Quantum Mechanics As A Limiting Case of Classical Mechanics

Partha Ghose

S. N. Bose National Centre for Basic Sciences

Block JD, Sector III, Salt Lake, Calcutta 700 091, India

In spite of its popularity, it has not been possible to vindicate the conventional wisdom that classical mechanics is a limiting case of quantum mechanics. The purpose of the present paper is to offer an alternative point of view in which quantum mechanics emerges as a limiting case of classical mechanics in which the classical system is decoupled from its environment.

PACS no. 03.65.Bz

I. INTRODUCTION

One of the most puzzling aspects of quantum mechanics is the quantum measurement problem which lies at the heart of all its interpretations. Without a measuring device that functions classically, there are no 'events' in quantum mechanics which postulates that the wave function contains *complete* information of the system concerned and evolves linearly and unitarily in accordance with the Schrödinger equation. The system cannot be said to 'possess' physical properties like position and momentum irrespective of the context in which such properties are measured. The language of quantum mechanics is not that of realism.

According to Bohr the classicality of a measuring device is *fundamental* and cannot be *derived* from quantum theory. In other words, the process of measurement cannot be analyzed within quantum theory itself. A similar conclusion also follows from von Neumann's approach [1]. In both these approaches the border line between what is to be regarded as quantum or classical is, however, arbitrary and mobile. This makes the theory intrinsically ill defined.

Some recent approaches have attempted to *derive* the classical world from a quantum substratum by regarding quantum systems as open. Their interaction with their 'environment' can be shown to lead to effective *decoherence* and the emergence of quasi- classical behaviour [2], [3]. However, the very concepts of a 'system' and its 'environment' already presuppose a clear cut division between them which, as we have remarked, is mobile and ambiguous in quantum mechanics. Moreover, the reduced density matrix of the 'system' evolves to a diagonal form only in the pointer basis and not in the other possible bases one could have chosen. This shows that this approach does not lead to a real solution of the measurement problem, as claimed by Zurek [4], though it is an important development that sheds new light on the emergence of quasi-classical behaviour from a quantum susbstratum.

The de Broglie-Bohm approach [5], on the other hand, does not accept the wave function description as complete. Completeness is achieved by introducing the position of the particle as an additional variable (the so-called 'hidden variable') with an *ontological* status. The wave function at a point is no longer just the probability amplitude that a particle will be *found* there if a measurement were to be made, but the probability amplitude that a particle *is* there even if no measurement is made. It is a realistic description, and measurements are reduced to ordinary interactions and lose their mystique. Also, the classical limit is much better defined in this approach through the 'quantum potential' than in the conventional approach. As a result, however, a new problem is unearthed, namely, it becomes quite clear that classical theory admits ensembles of a more general kind than can be reached from standard quantum ensembles. The two theories are really disparate while having a common domain of application [6].

Thus, although it is tacitly assumed by most physicists that classical physics is a limiting case of quantum theory, it is by no means so. Most physicists would, of course, scoff at the suggestion that the situation may really be the other way round, namely, that quantum mechanics is contained in a certain sense in classical theory. This seems impossible because quantum mechanics includes totally new elements like $\hbar$ and the uncertainty relations and the host of new results that follow from them. Yet, a little reflection shows that if true classical behaviour of a system were really to result from a quantum substratum through some process analogous to 'decoherence', its quantum behaviour ought also to emerge on isolating it sufficiently well from its environment, i.e., by a process which is the 'reverse of decoherence'. In practice, of course, it would be impossible to reverse decoherence once it occurs for a system. Nevertheless, it should still be possible to prepare a system sufficiently well isolated from its environment so that its quantum behaviour can be observed. If this were not possible, it would have been impossible ever to observe the quantum features of any system.

So, let us examine what the opposite point of view implies, namely that classical theory is more fundamental than quantum theory (in a sense to be defined more precisely). This would, in fact, be consistent with Bohr's position that the classicality of measuring devices is fundamental (nonderivable), leading to his preferred solution to the quantum measurement problem. At the same time, the approach of de Broglie and Bohm *coupled with the notion of decoherence as an environmental effect that can be switched on* would fall into place, but the non-realist Copenhagen interpretation would have to be abandoned.

II. THE HAMILTON-JACOBI THEORY

Our starting point is the non-relativistic Hamilton-Jacobi equation

$$\partial S_{cl}/\partial t + \frac{(\nabla S_{cl})^2}{2m} + V(x) = 0 \tag{1}$$

for the action S_{cl} of a classical paticle in an external potential V, together with the definition of the momentum

$$\mathbf{p} = m\,\frac{d\mathbf{x}}{dt} = \nabla S_{cl} \tag{2}$$

and the continuity equation

$$\frac{\partial \rho_{cl}(\mathbf{x}, t)}{\partial t} + \nabla\,.\,(\,\rho_{cl}\,\frac{\nabla\,S_{cl}}{m}) = 0 \tag{3}$$

for the position distribution function $\rho_{cl}(\mathbf{x}, t)$ of the ensemble of trajectories generated by solutions of equation (1) with different initial conditions (position or momentum). Suppose we introduce a complex wave function

$$\psi_{cl}\,(\,\mathbf{x}\,,\,t\,) = R_{cl}\,(\,\mathbf{x}\,,\,t\,)\,exp\,(\,\frac{i}{\hbar}\,S_{cl}) \tag{4}$$

into the formalism by means of the equation

$$\rho_{cl}\,(\,\mathbf{x}\,,\,t\,) = \psi^*_{cl}\,\psi_{cl} = R^2_{cl}\,. \tag{5}$$

What is the equation that this wave function must satisfy such that the fundamental equations (1) and (3) remain unmodified? The answer turns out to be the modified Schrödinger equation [6]

$$i\hbar\,\frac{\partial\psi_{cl}}{\partial t} = \left(-\frac{\hbar^2}{2m}\,\nabla^2 + V(x)\right)\,\psi_{cl} - Q_{cl}\,\psi_{cl} \tag{6}$$

where

$$Q_{cl} = -\frac{\hbar^2}{2m}\,\frac{\nabla^2 R_{cl}}{R_{cl}} \tag{7}$$

Thus, a system can behave classically in spite of it having an associated wave function that satisfies this modified Schrödinger equation.

