Positive Maps Which Are Not Completely Positive

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The concept of the *half density matrix* is proposed. It unifies the quantum states which are described by density matrices and physical processes which are described by completely positive maps. With the help of the half-density-matrix representation of Hermitian linear map, we show that every positive map which is not completely positive is a *difference* of two completely positive maps. A necessary and sufficient condition for a positive map which is not completely positive is also presented, which is illustrated by some examples.

Entanglement has become one of the central concept in quantum mechanics, specially in quantum information. A quantum state of a bipartite system is *entangled* if it cannot be prepared locally or it cannot be expressed as a convex combination of direct product states of two subsystems. This kind of state is also called *inseparable*. Though easily defined, it is very hard to recognize the inseparability of a mixed state of a bipartite system.

An operational-friendly criterion of separability was proposed by Peres [1]. This criterion is based on the observation that the partial transposition of a separable density matrix remains positive semidefinite. That the partial transposition of a density matrix is *not* positive semidefinite infers the inseparability of the density matrix. This provides a necessary condition for the separability. There exist entangled states with positive partial transposition, which exhibit bound entanglement [2]. Examples of such kind were first provided in Ref. [3] and then constructed in Ref. [4] systematically with the help of unextendible product basis.

Later on, by noticing that the transposition is a *positive map* (to be described later in details), a necessary and sufficient condition of the separability was proposed in Ref. [5]: A bipartite state is separable iff it is still positive semidefinite under all positive maps acting on a subsystem. In other words, a density matrix of a bipartite system is inseparable iff there exists a positive map acting on a subsystem such that the image of the density matrix is not positive semidefinite. Hence the inseparability can be recognized by positive maps which are not *completely positive*.

Completely positive maps, which are able to describe the most general physical process [6], are better understood than positive maps which are not completely positive. Positive maps from Hilbert space $\mathcal{H}_2$ (two-dimensional) to $\mathcal{H}_2$ or $\mathcal{H}_3$ are all decomposable [7], which are characterized by transposition and completely positive maps only. As a result in the cases of $\mathcal{H}_2 \times \mathcal{H}_2$ and $\mathcal{H}_2 \times \mathcal{H}_3$ the transposition criterion is also a sufficient condition for separability [5]. Therefore further understandings of positive maps which are not completely positive will facilitate the recognition and classification of the inseparable mixed states.

As a direct calculation will show, under an orthonormal and complete basis $\{|n\rangle\}_{n=0}^{L-1}$ the transposition of an $L \times L$ matrix

ρ can be expressed as

$$\rho^T = \mathrm{Tr}\rho - \sum_{m,n=0}^{L-1} \sigma_{mn}\, \rho\, \sigma_{mn}^\dagger, \qquad (1)$$

where $\sigma_{mn} = (|m\rangle\langle n| - |n\rangle\langle m|)/\sqrt{2}$. We see immediately that the transposition is a difference of two completely positive maps. And this statement will be proved to hold true for all positive maps which are not completely positive, which will be also characterized by a necessary and sufficient condition in this Letter.

For this purpose we shall first develop an extremely useful tool — *half density matrix* that unifies the description of the quantum states and physical processes. And then we derive a half-density-matrix representation of an arbitrary Hermitian linear map from which our main results are obtained. Along with the introduction of the concept of half density matrix, its relations to the ensembles and the purifications of mixed states are clarified and its applications in the field of quantum information such as quantum teleportation [8] are also presented.

Normally, quantum states, pure or mixed, are described by density matrices, positive semidefinite operators (whose eigenvalues are all nonnegative) on the Hilbert space of the system. Because of its property of positive semidefinite the density matrix ρ can always be written as $\rho = TT^\dagger$ where matrix T is called here as the *half density matrix* (HDM) for a quantum state.

Obviously, the half density matrix for a given density matrix is not unique. For example TU and T are corresponding to the same mixed state $\rho = TT^\dagger$ whenever U is unitary. Generally, the half density matrix T for a mixed state ρ of an s-level system is an $s \times L$ rectangular matrix with $L \geq r = \mathrm{Rank}(\rho)$, i.e., a linear map from an L-dimensional Hilbert space $\mathcal{H}_L$ to an s-dimensional Hilbert space $\mathcal{H}_s$. The rank r of the density matrix equals to the rank of the half density matrix T and $r = 1$ for pure state.

