An identity of Jack polynomials

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Abstract

In this work it is propose an alterative proof of one of basic properties of the zonal polynomials. This identity is generalised for the Jack polynomials.

1 Introduction

Many results in multivariate distribution theory have been obtained using zonal and invariant polynomials. Moreover, these results, in their final version, have been derived in very compact form, using hypergeometric functions with one or two matrix arguments, see Constantine [4], James [14], Chikuse and Davis [3], Davis [5] and Muirhead [17], among many others.

Many of these results obtained in the real case have also been studied with respect to complex, quaternion and octonion cases, see James [14], Li and Xue [16] and Forrester [9], and although many properties of real and complex zonal polynomials have been extended to the quaternion and octonion cases, many others remain unstudied.

In this paper, we are interested in particular in the basic property of real zonal polynomials, examined in James [13, Theorem 5, eq. (27)] (see also James [14, eq. (22)]), and proved by James [13], in terms of group representation theory. This property plays a fundamental role in the context of matrix multivariate elliptical distributions and specifically in that of related noncentral matrix multivariate distributions, such as generalised noncentral Wishart and beta distributions, and also in the context of generalised shape theory, see Díaz-García and González-Farías [7], Díaz-García and Gutiérrez-Jáimez [6] and Caro-Lopera *et al.* [2].

Section 2 proposes an alternative proof of one of the basic properties of zonal polynomials established by James [13, Theorem 5, eq. (27)] (see also James [14, eq. (22)]). The proof is given in terms of the results in Herz [11] and Constantine [4], and as the main result, this property is generalised for real normed division algebras.

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2 Main result

A detailed discussion of real normed division algebras may be found in Baez [1] and Gross and Richards [10], and of Jack polynomials and hypergeometric functions in Sawyer [18], Gross and Richards [10] and Koev and Edelman [15]. For convenience, we shall introduce some notations, although in general we adhere to standard notations.

There are exactly four real finite-dimensional normed division algebras: real numbers, complex numbers, quaternions and octonions, these being denoted generically as $\mathfrak{F}$, see Baez [1]. All division algebras have a real dimension of $1, 2, 4$ or 8, respectively, whose dimension is denoted by β, see Baez [1, Theorems 1, 2 and 3].

Let $\mathcal{L}^{\beta}_{m,n}$ be the linear space of all $n \times m$ matrices of rank $m \leq n$ over $\mathfrak{F}$ with m distinct positive singular values, where $\mathfrak{F}$ denotes a *real finite-dimensional normed division algebra*. Let $\mathfrak{F}^{n\times m}$ be the set of all $n \times m$ matrices over $\mathfrak{F}$, and let $\mathbf{A} \in \mathfrak{F}^{n\times m}$. Then $\mathbf{A}^{*} = \overline{\mathbf{A}}^{T}$ denotes the usual conjugate transpose.

The set of matrices $\mathbf{H}_1 \in \mathfrak{F}^{n\times m}$ such that $\mathbf{H}_1^{*}\mathbf{H}_1 = \mathbf{I}_m$ is a manifold denoted $\mathcal{V}^{\beta}_{m,n}$, termed the *Stiefel manifold*. In particular, $\mathcal{V}^{\beta}_{m,m}$, is the maximal compact subgroup $\mathfrak{U}^{\beta}(m)$ of $\mathcal{L}^{\beta}_{m,m}$ and consists of all matrices $\mathbf{H} \in \mathfrak{F}^{m\times m}$ such that $\mathbf{H}^{*}\mathbf{H} = \mathbf{I}_m$. If $\mathbf{H}_1 \in \mathcal{V}^{\beta}_{m,n}$ then

$$(\mathbf{H}_1^{*} d\mathbf{H}_1) = \bigwedge_{i=1}^{n} \bigwedge_{j=i+1}^{m} \mathbf{h}_j^{*} d\mathbf{h}_i.$$

where $\mathbf{H} = (\mathbf{H}_1|\mathbf{H}_2) = (\mathbf{h}_1, \ldots, \mathbf{h}_m|\mathbf{h}_{m+1}, \ldots, \mathbf{h}_n) \in \mathfrak{U}^{\beta}(m)$. The surface area or volume of the Stiefel manifold $\mathcal{V}^{\beta}_{m,n}$ is

$$\operatorname{Vol}(\mathcal{V}^{\beta}_{m,n}) = \int_{\mathbf{H}_1 \in \mathcal{V}^{\beta}_{m,n}} (\mathbf{H}_1^{*} d\mathbf{H}_1) = \frac{2^{m} \pi^{mn\beta/2}}{\Gamma^{\beta}_{m}[n\beta/2]}, \tag{1}$$

where $\Gamma^{\beta}_{m}[a]$ denotes the multivariate Gamma function for the space of hermitian matrices, see Gross and Richards [10].