Notice that the last term in this equation is nonlinear in $|\psi_{cl}|$, and is *uniquely* determined by the requirement that all quantum mechanical effects such as superposition, entanglement and nonlocality be eliminated. It is therefore to be sharply distinguished from certain other types of nonlinear terms that have been considered in constructing nonlinear versions of quantum mechanics [7]. An unacceptable consequence of such nonlinear terms (which are, unlike Q_{cl}, bilinear in the wave function) is that superluminal signalling using quantum entanglement becomes possible in such theories [8]. Since Q_{cl} eliminates quantum superposition and entanglement, it cannot imply any such possibility. Usual action-at-a-distance is, of course, implicit in non-relativistic mechanics, and can be eliminated in a Lorentz invariant version of the theory, as we will see later.

Deterministic nonlinear terms with arbitrary parameters have also been introduced in the Schrödinger equation to bring about collapse of quantum correlations [9] for isolated macroscopic systems. Such terms also imply superluminal signals via quantum entanglement. The term Q_{cl} is different from such terms as well in that it has no arbitrary parameters in it and eliminates quantum correlations for all systems deterministically, irrespective of their size.

Most importantly, it is clear from the above analysis that none of the other types of nonlinearity can guarantee strictly classical behaviour described by equations (1) and (3).

Let us now consider the classical version of the density matrix which must be of the form

$$\rho_{cl}(x, x^{'}, t) = R_{cl}(x,t) exp\left(\frac{i}{\hbar} S_{cl}(x,t)\right) R_{cl}(x^{'},t) exp\left(\frac{i}{\hbar} S_{cl}(x^{'},t)\right) \tag{8}$$

$$= R^2(x,t)\delta^3(x - x^{'}) \tag{9}$$

in order to satisfy the Pauli master equation. The absence of off-diagonal terms is a consequence of the absence of quantum correlations between spatially separated points. This implies that the classical wave function can be written as

$$\psi_{cl}(x,t) = \frac{1}{\sqrt{\pi^3}} \lim_{\epsilon \to 0} \sqrt{\frac{\epsilon}{(x - x(t))^2 + \epsilon^2}}\, exp(\frac{i}{\hbar} S_{cl})\,. \tag{10}$$

Such a function has only point support on the particle trajectory $x = x(t)$ determined by equation (2). It can also be written as a linear superposition of the delta function and its derivatives [11]. All this ensures a classical phase space.

The wave function ψ_{cl} is therefore entirely dispensable and "sterile" as long as we consider strictly classical systems. Conceptually, however, it acquires a special significance in considering the transition between quantum and classical mechanics, as we will see.

The wave function ψ of a quantum mechanical system, on the other hand, must of course satisfy the Schrödinger equation

$$i\hbar \frac{\partial\psi}{\partial t} = -\frac{\hbar^2}{2m}\nabla^2\psi + V\psi\ . \tag{11}$$

Using a polar representation similar to (4) for ψ in this equation and separating the real and imaginary parts, one can now derive the *modified* Hamilton-Jacobi equation

$$\partial S/\partial t + \frac{(\nabla S)^2}{2m} + Q + V = 0 \tag{12}$$

for the phase S of the wave function, where Q is given by

$$Q = -\frac{\hbar^2}{2m}\frac{\nabla^2 R}{R}\,, \tag{13}$$

and the continuity equation

$$\frac{\partial\rho(\mathbf{x},t)}{\partial t} + \nabla\,.\,(\rho\,\frac{\nabla S}{m}) = 0 \tag{14}$$

These differential equations ((12) and (14)) now become coupled differential equations which determine S and $\rho = R^2$. Note that *the phase S of a quantum mechanical system satisfies a modified Hamilton-Jacobi equation with an additional potential Q called the "quantum potential".* Its properties are therefore different from those of the classical action S_{cl} which satisfies equation (1) . Applying the operator ∇ on equation (12) and using the definition of the momentum (2), one obtains the equation of motion

$$\frac{d\mathbf{p}}{dt} = m\,\frac{d^2\,\mathbf{x}}{dt^2} = -\,\nabla\,(\,V + Q) \tag{15}$$

for the quantum particle. Integrating this equation or, equivalently equation (2), one obtains the Bohmian trajectories $x(t)$ of the particle corresponding to different initial positions. The departure from the classical Newtonian equation due to the presence of the "quantum potential" Q gives rise to all the quantum mechanical phenomena such as the existence of discrete stationary states, interference phenomena, nonlocality and so on. This agreement with quantum mechanics is achieved by requiring that the initial distribution P

of the particle is given by $R^2(\,x(t)\,,0\,)$. The continuity equation (14) then guarantees that it will agree with R^2 at all future times. This guarantees that the averages of all dynamical variables of the particle taken over a Gibbs ensemble of its trajectories will always agree with the expectation values of the corresponding hermitian operators in standard quantum mechanics. This is essentially the de Broglie-Bohm quantum theory of motion. For further details about this theory and its relationship with standard quantum mechanics, the reader is referred to the comprehensive book by Holland [6] and the one by Bohm and Hiley [5].