Under an orthonormal and complete bases $\{|m\rangle\}_{m=0}^{s-1}$ and $\{|n\rangle\}_{n=0}^{L-1}$ of Hilbert spaces $\mathcal{H}_s$ and $\mathcal{H}_L$, a typical half density matrix of dimension $s \times L$ can be constructed as $T_e = V^\dagger(\Delta_s, 0_{s\times(L-s)})$, where Δ_s is a diagonal $s \times s$ matrix formed by all the square roots of the eigenvalues of ρ (the singular numbers of T_e) and V is an $s \times s$ unitary matrix diagonalizing the density matrix ρ. Obviously we have $\rho = T_e T_e^\dagger$. As a direct result of the *singular number decomposition* of an arbitrary matrix [9] we have the following

Lemma: Given a density matrix ρ of an s-level system, an $s \times L$ matrix T is a half density matrix for ρ, i.e., $\rho = TT^\dagger$, if and only if there exists an $L \times L$ unitary matrix U such that $T = T_e U$.

When written explicitly in the established basis, the relation $\rho = T_e T_e^\dagger$ results in exactly an ensemble formed by all the eigenvectors $V^\dagger|m\rangle$ of the mixed state, which is referred to as *eigen-ensemble* here. In this way every half density matrix T of a mixed state ρ corresponds to an ensemble of the mixed state. The above *Lemma* tells us that every ensemble of a given mixed state is related to the eigen-ensemble by a unitary matrix which has been proved by other means [10]. Therefore the half density matrix of a density matrix is physically equivalent to an ensemble of the corresponding mixed state.

Every mixed state ρ admits a *purification* [11], a pure state $|\phi\rangle$ of a bipartite system including this system as a subsystem such that $\rho = \text{Tr}_2|\phi\rangle\langle\phi|$. Under the established basis, a general pure state in $\mathcal{H}_s \otimes \mathcal{H}_L$ is

$$|\phi\rangle = \sum_{m=0}^{s-1}\sum_{n=0}^{L-1} C_{mn}|m\rangle_1|n\rangle_2 := T|\Phi_L\rangle. \qquad (2)$$

Here pure state $|\Phi_L\rangle = \sum_{n=0}^{L-1}|n\rangle_1|n\rangle_2$ lives in Hilbert space $\mathcal{H}_L \otimes \mathcal{H}_L$ and T is a linear map from $\mathcal{H}_L$ to $\mathcal{H}_s$ acting on the first L-dimensional Hilbert space $\mathcal{H}_L$. Under the given bases linear map T is represented by an $s \times L$ rectangular matrix with matrix elements given by $\langle m|T|n\rangle = C_{mn}$. When the pure state $|\phi\rangle$ is normalized we have $\text{Tr}(T^\dagger T) = 1$. Alternatively, we also have $|\phi\rangle = T^T|\Phi_s\rangle$ with state $|\Phi_s\rangle$ defined in $\mathcal{H}_s \otimes \mathcal{H}_s$ similar to state $|\Phi_L\rangle$. The linear map $T^T : \mathcal{H}_s \mapsto \mathcal{H}_L$ acts on the second $\mathcal{H}_s$ and it is represented by the transposition of T under the established basis.

Tracing out the second system we obtain a reduced density matrix of the first subsystem $\rho_s = TT^\dagger$ and similarly $\rho_L = T^T T^*$ for the second subsystem. That is to say T is the HDM for the reduced half density matrix $\rho_s = \text{Tr}_L|\phi\rangle\langle\phi|$ of the first subsystem and its transposition T^T for $\rho_L = \text{Tr}_s|\phi\rangle\langle\phi|$. Thus a one-to-one correspondence between a normalized pure state $|\phi\rangle$ of a bipartite system, a purification, and a linear map T satisfying $\text{Tr}(T^\dagger T) = 1$, a half density matrix, is established. Therefore a half density matrix T is also equivalent to a purification of the mixed state. The linear map T is also referred to as the half density matrix of a bipartite pure state, which is unique by definition. If $s = L$ the polar decomposition of T will result in the useful Schmit-decomposition.

The pure bipartite state is separable iff the rank of its half density matrix is one. For a pure product state $|v\rangle_s|w\rangle_L$ the half density matrix is $|v\rangle\langle w^*|$ where $|w^*\rangle$ is the *index state* of state $|w\rangle$ defined by $|w^*\rangle = \langle w|\Phi_L\rangle$ [11]. For later use we define a *mirror operator* $M_L = |\Phi_L\rangle\langle\Phi_L|$ in the Hilbert space $\mathcal{H}_L \otimes \mathcal{H}_L$, which has the property $\langle w^*|M_L|w^*\rangle = |w\rangle\langle w|$. The partial transposition of the mirror operator $X = M_L^{T_1}$ is in fact the exchanging (or swapping) operator introduced by Werner [12] (denoted as V there).