Let $C^{\beta}_{\kappa}(\mathbf{B})$ be the Jack polynomials of $\mathbf{B} = \mathbf{B}^{*}$, corresponding to the partition $\kappa = (k_1, \ldots k_m)$ of k, $k_1 \geq \cdots \geq k_m \geq 0$ with $\sum_{i=1}^{m} k_i = k$, see Sawyer [18] and Koev and Edelman [15]. In addition,

$${}_pF^{\beta}_{q}(a_1, \ldots, a_p; b_1, \ldots, b_q; \mathbf{B}) = \sum_{k=0}^{\infty} \sum_{\kappa} \frac{[a_1]^{\beta}_{\kappa}, \ldots, [a_p]^{\beta}_{\kappa}}{[b_1]^{\beta}_{\kappa}, \ldots, [b_p]_{\kappa}}^{\beta} \frac{C^{\beta}_{\kappa}(\mathbf{B})}{k!},$$

defines the hypergeometric function with one matrix argument on the space of hermitian matrices, where $[a]^{\beta}_{\kappa}$ denotes the generalised Pochhammer symbol of weight κ, defined as

$$[a]^{\beta}_{\kappa} = \prod_{i=1}^{m} (a - (i-1)\beta/2)_{k_1}$$

where $\Re(a) > (m-1)\beta/2 - k_m$ and $(a)_i = a(a+1)\cdots(a+i-1)$, see Gross and Richards [10], Koev and Edelman [15] and Díaz-García [8].

Now, we clarify an apparent discrepancy between the results obtained by the different approaches. From Muirhead [17, Lemma 9.5.3, p. 397], it is easy to see that equality (3.5'), proved via a Laplace transform by Herz [11, p. 494], and equality (27) in James [14], proved via group representation theory by James [13, Theorem 5], coincide.

Then, from James [13, eq. (27)] (see also James [14, eq. (22)]) we have the following.

Lemma 2.1. *If* $\mathbf{X} \in \mathfrak{L}^1_{n,m}$*, then*

$$\int_{\mathbf{H}_1 \in \mathcal{V}^1_{m,n}} (\operatorname{tr}(\mathbf{X}\mathbf{H}_1))^{2k} (d\mathbf{H}_1) = \sum_{\kappa} \frac{\left(\frac{1}{2}\right)_k}{[n/2]^1_\kappa} C^1_\kappa(\mathbf{X}\mathbf{X}^*). \tag{2}$$

Proof. From Herz [11, eq. (3.5'), p. 494], and expanding in series of powers

$$\begin{aligned} {}_0F^1_1(n/2, \mathbf{X}\mathbf{X}^*/4) &= \int_{\mathbf{H}_1 \in \mathcal{V}^1_{m,n}} \operatorname{etr}\{\mathbf{X}\mathbf{H}_1\}(d\mathbf{H}_1) \\ &= \sum_{k=0}^{\infty} \frac{1}{k!} \int_{\mathbf{H} \in \mathcal{V}^1_{m,n}} (\operatorname{tr}(\mathbf{X}\mathbf{H}_1))^k (d\mathbf{H}_1). \end{aligned}$$

Recalling that if one or more parts $k_1, \dots, k_m$ of partition k is odd, then

$$\int_{\mathbf{H} \in \mathcal{V}^1_{m,n}} (\operatorname{tr}(\mathbf{X}\mathbf{H}_1))^k (d\mathbf{H}_1) = 0,$$

see James [12] and James [14]. Therefore

$${}_0F^1_1(n/2, \mathbf{X}\mathbf{X}^*/4) = \sum_{k=0}^{\infty} \frac{1}{(2k)!} \int_{\mathbf{H} \in \mathcal{V}^1_{m,n}} (\operatorname{tr}(\mathbf{X}\mathbf{H}_1))^{2k} (d\mathbf{H}_1). \tag{3}$$

Now, by the definition of hypergeometric functions with one matrix argument in terms of zonal polynomials, we have (see Constantine [4]),

$${}_0F^1_1(n/2, \mathbf{X}\mathbf{X}^*/4) = \sum_{k=0}^{\infty} \sum_{\kappa} \frac{1}{[n/2]^1_\kappa} \frac{C^1_\kappa(\mathbf{X}\mathbf{X}^*/4)}{k!}. \tag{4}$$

Then, equalling and comparing term-by-term the series on the right side of (3) and (4) we obtain

$$\sum_{\kappa} \frac{1}{[n/2]^1_\kappa} \frac{C^1_\kappa(\mathbf{X}\mathbf{X}^*/4)}{k!} = \frac{1}{(2k)!} \int_{\mathbf{H} \in \mathcal{V}^1_{m,n}} (\operatorname{tr}(\mathbf{X}\mathbf{H}_1))^{2k} (d\mathbf{H}_1).$$

Finally, observing that $4^k(1/2)_k/(2k)! = 1/k!$ and that $C^1_\kappa(a\mathbf{B}) = a^k C^1_\kappa(\mathbf{B})$, the desired result is obtained. □ □

Property (2) was also proved in an alternative way by Takemura [19, Lemma 1, p. 40], for the real case. Now, under our approach, property (2) is easily extended to the Jack polynomial case for real normed division algebras.