Now, let us for the time being assume that quantum mechanics is the more fundamental theory from which classical mechanics follows in some limit. Consider a quantum mechanical system interacting with its environment. It evolves according to the Schrödinger equation

$$i\,\hbar\,\frac{\partial\psi}{\partial t}=\left(-\frac{\hbar^2}{2m}\,\nabla^2+V(x)+W\right)\,\psi \tag{16}$$

where W is the potential due to the environment experienced by the system. For a complex enough environment such as a heat bath, the density matrix of the system in the position representation quickly evolves to a diagonal form. In a special model in which a particle interacts only with the thermal excitations of a scalar field in the high temperature limit, the density matrix evolves according to the *master equation* [12]

$$\frac{d\rho}{dt}=-\gamma(x-x^{'})(\partial_x-\partial_{x^{'}})\rho-\frac{2m\gamma k_B T}{\hbar^2}(x-x^{'})^2\rho \tag{17}$$

where γ is the relaxation rate, k_B is the Boltzmann constant and T the temperature of the field. It follows from this equation that quantum coherence falls off at large separations as the square of $\Delta x=(x-x^{'})$. The decoherence time scale is given by

$$\tau_D\approx\tau_R\frac{\hbar^2}{2mk_B(\Delta x)^2}=\gamma^{-1}\left(\frac{\lambda_T}{\Delta x}\right)^2 \tag{18}$$

where $\lambda_T = \hbar/\sqrt{2mk_BT}$ is the thermal de Broglie wavelength and $\tau_R = \gamma^{-1}$. For a macroscopic object of mass $m = 1$ g at room temperature ($T = 300K$) and separation $\Delta x = 1$ cm, the ratio $\tau_D/\tau_R = 10^{-40}$! Thus, even if the relaxation time was of the order of the age of the universe, $\tau_R \simeq 10^{17}$ sec, quantum coherence would be destroyed in $\tau_D \simeq 10^{-23}$ sec. For an electron, however, τ_D can be much more than τ_R on atomic and larger scales.

However, the diagonal matrix does not become diagonal in, for example, the momentum representation, showing that coherence has not really been destroyed. The FAPP diagonal density matrix does not therefore represent a *proper* mixture of mutually exclusive alternatives, the classical limit is not really achieved and the measurement problem remains [10].

This is not hard to understand once one realizes that a true classical system must be governed by a Schrödinger equation that is *modified* by the addition of a unique term that is nonlinear in $|\psi|$ (equation (6)), and that *such a nonlinear term cannot arise from unitary Schrödinger evolution.* On the contrary, it is not unnatural to expect a linear equation of the Schrödinger type to be the limiting case of a nonlinear equation like equation (6). It is therefore tempting to interpret the last term in equation (6) as an 'effective' potential that represents the coupling of the classical system to its environment. It is important to bear in mind that in such an interpretation, the potential Q_{cl} must obviously be regarded as *fundamentally given* and *not derivable from a quantum mechanical substratum*, being uniquely and solely determined by the requirement of classicality, as shown above.

Let us now consider a quantum system which is inserted into a thermal bath at time $t = 0$. If it is to evolve into a genuinely classical system after a sufficient lapse of time Δt, its wave function ψ must satisfy the equation of motion

$$i\hbar \frac{\partial\psi}{\partial t} = \left(-\frac{\hbar^2}{2m}\nabla^2 + V(x) - \lambda(t)Q_{cl}\right)\psi \tag{19}$$

where $\lambda(0) = 0$ in the purely quantum limit and $\lambda(\Delta t) = 1$ in the purely classical limit. (Here $\Delta t \gg \tau_D$ where τ_D is typically given by $\gamma^{-1}(\lambda_T/\Delta x)^2$ (18).) Thus, for example, if $= \lambda(t) = 1 - exp(-t/\tau_D)$, a macroscopic system would very rapidly behave like a true classical system at sufficiently high temperatures, whereas a mesoscopic system would behave neither fully like a classical system nor fully like a quantum mechanical system at appropriate temperatures for a much longer time. What happens is that the reduced density operator of the system evolves according to the equation

$$\rho(x, x^{'}, \Delta t) = exp(-i\int_0^{\Delta t} \lambda Q_{cl} dt/\hbar)\rho(x, x^{'}, 0)exp(i\int_0^{\Delta t} \lambda Q_{cl} dt/\hbar) \tag{20}$$

$$= R^2(x, \Delta t)\delta^3(x - x^{'}) \tag{21}$$

during the time interval Δt during which the nonlinear interaction λQ_{cl} completely destroys all superpositions, so that at the end of this time interval the system is fully classical and the equation for the density operator reduces to the Pauli master equation for a classical system.

A variety of functions $\lambda(t)$ would satisfy the requirement $\lambda = 0$ and $\lambda = 1$. This is not surprising and is probably a reflection of the diverse ways in which different systems decohere in different environments.

It is clear that a system must be extremely well isolated ($\lambda = 0$) for it to behave quantum mechanically. Such a system, however, would inherit only a de Broglie-Bohm ontological and causal interpretation, not an interpretation of the Copenhagen type. The practical difficulty is that once a quantum system and its environment get coupled, it becomes FAPP impossible to decouple them in finite time because of the extremely large number of degrees of freedom of the environment. However, we know from experience that it is possible to *create* quantum states in the laboratory that are very well isolated from their environment. Microscopic quantum systems are, of course, routinely created in the laboratory (such as single atoms, single electrons, single photons, etc.,) and considerable effort is being made to create isolated macroscopic systems that would show quantum coherence, and there is already some evidence of the existence of mesoscopic 'cat states' which decohere when appropriate radiation is introduced into the cavity [13].

Equation (19) is a totally new equation that correctly bridges the gap between the quantum and the classical worlds. It should form a sound starting point for studying systems, parametrized by $\lambda(t)$, that lie anywhere in the continuous spectrum stretching between the quantum and classical limits.

Notice that if one defines the momentum by the relation $\pi = \nabla S - \int \nabla Q dt$, the equation of motion can be written in the classical form

$$\frac{d\pi}{dt} = -\nabla V\,. \tag{22}$$

This shows that it is π which is conserved in the absence of any external potential and not the particle momentum p. This is obviously due to the existence of the quantum potential.