As an application, we consider a state $|\phi\rangle_{12}|\psi\rangle_3$ of a tripartite system with all three subsystems 1,2 and 3 being s-level systems. Let T_ϕ denote the HDM of the bipartite state

$|\phi\rangle_{12}$ and $|k;l\rangle_{23} = T_{kl}|\Phi_s\rangle_{23}$ denote an orthonormal complete basis for systems 2 and 3 with HDMs T_{kl} satisfying $\text{Tr}T_{kl}T_{k'l'}^\dagger = \delta_{kk'}\delta_{ll'}$ for orthogonality and $\sum_{kl} T_{kl}\mathcal{O}T_{kl}^\dagger = \text{Tr}\mathcal{O}$ for completeness. We then have expansion

$$|\phi\rangle_{12}|\psi\rangle_3 = \sum_{k,l=0}^{s-1} T_\phi\, T_{kl}^*\, |\psi\rangle_1|k;l\rangle_{23}. \qquad (3)$$

This describes exactly a quantum teleportation of an unknown quantum state $|\psi\rangle$ from system 3 to system 1 when both T_ϕ and T_{kl} are unitary or state $|\phi\rangle_{12}$ and basis $|k;l\rangle_{23}$ are maximally entangled states [13].

The mixed state ρ_{sL} of an $(s \times L)$ bipartite system can also be equivalently and conveniently characterized by HDMs of pure bipartite states. Let $\{|\phi_i\rangle, p_i\}_{i=1}^R$ be an ensembles of ρ_{sL} we have

$$\rho_{sL} = \sum_{i=1}^{R} p_i|\phi_i\rangle\langle\phi_i| = \sum_{i=1}^{R} A_i M_L A_i^\dagger, \qquad (4)$$

where we have denoted A_i as the half density matrix of the pure state $\sqrt{p_i}|\phi_i\rangle$, i.e., $\sqrt{p_i}|\phi_i\rangle = A_i|\Phi_L\rangle$. Obviously HDMs defined by $\tilde{A}_i = \sum_j U_{ij} A_j$ characterize the same density matrix whenever U is a unitary matrix. And from the *Lemma* we know that given a density matrix this is the only freedom that the half density matrices can have.

The density matrix expressed in the form as in Eq.(4) can be easily manipulated by local operations. For example the density matrix under operation $U_s \otimes U_L^*$ is transformed to density matrix specified by half density matrices $U_s A_i U_L^\dagger$. The tilde operation $\rho \mapsto \tilde{\rho}$ introduced in Ref. [14] to obtain explicitly the entanglement of formation of two-qubit is simply an anti-linear transformation $\tilde{A}_i = \text{Tr}A_i^\dagger - A_i^\dagger$.

In the discussions above we have defined the half density matrices for the states of a single system, for pure bipartite states, and for mixed bipartite states. The physical processes can also be characterized by half density matrices. A general physical process which can include unitary evolutions, tracing out one system, and general measurements is described by *trace-preserving completely positive maps* [6, 11], which is a special kind of Hermitian linear map.

A Hermitian linear map sends linearly Hermitian operators to Hermitian operators that may live in different Hilbert spaces. Let $\mathcal{L}$ denote a general Hermitian linear map from Hilbert space $\mathcal{H}_L$ to $\mathcal{H}_s$. Because the map $\mathcal{L}$ is linear the map $\mathcal{L} \otimes \mathcal{I}_L$ is also a Hermitian linear map from $\mathcal{H}_L \otimes \mathcal{H}_L$ to $\mathcal{H}_s \otimes \mathcal{H}_L$, where $\mathcal{I}_L$ denotes the identity map on $\mathcal{H}_L$. Recalling that the mirror operator $M_L = |\Phi_L\rangle\langle\Phi_L|$ is defined on $\mathcal{H}_L \otimes \mathcal{H}_L$, its image

$$H_{sL} = \mathcal{L} \otimes \mathcal{I}_L(M_L) \qquad (5)$$

is therefore a Hermitian operator in $\mathcal{H}_s \otimes \mathcal{H}_L$. Let $|\psi_i^+\rangle = A_i|\Phi_L\rangle$ ($i \le i_+$) denote the eigenvectors corresponding to the positive eigenvalues of H_{sL} and $|\psi_i^-\rangle = B_i|\Phi_L\rangle$ ($i \le i_-$) to negative eigenvalues of H_{sL}, where $i_\pm$ is the number of the

