO(3)$ with the space $C^\infty(S^2)$ of all the C^∞ functions over S^2 excepting poles $\theta = 0, \pi$. The corresponding operators $\mathbf{S}_j$ in quantum mechanics to generators S_j become

$$\begin{aligned}
\left\langle \theta,\phi,\chi\left|\hat{\mathbf{S}}_1\right|\psi\right\rangle &= \left\{\frac{\hbar}{i}\hat{L}_1 + \frac{\hbar}{i}\cdot\frac{\cos\phi}{\sin\theta}\frac{\partial}{\partial\chi}\right\}\left\langle \theta,\phi,\chi\left|\psi\right.\right\rangle, && (209)\\
\left\langle \theta,\phi,\chi\left|\hat{\mathbf{S}}_2\right|\psi\right\rangle &= \left\{\frac{\hbar}{i}\hat{L}_2 + \frac{\hbar}{i}\cdot\frac{\sin\phi}{\sin\theta}\frac{\partial}{\partial\chi}\right\}\left\langle \theta,\phi,\chi\left|\psi\right.\right\rangle, && (210)\\
\left\langle \theta,\phi,\chi\left|\hat{\mathbf{S}}_3\right|\psi\right\rangle &= \frac{\hbar}{i}\hat{L}_3\ \left\langle \theta,\phi,\chi\left|\psi\right.\right\rangle . && (211)
\end{aligned}$$

which has the following reduced expression:

$$\begin{aligned}
\left\langle \theta,\phi\left|\hat{\mathbf{S}}_1\right|\psi\right\rangle &= \left\{\frac{\hbar}{i}\hat{L}_1 + \frac{\hbar}{2}\cdot\frac{\cos\phi}{\sin\theta}\right\}\left\langle \theta,\phi\left|\psi\right.\right\rangle, && (212)\\
\left\langle \theta,\phi\left|\hat{\mathbf{S}}_2\right|\psi\right\rangle &= \left\{\frac{\hbar}{i}\hat{L}_2 + \frac{\hbar}{2}\cdot\frac{\sin\phi}{\sin\theta}\right\}\left\langle \theta,\phi\left|\psi\right.\right\rangle, && (213)\\
\left\langle \theta,\phi\left|\hat{\mathbf{S}}_3\right|\psi\right\rangle &= \frac{\hbar}{i}\hat{L}_3\ \left\langle \theta,\phi\left|\psi\right.\right\rangle . && (214)
\end{aligned}$$

These operators induce the half-spin:

$$\left|\psi_t\right\rangle = c_+(t)\left|+\right\rangle + c_-(t)\left|-\right\rangle, \tag{215}$$

where the eigen states have the following expression:

$$\langle \theta,\phi,\chi\,|+\rangle = \frac{1}{\sqrt{2\pi}} e^{\frac{i}{2}(\phi+\chi)} \cos\frac{\theta}{2} \quad , \quad \langle \theta,\phi,\chi\,|-\rangle = \frac{1}{\sqrt{2\pi}} e^{-\frac{i}{2}(\phi-\chi)} \sin\frac{\theta}{2}. \tag{216}$$

whose reduced version is

$$\langle \theta,\phi\,|+\rangle = \frac{1}{\sqrt{2\pi}} e^{-is} e^{i\frac{\phi}{2}} \cos\frac{\theta}{2} \quad , \quad \langle \theta,\phi\,|-\rangle = \frac{1}{\sqrt{2\pi}} e^{-is} e^{-i\frac{\phi}{2}} \sin\frac{\theta}{2}. \tag{217}$$

They satisfy

$$\hat{\mathbf{S}}\cdot\hat{\mathbf{S}}\,\Big|\pm\Big\rangle = \frac{3}{4}\hbar^2\,|\pm\rangle \quad , \quad \hat{\mathbf{S}}_3\,\Big|\pm\Big\rangle = \pm\frac{\hbar}{2}\,|\pm\rangle\,. \tag{218}$$