A look at the modified Hamilton-Jacobi equation (12) also shows that the quantity conserved by it is not the classical energy but this energy plus the quantum potential. Also notice that the equation of motion (15) implies that a quantum mechanical particle is not free even in the absence of an external potential. It is obvious therefore that the interaction of the corresponding classical system with its environment must serve to cancel this purely quantum force and restore the classical laws of motion. Once the form of the classical Hamilton-Jacobi equation is restored, conservation of energy is mathematically inevitable.

Notice that the additional interaction of a classical system with its environment in the form of the effective potential Q_{cl} becomes manifest only when the Hamilton-Jacobi equation is recast in terms of the classical wave function (equations (6) and (7)). This is why the Hamilton-Jacobi equation can be written without ever knowing about this interaction. The wave function approach reveals what lies hidden and sterile in the traditional classical approach. This is a significant new insight offered by the wave function approach.

It is important to point out a fundamental difference between the two potentials $V(x)$ and Q in (12). $V(x)$ is a given external potential whereas Q is not so—it depends on the modulus of the wave function of the system, and is therefore nonlocal in character.

This leads to a fundamental difference of the approach advocated in this paper from the conventional de Broglie-Bohm theory in which quantum mechanics rather than classical mechanics is regarded as being more fundamental. In the de Broglie-Bohm theory the quantum potential must necessarily vanish in the classical limit, and the quantum system *appears* to behave classically. On the other hand, in the present approach there is no need for the quantum potential to vanish in the classical limit—only its effects must be completely *cancelled* by *nonlinear* environment-induced decoherence of a very special type. Furthermore, besides the wave function, de Broglie and Bohm must also introduce the position of the particle as an additional variable to complete the description of the system. If classical mechanics happens to be more fundamental than quantum mechanics, there is no need to do this as the position and trajectory are already present in the fundamental description. It is , in fact, the wave function that acquires a subsidiary role in this approach. It is interesting that some circumstantial evidence already seems to exist indicating that the position of a quantum system plays a more fundamental role than its wave function [14].

There is therefore a fairly strong case in favour of the possibility that quantum theory might be the limiting case of classical mechanics in which the interaction of the system with its environment (nonlinear in $|\psi|$) is completely switched off. It is difficult to see how such a situation can be accommodated within the standard Copenhagen philosophy. The wave function also acquires a new significance—it is sterile and dispensable in the classical limit but becomes potent and indispensable in the quantum limit.

III. THE KLEIN-GORDON EQUATION

Let the Hamilton-Jacobi equation for free relativistic classical particles be

$$\frac{\partial S_{cl}}{\partial t} + \sqrt{(\partial_i S_{cl})^2\, c^2 + m_0^2\, c^4} = 0\,. \tag{23}$$

Then, using the relation $p_\mu = -\partial_\mu S_{cl} = m_0\, u_\mu$ where $u_\mu = d\,x_\mu/d\,\tau$ with $\tau = \gamma^{-1}\, t$, $\gamma^{-1} = \sqrt{1 - v^2/c^2}, v_i = d\,x_i/d\,t$, the particle equation of motion is postulated to be

$$m_0\, \frac{du_\mu}{d\tau} = 0 = \frac{d\,p_\mu}{d\,\tau}\,. \tag{24}$$

It is quite easy to show that the classical equations (23) and (3) continue to hold if one describes the system in terms of a complex wave function $\psi_{cl} = R_{cl}\, exp\, (\, \frac{i}{\hbar} S_{cl}\,)$ that satisfies the modified Klein-Gordon equation

$$\left(\Box + \frac{m_0^2\, c^2}{\hbar^2} - \frac{Q_{cl}}{\hbar^2} \right) \psi_{cl} = 0 \tag{25}$$

with

$$Q_{cl} = \hbar^2 \frac{\Box R_{cl}}{R_{cl}}\,. \tag{26}$$

As in the non-relativistic case, Q_{cl} may be interpreted as an effective potential in which the system finds itself when described in terms of the wave function ψ_{cl}. If this potential goes to zero in some limit, one obtains the free Klein-Gordon equation which is the quantum limit.

On the other hand, using $\psi = R\, exp\, (\frac{i}{\hbar} S)$ in the Klein-Gordon equation and separating the real and imaginary parts, one obtainds respectively the equation

$$\frac{1}{c^2}\left(\frac{\partial S}{\partial t}\right)^2 - (\partial_i S)^2 - m_0^2\, c^2 - Q = 0 \tag{27}$$

which is equivalent to the modified Hamilton-Jacobi equation

$$\left(\frac{\partial S}{\partial t}\right) + \sqrt{(\partial_i S)^2\ c^2 + m_0^2\, c^4 + c^2\, Q} = 0 \tag{28}$$

and the continuity equation

$$\partial^\mu\, (\, R^2 \partial_\mu\, S\,) = 0\, . \tag{29}$$

One can then identify the four-current as $j_\mu = -R^2 \partial_\mu S$ so that $\rho = j_0 = R^2 E/c$ which is not positive definite because E can be either positive or negative, and therefore, as is well known, it is not possible to interpret it as a probability density.