Theorem 2.1. *Let* $\mathbf{X} \in \mathfrak{L}^\beta_{n,m}$*, then*

$$\int_{\mathbf{H}_1 \in \mathcal{V}^\beta_{m,n}} (\operatorname{tr}(\mathbf{X}\mathbf{H}_1))^{2k} (d\mathbf{H}_1) = \sum_{\kappa} \frac{\left(\frac{1}{2}\right)_k}{[\beta n/2]^\beta_\kappa} C^\beta_\kappa(\mathbf{X}\mathbf{X}^*). \tag{5}$$

Proof. Observe that by Gross and Richards [10] and Koev and Edelman [15],

$${}_0F^\beta_1(\beta n/2, \mathbf{X}\mathbf{X}^*/4) = \sum_{k=0}^{\infty} \sum_{\kappa} \frac{1}{[\beta n/2]^\beta_\kappa} \frac{C^\beta_\kappa(\mathbf{X}\mathbf{X}^*/4)}{k!},$$

and by Díaz-García [8],

$${}_0F^\beta_1(\beta n/2, \mathbf{X}\mathbf{X}^*/4) = \int_{\mathbf{H}_1 \in \mathcal{V}^\beta_{m,n}} \operatorname{etr}\{\mathbf{X}\mathbf{H}_1\}(d\mathbf{H}_1),$$

whose equality was found by James [14], for the complex case, and by Li and Xue [16], for the quaternion case. Then, the desired result is obtained following the proof of Lemma 2.1. □ □

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References

[1] J. C. Baez, The octonions, Bulletin of the American Mathematical Society 39(2002) 145–205.

[2] F. J. Caro-Lopera, J. A. Díaz-García, G. González-Farías, Noncentral elliptical configuration density, Journal of Multivariate Analysis 101(1) (2009) 32–43.

[3] T. Chikuse, A. W. Davis, Invariant polynomials with two matrix arguments. Extending the zonal polynomials: Applications to multivariate distribution theory, Annals of the Institute of Statistical Mathematics Part A 31(1979) 465–485.

[4] A. C. Constantine, Noncentral distribution problems in multivariate analysis, The Annals of Mathematical Statistics 34(1963) 1270–1285.

[5] A. W. Davis, Invariant polynomials with two matrix arguments, extending the zonal polynomials, In: Krishnaiah P R (ed.) Multivariate Analysis V, North-Holland Publishing Company, 287–299, 1980.

[6] J. A. Díaz-García, R. Gutiérrez-Jáimez, Wishart and Pseudo-Wishart distributions under elliptical laws and related distributions in the shape theory context, Journal of Statistical Planning and Inference 136(12)(2006) 4176–4193.

[7] J. A. Díaz-García, G. González-Farías, Singular Random Matrix decompositions: Distributions, Journal of Multivariate Analysis 94(1)(2005) 109–122.

[8] J. A. Díaz-García, Special functions: Integral properties of Jack polynomials, hypergeometric functions and Invariant polynomials, http://arxiv.org/abs/0909.1988, 2009. Also submitted.

[9] P. J. Forrester, Log-gases and random matrices, To appear. (Available in: `http://www.ms.unimelb.edu.au/~matpjf/matpjf.html`, 2009.

[10] K. I. Gross, D. ST. P. Richards, Special functions of matrix argument I: Algebraic induction zonal polynomials and hypergeometric functions, Transactions of the American Mathematical Society 301(2)(1987) 475–501.

[11] C. S. Herz, Bessel functions of matrix argument, Annals of Mathematics 61(3)(1955) 474-523.

[12] A. T. James, The distribution of noncentral means with known covariance, The Annals of Mathematical Statistics 32(3)(1954) 874–882.

[13] A. T. James, Zonal polynomials of the real positive definite symmetric matrices, Annals of Mathematics 35(1961) 456–469.

[14] A. T. James, Distribution of matrix variate and latent roots derived from normal samples, The Annals of Mathematical Statistics 35(1964) 475–501.

[15] P. Koev, A. Edelman, The efficient evaluation of the hypergeometric function of a matrix argument, Mathematics of Computation 75(2006) 833–846.

[16] F. Li, Y. Xue, Zonal polynomials and hypergeometric functions of quaternion matrix argument, Communications in Statistics, Theory Methods 38(8)(2009) 1184-1206.

[17] R. J. Muirhead, Aspects of Multivariate Statistical Theory, John Wiley & Sons, New York, 1982.

[18] P. Sawyer, Spherical Functions on Symmetric Cones, Transactions of the American Mathematical Society 349(1997) 3569–3584.

[19] A. Takemura, Zonal polynomials, Lecture Notes-Monograph Series, 4, Institute of Mathematical Statistics, Hayward, California, 1984.