In addition, we can introduce the increasing operator and the decreasing one $\hat{\mathbf{S}}_\pm = \hat{\mathbf{S}}_1 \pm i\hat{\mathbf{S}}_2$:

$$\left\langle \theta,\phi,\chi \left|\hat{\mathbf{S}}_\pm\right| \psi\right\rangle = \hbar e^{\pm i\phi}\left\{\pm\frac{\partial}{\partial\theta} + icot\theta\frac{\partial}{\partial\phi} - \frac{i}{sin\theta}\frac{\partial}{\partial\chi}\right\}\,\langle \theta,\phi,\chi\,|\psi\rangle\,, \tag{219}$$

which proves the following relations:

$$\hat{\mathbf{S}}_\pm\,|\mp\rangle = |\pm\rangle \quad , \quad \hat{\mathbf{S}}_\pm\,|\pm\rangle = 0. \tag{220}$$

As in the usual expression [7] originated by Pauli, if ketvectors $|\pm\rangle$ are denoted as

$$|+\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad , \quad |-\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \tag{221}$$

then, $\hat{\mathbf{S}}_j = \frac{\hbar}{2}\sigma_j$ for the Pauli matrices:

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad , \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \quad and \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}; \tag{222}$$

$$\sigma_+ = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \quad and \quad \sigma_- = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}. \tag{223}$$

A general state of the half-integer spin of a particle has the following expression:

$$|\psi_t\rangle = \sum_{m=-l-1}^{l} c_m^{l+1/2}(t)\,|l+1/2, m+1/2\rangle\,, \tag{224}$$

where, for the normalization constant $N_{l+1/2}^{m+1/2}$,

$$\langle \theta,\phi\,|l+1/2; m+1/2\rangle = N_{l+1/2}^{m+1/2}\sqrt{\frac{l+m+1}{2l+1}} e^{-is} e^{i\frac{\phi}{2}} \cos\frac{\theta}{2}\;Y_l^m(\theta,\phi) \tag{225}$$

$$+N_{l+1/2}^{m+1/2}\sqrt{\frac{l-m}{2l+1}} e^{-is} e^{-i\frac{\phi}{2}} \sin\frac{\theta}{2}\;Y_l^{m+1}(\theta,\phi); \tag{226}$$

and the eigen states satisfy

$$\hat{\mathbf{S}}\cdot\hat{\mathbf{S}}\,\Big|l+1/2, m+1/2\Big\rangle = \hbar^2(l+1/2)(l+3/2)\,|l+1/2; m+1/2\rangle \quad , \tag{227}$$

$$\hat{\mathbf{S}}_3\,\Big|l+1/2, m+1/2\Big\rangle = \hbar(m+1/2)\,|l+1/2; m+1/2\rangle\,. \tag{228}$$

Let us assume the classical motion of a rigid rotor has the following Hamiltonian:

$$H = I^{-1} S \cdot S + S \cdot B. \tag{229}$$

To elucidate that Hamiltonian (231) has no trouble in the operator-ordering problem, we can introduce

$$H = I^{-1} r^2 (p^2 + C^2 p'^2) + x \cdot \{p \times C p'\} + x \cdot \{p \times (B - C p')\}, \tag{230}$$

where the induced motion preserve the following initial conditions:

$$x \cdot p = 0 \ \ and \ \ p' = \frac{\hbar}{2}. \tag{231}$$

For Hamiltonian (231), the infinitesimal generator of motion is equivalent to the following observable:

$$\hat{\mathbf{H}} = I^{-1} \hat{\mathbf{S}} \cdot \hat{\mathbf{S}} + \frac{1}{2} \left\{ \hat{\mathbf{S}} \cdot B + B \cdot \hat{\mathbf{S}} \right\}, \tag{232}$$

where

$$C = \frac{\hbar}{2} \left(\frac{x}{2\,(x^2 + y^2)}, \frac{y}{2\,(x^2 + y^2)}, 0 \right) + x \times \nabla s. \tag{233}$$

Now, we can investigate the internal structure of such a half-integer spin particle, an quark or lepton as an electron or a constituted particle as a nucleus, which would have the following spin for the internal three-dimensional Euclid space:

$$S(x, p) = x \times (p + \nabla s) + \frac{\hbar}{2} \left(\frac{x}{2\,(x^2 + y^2)}, \frac{y}{2\,(x^2 + y^2)}, 0 \right). \tag{234}$$