Nevertheless, let us note in passing that, if use is made of the definition $p_\mu = -\partial_\mu\, S$ of the particle four-momentum, (27) implies

$$p_\mu\, p^\mu = m_0\, c^2 + Q \tag{30}$$

and $p_\mu = M_0\, u_\mu$ where $M_0 = m_0\, \sqrt{1 + Q/m_0^2\, c^2}$. Thus, the quantum potential Q acts on the particles and contributes to their energy-momentum so that they are off their mass-shell. * Applying the operator ∂_μ on equation (27), we get the equation of motion

$$\frac{d\, p_\mu}{d\, \tau} = \frac{\partial_\mu\, Q}{2\, M_0} \tag{31}$$

which has the correct non-relativistic limit. The equation for the acceleration of the particle is therefore given by [6]

---

*The author is grateful to E. C. G. Sudarshan for drawing his attention to this important point.

$$\frac{d\,u_\mu}{d\,\tau} = \frac{1}{2\,m_0^2}\,(\,c^2\,g_{\mu\nu} - u_\mu\,u_\nu\,)\,\partial^\nu\,log\,(1 + \frac{Q}{m_0^2\,c^2}\,)\,. \tag{32}$$

If, on the other hand, one uses the modified Klein-Gordan equation (25) and the corresponding Hamilton-Jacobi equation (23), the particles are on their mass-shell and the free particle classical equation (24) is satisfied.

IV. RELATIVISTIC SPIN 1/2 PARTICLES

Let us now examine the Dirac equation for relativistic spin 1/2 particles,

$$(\,i\hbar\gamma_\mu\partial^\mu + m_0\,c)\,\psi = 0. \tag{33}$$

Let us write the components of the wave function ψ as $\psi^a = R\,\theta^a\,exp\,(\,\frac{i}{\hbar}\,S^a)$, θ^a being a spinor component. It is not straightforward here to separate the real and imaginary parts as in the previous cases. One must therefore follow a different method for relativistic fermions.

It is well known that every component ψ^a of the Dirac wave function satisfies the Klein-Gordan equation. It follows therefore, by putting $\psi^a = R\theta^a\,exp\,(\,i\,S^a/\hbar\,)$, that S^a must satisfy the modified Hamilton-Jacobi equation

$$\partial_\mu\,S^a\,\partial^\mu\,S^a - m_0^2\,c^2 - Q^a = 0\,. \tag{34}$$

where $Q^a = \hbar^2\,\Box\,R\,\theta^a/R\,\theta^a$. Summing over a, we get

$$\sum_a \partial_\mu\,S^a\,\partial^\mu\,S^a - 4\,m_0^2\,c^2 - \sum_a Q^a = 0\,. \tag{35}$$

Defining

$$\partial_\mu\,S\,\partial^\mu\,S = \frac{1}{4}\sum_a \partial_\mu\,S^a\,\partial^\mu\,S^a \tag{36}$$

$$Q = \frac{1}{4}\sum_a Q^a\,, \tag{37}$$

we have

$$\partial_\mu\,S\,\partial^\mu\,S - m_0^2\,c^2 - Q = 0\,. \tag{38}$$

Then, defining the particle four-momentum by $p_\mu = -\partial_\mu S$, one has $p_\mu p^\mu = m_0^2 c^2 + Q$. Therefore, one has the equation of motion

$$\frac{d\,p_\mu}{d\,\tau} = \frac{\partial_\mu Q}{2\,M_0}\,. \tag{39}$$

The Bohmian 3-velocity of these particles is defined by the relation

$$v_i = \gamma^{-1}\,u_i = c\,\frac{u_i}{u_0} = c\,\frac{j_i}{j_0} = c\,\frac{\psi^\dagger\,\alpha_i\,\psi}{\psi^\dagger\,\psi}\,. \tag{40}$$

Then, it follows that

$$u_\mu = \gamma\,v_\mu = \gamma\,c\,\frac{j_\mu}{\rho} \tag{41}$$

where $\rho = \psi^\dagger\,\psi$. This relation is satisfied because $j_\mu\,j^\mu = \rho^2/\gamma^2$ if (40) holds.

As we have seen, for a classical theory of spinless particles, the correct equation for the associated wave function is the modified Klein-Gordon equation (25). Let the corresponding modified wave equation for classical spin 1/2 particles be of the form

$$(\,i\,\hbar\,\gamma_\mu\,D^\mu + m_0\,c\,)\ \psi_{cl} = 0 \tag{42}$$

where $D^\mu = \partial^\mu + (i/\hbar)\,Q^\mu$. Then we have

$$(\,D_\mu\,D^\mu + \frac{m_0^2\,c^2}{\hbar^2}\,)\,\psi^a_{cl} = 0\,. \tag{43}$$

Writing $\psi^a_{cl} = R_{cl}\,\theta^a\,exp\,(\frac{i}{\hbar}\,S^a_{cl})$, one obtains

$$\partial_\mu\,S^a_{cl}\,\partial^\mu\,S^a_{cl} - m_0^2\,c^2 - Q^a_{cl} + Q_\mu\,Q^\mu - 2\,Q_\mu\,\partial^\mu\,S^a_{cl} = 0 \tag{44}$$

where

$$Q^a_{cl} = \frac{\hbar^2\Box R_{cl}\theta^a}{R_{cl}\theta^a}. \tag{45}$$

Define a diagonal matrix $B^{a\,b}_\mu \equiv \partial_\mu\,S^a_{cl}\,\delta^{a\,b}$ such that

$$\frac{1}{2}\,TrB_\mu = \frac{1}{2}\sum_a\,\partial_\mu\,S^a_{cl} \equiv \partial_\mu\,S_{cl}\,. \tag{46}$$

Then

$$\partial_\mu S_{cl} \, \partial^\mu S_{cl} = \frac{1}{4} Tr \, B_\mu \, Tr \, B^\mu = \frac{1}{4} Tr \, (B_\mu \, B^\mu) \tag{47}$$

$$= \frac{1}{4} \sum_a \partial_\mu S^a_{cl} \, \partial^\mu S^a_{cl} \, . \tag{48}$$