positive/negative eigenvalues of H_{sL}. We then have

$$H_{sL} = \sum_{i=1}^{i_+} A_i M_L A_i^\dagger - \sum_{i=1}^{i_-} B_i M_L B_i^\dagger, \tag{6}$$

in which the norms of the eigenvectors $|\psi_i^\pm\rangle$ have been taken to be the absolute values of corresponding eigenvalues. Because the eigenvectors corresponding to different eigenvalues are orthonormal we have $\text{Tr}(A_i B_j^\dagger) = 0$ for all i and j. In this sense two families of half density matrices $\{A_i\}$ and $\{B_i\}$ are *orthogonal* to each other.

For a pure state $P_w = |w\rangle\langle w|$ in the Hilbert space $\mathcal{H}_L$ we have $P_w = \langle w^*|M_L|w^*\rangle$ where $|w^*\rangle$ is the index state of $|w\rangle$. As a result we have $\mathcal{L}(P_w) = \langle w^*|H_{sL}|w^*\rangle$. Taking into account of the linearity of the Hermitian map $\mathcal{L}$, we finally obtain

$$\mathcal{L}(H) = \sum_{i=1}^{i_+} A_i H A_i^\dagger - \sum_{i=1}^{i_-} B_i H B_i^\dagger, \tag{7}$$

where H is an arbitrary Hermitian matrix in $\mathcal{H}_L$. This is called the *half-density-matrix representation* of a Hermitian linear map. As one result we have

$$\langle \Phi_s | \mathcal{I}_s \otimes \mathcal{L}(\Sigma_{sL}) | \Phi_s \rangle = \text{Tr}(H_{sL} \Sigma_{sL}^T) \tag{8}$$

for an arbitrary Hermitian matrix Σ_{sL} in $\mathcal{H}_s \otimes \mathcal{H}_L$. As another consequence, a one-to-one correspondence between the Hermitian maps $\mathcal{L} : \mathcal{H}_L \mapsto \mathcal{H}_s$ and Hermitian matrix H_{sL} (an observable) in $\mathcal{H}_s \otimes \mathcal{H}_L$ can be established

$$\mathcal{L}(H) = \text{Tr}_L(H_{sL} H^T) \tag{9}$$

in addition to Eq.(5).

The HDM representation of Hermitian linear map is not unique. Suppose two integers $M \geq i_+$ and $N \geq i_-$ and let $SU(M, N)$ denote the pseudo-unitary group formed by $(M + N) \times (M + N)$ matrices satisfying $S\eta S^\dagger = \eta$ where $\eta = I_M \oplus (-I_N)$ and $I_{M(N)}$ is the $M \times M$ ($N \times N$) identity matrix. If we define a family of HDMs $\{T_i\}_{i=1}^{M+N}$ as $T_i = A_i$ ($1 \leq i \leq i_+$), $T_i = B_i$ ($M + 1 \leq i \leq M + i_-$) and $T_i = 0$ otherwise and take an arbitrary element S of $SU(M, N)$, a new family of HDMs $\{\tilde{T}_i\}_{i=1}^{M+N}$ defined by $\tilde{T}_i = \sum_j S_{ij} T_j$ represents the same Hermitian linear map

$$\mathcal{L}(H) = \sum_{i=1}^{M} \tilde{T}_i H \tilde{T}_i^\dagger - \sum_{j=1}^{N} \tilde{T}_j H \tilde{T}_j^\dagger. \tag{10}$$

A positive map is a special Hermitian linear map which maps any positive semidefinite operator to a positive semidefinite operator. A Hermitian linear map $\mathcal{S} : \mathcal{H}_L \mapsto \mathcal{H}_s$ is positive if and only if $\text{Tr}(\mathcal{S}(Q_L) P_s) = \text{Tr}(H_{sL}^T P_s \otimes Q_L) \geq 0$ for all pure product state $P_s \otimes Q_L$ where $H_{sL} = \mathcal{S} \otimes \mathcal{I}_L(M_L)$. In the following $\mathcal{S}$ is always a positive map.

A completely positive (CP) map is a positive map which keeps its positivity when the system it acts on is embedded as a subsystem in an arbitrary larger system. That is, for a CP map $\mathcal{S} : \mathcal{H}_L \mapsto \mathcal{H}_s$ and an arbitrary positive integer k the induced map $\mathcal{S} \otimes \mathcal{I}_k$ from $\mathcal{H}_L \otimes \mathcal{H}_k$ to $\mathcal{H}_s \otimes \mathcal{H}_k$ is positive.