Such an interpretation of half-integer spin allows us to describe the Dirac equation as the equation of the motion for the following Hamiltonian:

$$H(x, p, \alpha, \beta) = \alpha_1 \beta \cdot \left(p - \frac{e}{c} A \right) + mc^2 \alpha_3 - eA_0, \tag{235}$$

where α and β are the internal spins expressed as relation (234). Since the obtained Hamiltonian is also canonical as discussed in the previous subsection, it has the following infinitesimal generator:

$$\hat{\mathbf{H}} = \left(\hat{\gamma}_j \left(\hat{\mathbf{p}}^j - \frac{e}{c} A \right) + mc^2 \right) \hat{\gamma}_0 - eA_0, \tag{236}$$

where $\hat{\gamma}$ is the Dirac matrices. In the same way, the internal freedom like the isospins of a particle can be expressed as the invariance of motion, if its Lie group is a subset of the infinite-dimensional semidirect-product group $S(M)$. More detailed consideration on the relativistic quantum mechanics will be held elsewhere.

5 CONCLUSION

The present paper proved that SbM and then protomechanics deduces both classical mechanics and quantum mechanics in its natural consequence, and supported the rigid-body interpretation of a half-integer spin. The next paper [8] will discuss the intimate relationships between the present theory and the other quantization methods known in twentieth century; and it will reveal that the new interpretation of the measurement process is compatible with reality and causality.

APPENDIX: INTEGRATION ON MANIFOLD

Let us here determine the properties of the manifold M that is the three-dimensional physical space for the particle motion in classical or quantum mechanics, or the space of graded field variables for the field motion in classical or the quantum field theory.

Let $(M, \mathcal{O}_M)$ be a Hausdorff space for the family $\mathcal{O}_M$ of its open subsets, and also a N-dimensional oriented C^∞ manifold that is modeled by the N-dimensional Euclid space $\mathbf{R}^N$ and thus it has an atlas $(U_\alpha, \varphi_\alpha)_{\alpha\in\Lambda_M}$ (the set of a local chart of M) for some countable set Λ_M such that

1. $M = \bigcup_{\alpha\in\Lambda_M} U_\alpha$,
2. $\varphi_\alpha : U_\alpha \to V_\alpha$ is a C^∞ diffeomorphism for some $V_\alpha \subset \mathbf{R}^N$ and
3. if $U_\alpha \cap U_\beta \neq \emptyset$, then $g^\varphi_{\alpha\beta} = \varphi_\beta \circ \varphi_\alpha^{-1} : V_\alpha \cap V_\beta \to V_\alpha \cap V_\beta$ is a C^∞ diffeomorphism.

The above definition would be extended to include that of the infinite-dimensional manifolds called ILH-manifolds. A ILH-manifold that is modeled by the infinite-dimensional Hilbert space having an inverse-limit topology instead of $\mathbf{R}^N$ [4]. We will, however, concentrate ourselves on the finite-dimensional cases for simplicity. Let us further assume that M has no boundary $\partial M = \emptyset$ for the smoothness of the C^∞ diffeomorphism group $D(M)$ over M, i.e., in order to consider the mechanics on a manifold N that has the boundary $\partial N \neq \emptyset$, we shall substitute the doubling of N for M: $M = N \cup \partial N \cup N$.

Now, manifold M is the topological measure space $M = (M, \mathcal{B}(\mathcal{O}_M), vol)$ that has the volume measure vol for the topological σ-algebra $\mathcal{B}(\mathcal{O}_M)$. For the Riemannian manifold M, the (psudo-)Riemannian structure induces the volume measure vol.

Second, we assume that the particle moves on manifold M and has its internal freedom represented by a oriented manifold $F = (F, \mathcal{O}_F)$, where $\mathcal{O}_F$ is the family of open subsets of F. Let $F = (F, \mathcal{B}(\mathcal{O}_F), m_F)$ be the topological measure space with the invariant measure m_F under the group transformation G_F: $\tilde{g}_* m_F = m_F$ for $\tilde{g} \in G_F$ where $\tilde{g}_* m_F(\tilde{g}(A)) = m_F(A)$ for $A \in \mathcal{B}(\mathcal{O}_F)$. In this case, the state of the particle can be represented as a position on the locally trivial, oriented fiber bundle $E = (E, M, F, \pi)$ with fiber F over M with a canonical projection $\pi : E \to M$, i.e., for every $x \in M$, there is an open neighborhood $U(x)$ and a C^∞ diffeomorphism $\phi_U : \pi^{-1}(U(x)) \to U(x) \times F$ such that $\pi = \pi_U \circ \phi_U$ for $\pi_U : U(x) \times F \to U(x) : (x, s) \to x$. Let G_F be the structure group of fiber bundle E: the mapping $\tilde{g}_{\alpha\beta} = \phi_{U_\alpha} \circ \phi_{U_\beta}^{-1} : U_\alpha \cap U_\beta \times F \to U_\alpha \cap U_\beta \times F$ satisfies $\tilde{g}_{\alpha\beta}(x, s) \in G_F$ for $(x, s) \in U_\alpha \cap\ U_\beta \times F$ and the cocycle condition:

$$\tilde{g}_{\alpha\beta}(x, s) \cdot \tilde{g}_{\beta\gamma}(x, s) = \tilde{g}_{\alpha\gamma}(x, s) \qquad for \quad (x, s) \in U_\alpha \cap U_\beta \cap U_\gamma \times F, \tag{A1}$$

where $\alpha,\ \beta,\ \gamma \in \Lambda_M$; and condition (A1) includes the following relations:

$$\tilde{g}_{\alpha\alpha}(x, s) = id. \quad for \quad x \in U_\alpha, \quad and \quad \tilde{g}_{\alpha\beta}(x, s) = \tilde{g}_{\beta\alpha}(x, s)^{-1} \quad for \quad (x, s) \in U_\alpha \cap U_\beta \times F. \tag{A2}$$

Thus, $(E, \mathcal{O}_E)$ is the Hausdorff space for the family $\mathcal{O}_E$ of the open subsets of E such that $\tilde{U} \in \mathcal{O}_E$ satisfies $\phi_{U_\alpha}\left(\tilde{U}\right) = U_\alpha \times U'_\alpha$ for some U_α ($\alpha \in \Lambda_M$) and $U'_\alpha \in \mathcal{O}_F$.

Now, $(E, \mathcal{B}(\mathcal{O}_E), m_E)$ becomes the topological measure space with the measure m_E induced by the measures vol and m_F as follows. For $A \in \mathcal{B}(\mathcal{O}_E)$, there exists the following disjoint union corresponding to the covering $M = \bigcup_{\alpha\in\Lambda_M} U_\alpha$ such that

1. $A = \bigcup_{\alpha\in\Lambda_M} A_\alpha$ where $\pi(A_\alpha) \subset U_\alpha$, and
2. $A_\alpha \cap A_\beta = \emptyset$ for $\alpha \neq \beta$.

Thus, the measure m_E can be defined as

$$m_E(A) = \sum_{\alpha\in\Lambda_M} (vol \otimes m_F) \circ \phi_{U_\alpha}(A_\alpha). \tag{A3}$$

Notice that the above definition of m_E is independent of the choice of $\{A_\alpha\}_{\alpha\in\Lambda_M}$ such that $A = \bigcup_{\alpha\in\Lambda_M} A_\alpha$ is a disjoint union since m_F is the invariant measure on F for the group transformation of G_F.

Let us introduce the space $\mathcal{M}(E)$ of all the possible *probability Radon measures* for the particle positions on E defined as follows:

1. every $\nu \in \mathcal{M}(E)$ is the linear mapping $\nu : C^\infty(E) \oplus \mathbf{M} \to \mathbf{R}$ such that $\nu(F) < +\infty$ for $F \in C^\infty(E)$, and
2. for every $\nu \in \mathcal{M}(E)$, there exists a σ-additive positive measure P such that

$$\nu(F) = \int_E dP(y)\,(F(y)) \tag{A4}$$

and that $P(M) = 1$, i.e., $\nu(1) = 1$.

For every $\nu \in \mathcal{M}(E)$, the probability density function (PDF) $\rho \in L^1(E, \mathcal{B}(\mathcal{O}_E))$ is the positive-definite, and satisfies

$$\begin{aligned} \nu(F) &= \int_{E=\cup_{\alpha\in\Lambda_M} A_\alpha} dm_E(y)\ \rho(y)\,(F(y)) && \text{(A5)} \\ &= \sum_{\alpha\in\Lambda_M} \int_{\phi_{U_\alpha}(A_\alpha)} dvol(x)\ dm_F(\vartheta)\ \rho\circ\phi^{-1}_{U_\alpha}(x,\vartheta)\left(F\circ\phi^{-1}_{U_\alpha}(x,\vartheta)\right), && \text{(A6)} \end{aligned}$$

where $dP = dm_E \otimes \rho$.

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