Therefore, taking equation (44) and summing over a, we have

$$\partial_\mu S_{cl} \, \partial^\mu S_{cl} - m_0^2 \, c^2 - Q_{cl} + Q_\mu \, Q^\mu - Q_\mu \, \partial^\mu S_{cl} = 0 \tag{49}$$

where

$$Q_{cl} = \frac{1}{4} \sum_a Q^a_{cl} \, . \tag{50}$$

In order that the classical free particle equation is satisfied, the effects of the quantum potential must be cancelled by this additional interaction, and one must have

$$Q_\mu \, (\, Q^\mu - \partial^\mu \, S_{cl} \,) = Q_{cl} \, . \tag{51}$$

A solution is given by

$$p_\mu = -\partial_\mu \, S_{cl} = m_0 \, u_\mu \, , \tag{52}$$

$$Q_\mu = \alpha \, m_0 \, u_\mu \tag{53}$$

with

$$\alpha = \frac{1}{2} \pm \frac{1}{2} \sqrt{1 + 4 \, Q_{cl} / m_0^2 \, c^2} \, . \tag{54}$$

V. RELATIVISTIC SPIN 0 AND SPIN 1 PARTICLES

It has been shown [15] that a consistent relativistic quantum mechanics of spin 0 and spin 1 bosons can be developed using the Kemmer equation [16]

$$(\, i \, \hbar \, \beta_\mu \, \partial^\mu + m_0 \, c \,) \, \psi = 0 \tag{55}$$

where the matrices β satisfy the algebra

$$\beta_\mu\,\beta_\nu\,\beta_\lambda + \beta_\lambda\,\beta_\nu\,\beta\mu = \beta_\mu\,g_{\nu\lambda} + \beta_\lambda\,g_{\nu\mu}\,. \tag{56}$$

The 5×5 dimensional representation of these matrices describes spin 0 bosons and the 10×10 dimensional representation describes spin 1 bosons. Multiplying (55) by β_0, one obtains the Schrödinger form of the equation

$$i\,\hbar\,\frac{\partial\psi}{dt} = [\,-i\,\hbar\,c\,\tilde{\beta}_i\,\partial_i - m_0\,c^2\,\beta_0\,]\,\psi \tag{57}$$

where $\tilde{\beta}_i \equiv \beta_0\,\beta_i - \beta_i\,\beta_0$. Multiplying (55) by $1-\beta_0^2$, one obtains the first class constraint

$$i\,\hbar\,\beta_i\,\beta_0^2\,\partial_i\,\psi = -m_0\,c\,(\,1-\beta_0^2\,)\,\psi. \tag{58}$$

The reader is referred to Ref. [15] for further discussions regarding the significance of this constraint.

If one multiplies equation (57) by $\psi^\dagger$ from the left, its hermitian conjugate by ψ from the right and adds the resultant equations, one obtains the continuity equation

$$\frac{\partial\,(\psi^\dagger\,\psi)}{\partial t} + \partial_i\,\psi^\dagger\,\tilde{\beta}_i\,\psi = 0\,. \tag{59}$$

This can be written in the form

$$\partial^\mu\,\Theta_{\mu0} = 0 \tag{60}$$

where $\Theta_{\mu\nu}$ is the symmetric energy-momentum tensor with $\Theta_{00} = -m_0\,c^2\,\psi^\dagger\,\psi < 0$. Thus, one can define a wavefunction $\phi = \sqrt{m_0\,c^2/E}\,\psi$ (with $E = -\int\,\Theta_{00}\,dV$) such that $\phi^\dagger\,\phi$ is non-negative and normalized and can be interpreted as a probability density. The conserved probability current density is $s_\mu = -\Theta_{\mu0}/E = (\,\phi^\dagger\,\phi, -\phi^\dagger\,\tilde{\beta}_i\,\phi)$ [15].

Notice that according to the equation of motion (57), the velocity operator for massive bosons is $c\,\tilde{\beta}_i$, so that the Bohmian 3-velocity can be defined by

$$v_i = \gamma^{-1}\, u_i = c\, \frac{u_i}{u_0} = c\, \frac{s_i}{s_0} = c\, \frac{\psi^\dagger\, \tilde{\beta}_i\, \psi}{\psi^\dagger\, \psi}\,. \tag{61}$$

Exactly the same procedure can be followed for massive bosons as for massive fermions to determine the quantum potential and the Bohmian trajectories, except that the sum over a has to be carried out only over the independent degrees of freedom (six for ψ and six for $\bar{\psi}$ for spin-1 bosons). The constraint (58) implies the four conditions $\vec{A} = \vec{\nabla} \times \vec{B}$ and $\vec{\nabla}\,.\,\vec{E} = 0$.

The theory of massless spin 0 and spin 1 bosons cannot be obtained simply by taking the limit m_0 going to zero. One has to start with the equation [17]

$$i\,\hbar\,\beta_\mu \partial^\mu\,\psi + m_0\,c\,\Gamma\,\psi = 0 \tag{62}$$

where Γ is a matrix that satisfies the following conditions:

$$\Gamma^2 = \Gamma \tag{63}$$

$$\Gamma\,\beta_\mu + \beta_\mu\,\Gamma = \beta_\mu\,. \tag{64}$$

Multiplying (62) from the left by $1 - \Gamma$, one obtains

$$\beta_\mu\,\partial^\mu\,(\,\Gamma\,\psi\,) = 0\,. \tag{65}$$

Multiplying (62) from the left by $\partial_\lambda\,\beta^\lambda\,\beta^\nu$, one also obtains

$$\partial^\lambda\,\beta_\lambda\,\beta_\nu\,(\,\Gamma\,\psi\,) = \partial_\nu\,(\,\Gamma\,\psi\,)\,. \tag{66}$$

It follows from (65) and (66) that

$$\Box\;(\,\Gamma\,\psi\,) = 0 \tag{67}$$

which shows that $\Gamma\,\psi$ describes massless bosons. The Schrödinger form of the equation

$$i\,\hbar\,\frac{\partial\,(\,\Gamma\,\psi\,)}{dt} = -i\,\hbar\,c\tilde{\beta}_i\,\partial_i\,(\Gamma\,\psi) \tag{68}$$

and the associated first class constraint

$$i\hbar\,\beta_i\,\beta_0^2\,\partial_i\,\psi + m_0\,c\,(\,1 - \beta_0^2\,)\,\Gamma\,\psi = 0 \tag{69}$$

follow by multiplying (62) by β_0 and $1 - \beta_0^2$ respectively. The rest of the arguments are analogous to the massive case. For example, the Bohmian 3-velocity v_i for massless bosons can be defined by equation (61).