However it is enough to check whether the image $H_{sL} = \mathcal{S} \otimes \mathcal{I}_L(M_L)$ of the mirror operator M_L is positive semidefinite or not. If it is positive semidefinite, then the negative part in the HDM representation Eq.(7) disappears, which yields exactly the *operator-sum representation* of a CP map [11]

$$\mathcal{S}(\rho) = \sum_{i=1}^{i_+} A_i \rho A_i^\dagger. \tag{11}$$

If the trace is preserved, we have further $\sum_i A_i^\dagger A_i = 1$. Therefore the operator-sum representation of a CP map can also be referred to as a *half-density-matrix representation*. Especially, if H_{sL} equals to the identity matrix $I_s \otimes I_L$, the corresponding CP map is simply the trace operation $\mathcal{S}_T(\rho) = I_s \text{Tr}\rho$.

A positive map which is not completely positive (non-CP) is nonetheless a Hermitian map so that it has a HDM representation as Eq.(7), from which we obtain $\mathcal{S} = \mathcal{S}_A - \mathcal{S}_B$ where two CP maps $\mathcal{S}_{A,B}$ are represented by HDMs $\{A_i\}$ and $\{B_i\}$ respectively. Two CP maps $\mathcal{S}_{A,B}$ are said to be orthogonal if their HDMs are orthogonal to each other, i.e., $\text{Tr}(A_i B_j^\dagger) = 0$ for all i, j. We see that H_{sL} can not be positive semidefinite.

Conversely, if the Hermitian matrix H_{sL} has at least one negative eigenvalue then it determines a non-CP positive map. Let $|\psi\rangle$ denote an eigenvector corresponding to one of the negative eigenvalues of H_{sL} and $P_\psi = |\psi\rangle\langle\psi|$. From identity (8) we see immediately that $\mathcal{I}_s \otimes \mathcal{S}(P_\psi^T)$ is not positive semidefinite, i.e., map $\mathcal{S}$ is not completely positive. We note that the eigenspace corresponding to the negative eigenvalues of H_{sL} contains no product state because of positivity. To summarize, we have the following

Theorem: Every positive map which is not completely positive is a difference of two orthogonal completely positive maps; A Hermitian linear map $\mathcal{S} : \mathcal{H}_L \mapsto \mathcal{H}_s$ is positive but not completely positive if and only if for all pure product state $P_s \otimes Q_L$ in $\mathcal{H}_L \otimes \mathcal{H}_s$ we have $\text{Tr}(H_{sL} P_s \otimes Q_L) \geq 0$ while $H_{sL} = \mathcal{S} \otimes \mathcal{I}_L(M_L)$ is *not* positive semidefinite.

This theorem provides an obvious way to construct a non-CP positive map form $\mathcal{H}_L$ to $\mathcal{H}_s$. First, we choose a proper Hermitian matrix H_{sL} in $\mathcal{H}_s \otimes \mathcal{H}_L$ satisfying the conditions specified in the above theorem. Then a non-CP positive map $\mathcal{S} : \mathcal{H}_L \mapsto \mathcal{H}_s$ is determined by $\mathcal{S}(\rho_L) = \text{Tr}_L(H_{sL} \rho_L^T)$.

As the first example we consider the the exchanging operator defined in $\mathcal{H}_L \otimes \mathcal{H}_L$ by $X = M_L^{T_1}$ or explicitly

$$X = \sum_{m,n=0}^{L-1} |m, n\rangle\langle n, m|. \tag{12}$$

The exchanging operator X has two eigenvalues ± 1 and $\sigma_{mn}|\Phi_L\rangle$ ($m > n$) are the eigenvectors corresponding to eigenvalue -1. Therefore X is not positive semidefinite and for any pure product states $|pp\rangle = |v\rangle|w\rangle$ we have $\langle pp|X|pp\rangle = |\langle v|w\rangle|^2 \geq 0$ as specified by the above theorem. In fact the resulting non-CP positive map on $\mathcal{H}_L$ is exactly the transposition $\rho^T = \text{Tr}_2(X\rho^T)$. By writing X in its diagonal

form we obtain $\rho^T = \mathcal{S}_T(\rho) - \mathcal{S}_\sigma(\rho)$, where the CP map $\mathcal{S}_\sigma$ is represented by HDMs $\{\sigma_{mn}\}$ and $\mathcal{S}_T$ is the trace operation.