Neutral massless spin-1 bosons have a special significance in physics. Their wavefunction is real, and so their charge current $j_\mu = \phi^T\,\beta_\mu\,\phi$ vanishes. However, their probability current density s_μ does not vanish. Furthermore, s_i turns out to be proportional to the Poynting vector, as it should.

Modifications to these equations can be introduced as in the massive case to obtain a classical theory of massless bosons.

VI. THE GRAVITATIONAL FIELD

Exactly the same procedure can also be applied to the gravitational field described by Einstein's equations

$$R_{\mu\nu} - \frac{1}{2}\,g_{\mu\nu}\,R = 0 \tag{70}$$

for the vacuum, where $R_{\mu\nu}$ is the Ricci tensor and R the curvature scalar. In this section, following [18], we will use the signature $-+++$ and the absolute system of units $\hbar = c = 16\,\pi\,G = 1$. The decompostion of the metric is given by [6]

$$\begin{aligned} ds^2 &= g_{\mu\nu}\,dx^\mu\,dx^\nu \\ &= (\,N_i\,N^i - N^2\,)\,dt^2 + 2\,N_i\,dx^i\,dt + g_{ij}\,dx^i\,dx^j \end{aligned} \tag{71}$$

with $g_{i\,j}(\mathbf{x})$, the 3-metric of a 3-surface embedded in space-time, evolving dynamically in superspace, the space of all 3-geometries.

By quantizing the Hamiltonian constraint, one obtains in the standard fashion the Wheeler-DeWitt equation [18]

$$\left[G_{i\,j\,k\,l} \frac{\delta^2}{\delta g_{i\,j}\, \delta g_{k\,l}} + \sqrt{g}\ \ {}^3R \right] \Psi = 0 \tag{72}$$

where $g = \det g_{i\,j}$, ${}^3R$ is the intrinsic curvature, $G_{i\,j\,k\,l}$ is the supermetric, and $\Psi[g_{i\,j}(x)]$ is a wave functional in superspace. Substituting $\Psi = A\ exp\ (i\,S)$, one obtains as usual a conservation law

$$G_{i\,j\,k\,l} \frac{\delta}{\delta g_{i\,j}} \left(A^2 \frac{\delta S}{\delta g_{k\,l}} \right) = 0 \tag{73}$$

and a modified Einstein-Hamilton-Jacobi equation

$$G_{i\,j\,k\,l} \frac{\delta S}{\delta g_{i\,j}} \frac{\delta S}{\delta g_{k\,l}} - \sqrt{g}\ \ {}^3R + Q = 0 \tag{74}$$

where

$$Q = -A^{-1}\, G_{i\,j\,k\,l}\, \delta^2\, A / \delta g_{i\,j}\, \delta g_{k\,l} \tag{75}$$

is the quantum potential. It is invariant under 3-space diffeomorphisms. The causal interpretation of this *field* theory (as distinct from particle mechanics considered earlier) assumes that the universe whose quantum state is governed by equation (72) has a definite 3-geometry at each instant, described by the 3-metric $g_{ij}(\mathbf{x}, t)$ which evolves according to the classical Hamilton-Jacobi equation

$$\frac{\partial g_{i\,j}(\mathbf{x}, t)}{\partial t} = \partial_i N_j + \partial_j N_i + 2\, N\, G_{i\,j\,k\,l} \frac{\delta S}{\delta g_{k\,l}} |_{g_{i\,j}(x) = g_{i\,j}(\mathbf{x}, t)} \tag{76}$$

but with the action S as a phase of the quantum wave functional. This equation can be solved if the initial data $g_{i\,j}(\mathbf{x}, 0)$ are specified. The metric in this field theory clearly corresponds to the position in particle mechanics, equation (76) being its guidance condition.

It is now clear that one can modify the Wheeler-DeWitt equation (72) to the form

$$\left[G_{i\,j\,k\,l} \frac{\delta^2}{\delta g_{i\,j}\, \delta g_{k\,l}} + \sqrt{g}\ \ {}^3R - Q_{cl} \right] \Psi_{cl} = 0 \tag{77}$$

where Q_{cl} is defined by an expression analogous to (75) with A and S replaced by the classical variables A_{cl} and S_{cl}. This leads to the classical Einstein-Hamilton-Jacobi equation

$$G_{ijkl}\frac{\delta S_{cl}}{\delta g_{ij}}\frac{\delta S_{cl}}{\delta g_{kl}} - \sqrt{g}\ {}^{3}R = 0\,. \tag{78}$$

The term Q_{cl} can then be interpreted, as before, as a potential arising due to the coupling of gravitation with other forms of energy. If this coupling could be switched off, quantum gravity effects would become important. The question arises as to whether this can at all be done for gravitation.

VII. CONCLUDING REMARKS

It is usually assumed that a classical system is in some sense a limiting case of a more fundamental quantum substratum, but no general demonstration for ensembles of systems has yet been given. That a quantum system may, on the other hand, be a part of a classical system in which its typical quantum features lie dormant is, however, clear from the above discussions. The part therefore naturally shares the ontology of the total classical system, and *the measurement problem does not even arise.* The nonlocal quantum potential that is responsible for self-organization and the creation of varied stable and metastable quantum structures, becomes active only when the coupling of the part to the whole is switched off. This is a clearly defined physical process that links the classical and quantum domains.

According to this view, therefore, every quantum system is a closed system and every classical system is an open system. The first Newtonian law of motion therefore acquires a new interpretation—the law of inertia holds for a system not when it is isolated from everything else but when it interacts with its environment to an extent that all its quantum aspects are quenched.

Various attempts to show that the classical limit of quantum systems is obtained in certain limits, like large quantum numbers and/or large numbers of constituents, have so far failed [19]. The reason is clear—a linear equation like the Schrödinger equation can never describe a classical system which is described by a *modified* Schrödinger equation with a nonlinear term. This nonlinear term must be generated through some mechanism like the coupling of the system to its environment. There are, of course, other purely *formal* limits too (like $\hbar$ going to zero, for example) in which a closed quantum system reduces to a classical system, as widely discussed in the literature.

It is clear from the usual 'decoherence' approach that the interaction of a quantum system with its environment in the form of some kind of heat bath is necessary to obtain a quasi-classical limit of quantum mechanics. This is usually considered to be a major advance in recent years. Such decoherence effects have already been measured in cavity QED experiments. Decoherence effects are very important to take into account in other critical experiments too, like the use of SQUIDs to demonstrate the existence of Schrödinger cat states. The failure to observe cat states so far in such experiments shows how real these effects are and how difficult it is to eliminate them even for mesoscopic systems. I have taken these advances in our knowledge seriously in a phenomenological sense and tried to incorporate them into a conceptually consistent scheme.

The usual decoherence approach however suffers from the following difficulty: it does neither solve the measurement problem nor does it lead to a truly classical phase space. The two problems seem to be intimately related. The density matrix becomes diagonal only in the coordinate representation. In other words, it does not represent a proper 'statistical mixture'. The use of the linear Schrödinger equation then automatically implies that the momentum space representation is necessarily non-diagonal. This does not happen in the approach advocated in this paper because of equation (19) which guarantees the emergence of classical phase space and a proper 'statistical mixture'. A clear empirical difference must therefore exist between the predictions of the usual decoherence approach and the approach advocated in this paper in the classical limit. It should be possible to test this by suitable experiments which are under consideration. The proposed conceptual frame! work is therefore falsifiable.

VIII. ACKNOWLEDGEMENTS

The basic idea that quantum mechanics may be contained within classical mechanics occurred during a discussion with C. S. Unnikrishnan in the context of an experiment carried out by R. K. Varma and his colleagues and Varma's theoretical explanation of it in terms of the Liouville equation written as a series of Schrödinger-like equations. Unnikrishnan should not, however, be held responsible for any defects of my particular formulation of the idea.

The first version of the non-relativistic part of this paper was presented at the "Werner Heisenberg Colloquium: The Quest for Unity: Perspectives in Physics and Philosophy" organized by the Max Muller Bhavan, New Delhi, the National Institute of Science, Technology and & Development Studies, CSIR, Government of India and the Indian Institute of Advanced Study, Shimla from 4th to 7th August, 1997 at Shimla.

The author is grateful to the Department of Science and Technology, Government of India, for a research grant that enabled this work to be completed.

[1] J. von Neumann, *Mathemetical Foundations of Quantum Mechanics*, translated by R. T. Beyer (Princeton University Press, 1955).

[2] E. Joos and H. D. Zeh, *Z. Phys.* **B59**, 223, 1985.

[3] W. H. Zurek, *Phys. Rev.* **D24**, 1516, 1981; **D26**, 1862, 1982.

[4] W. H. Zurek, *Phys. Today*, October 1991, p. 36.

[5] D. Bohm and B. J. Hiley, *The Undivided Universe* (Routledge & Chapman & Hall, 1993).

[6] P. R. Holland, *Quantum Theory of Motion* (Cambridge University Press, 1993), pp.224-229.

[7] S. Weinberg, *Phys. Rev. Letters* **62**, 485, 1989.

[8] N. Gisin, *Phys. Letters* **A 143**, 1, 1990;
J. Polchinski, *Phys. Rev. Letters* **66**, 397, 1991.

[9] G. C. Ghirardi, A. Rimini and T. Weber, *Phys. Rev.* **D34**, 470, 1986.

[10] See, for example, P. Pearle in *True Collapse and False Collapse* in *The Proceedings of the 4th Drexel Symposium on Quantum Nonintegrability*, eds. Da Hsuan Feng *et al.* (International Press, 1996).

[11] I. M. Gel'fand and G. E. Shilov, *Generalized Functions*, Volume II, Academic Press, New York and London, 1964.

[12] See, for example, ref. 4.

[13] M. Brune, E. Hagley, J. Dreyer, X. Maître, A. Maali, C. Wunderlich, J. M. Raimond and S. Haroche, *Phys. Rev. Letters* **77**, 4887, 1996.

[14] D. Home and R. Chattopadhyaya, *Phys. Rev. Letters* **76**, 2836, 1996.

[15] P. Ghose, *Found. of Physics* **26**, 1441, 1996.

[16] N. Kemmer, *Proc. Roy. Soc. A* **173**, 91, 1939.

[17] Harish-Chandra, *Proc. Roy. Soc. A* **186**, 502, 1946.

[18] J. A. Wheeler in *Battelle Rencontres*, eds. C. M. DeWitt & J. A. Wheeler, pp. 242-307 (Benjamin, New York 1968).

B. S. DeWitt, *Phys. Rev.* **160**, 1113, 1967.

[19] N. D. Mermin, *Phys. Rev. Letters* **65**, 1838 (1990).

S. M. Roy and V. Singh, *Phys. Rev. Letters* **67**, 2761, 1991.

N. Gisin and A. Peres, *Phys. Letters* **A162**, 15, 1992.

P. D. Drummond, *Phys. Rev. Letters* **50**, 1407, 1993.

D. Home and A. S. Majumdar, *Phys. Rev.* **A52**, 4959, 1995.