As the second example we consider a Hermitian matrix in $\mathcal{H}_L \otimes \mathcal{H}_L$ defined by $H_R = I_L \otimes I_L - M_L$. It is not positive semidefinite because $\langle \Phi_L | H_R | \Phi_L \rangle < 0$ and for every product states $|pp\rangle$ we have $\langle pp | H_R | pp \rangle = 1 - |\langle v | w^* \rangle|^2 \geq 0$. Accordingly, a non-CP positive map is defined on $\mathcal{H}_L$ as $\Lambda(\rho) = \text{Tr}\rho - \rho$, which provides the *reduction criterion* [15, 16]: Every inseparable state in $\mathcal{H}_L \otimes \mathcal{H}_L$ which loses its positivity under map $\mathcal{I}_L \otimes \Lambda$ is distillable and in the distillation procedure provided in Ref. [15] the HDM of pure bipartite state serves as the filtering operation. Because $\Lambda(\rho) = \mathcal{S}_\sigma(\rho^T)$, the reduction map Λ is a decomposable positive map, which is generally of form $\mathcal{S}_d(\rho) = \mathcal{S}_1(\rho) + \mathcal{S}_2(\rho^T)$ with $\mathcal{S}_{1,2}$ being two CP maps.

The last example makes use of an unextendible product basis [4], a set of orthonormal product basis $\{|\alpha_i\rangle|\beta_i\rangle\}_{i=1}^S$ of $\mathcal{H}_s \otimes \mathcal{H}_L$ where $S < sL$ and there is no other pure product state that is orthogonal to this set of basis. If we denote $P = \sum |\alpha_i\rangle\langle\alpha_i| \otimes |\beta_i\rangle\langle\beta_i|$ then $\tilde{\rho} = (1-P)/(sL-S)$ represents an inseparable states with positive partial transposition. If we define

$$\epsilon = \min_{|\alpha\rangle|\beta\rangle} \langle\alpha|\langle\beta|P|\alpha\rangle|\beta\rangle$$

it can be sure that $0 < \epsilon \leq S/sL$ [17]. Denoting ρ_0 as a normalized density matrix in $\mathcal{H}_s \otimes \mathcal{H}_L$ which has the property $\text{Tr}(\rho_0\tilde{\rho}) > 0$, we define a Hermitian matrix as $H_\epsilon = P - \epsilon d\rho_0$ where

$$\frac{1}{d} = \max_{|\alpha\rangle|\beta\rangle} \langle\alpha|\langle\beta|\rho_0|\alpha\rangle|\beta\rangle$$

and $1 \leq d \leq sL$. Matrix H_ϵ is not positive semidefinite since $\text{Tr}H_\epsilon\tilde{\rho} = -\epsilon d\text{Tr}(\rho_0\tilde{\rho}) < 0$ and for an arbitrary pure product state $\langle\alpha|\langle\beta|H_\epsilon|\alpha\rangle|\beta\rangle \geq 0$. If we choose $\rho_0 = I_s \otimes I_L/sL$ then a non-CP positive map is defined by

$$\mathcal{S}_\epsilon(\rho) = \sum_{i=1}^S T_i\rho T_i^\dagger - \epsilon\text{Tr}\rho, \qquad (13)$$

where $T_i = |\alpha_i\rangle\langle\beta_i^*|$ is the half density matrix of the product base $|\alpha_i\rangle|\beta_i\rangle$. In Ref. [17] ρ_0 is taken as a maximally entangle state and $d = \min(s, L)$. Positive map $\mathcal{S}_\epsilon$ is indecomposable because $\mathcal{I}_s \otimes \mathcal{S}_\epsilon(\tilde{\rho})$ is not positive semidefinite while $\mathcal{I}_s \otimes \mathcal{S}_d(\tilde{\rho})$ is positive semidefinite for any decomposable map.

In conclusion, the concept of the half density matrix was studied and its applications to the quantum information are discussed in some detail. Based on the half-density-matrix representation of a Hermitian linear map, we proved that every positive map which is not completely positive is a difference of two completely positive maps. A necessary and sufficient condition for a non-CP positive map is given, which provides a way of constructing such kind of maps. Some examples are also presented. Further applications of the half density matrix in the quantum information and other fields can be expected and the understandings of positive maps provided here may be helpful the recognition of the inseparable quantum states and to the quantification of the entanglement [18].

[1] A. Peres, Phys. Rev. Lett. **77**, 1413 (1996).

[2] M. Horodecki, P. Horodecki, and R. Horodecki, Phys. Rev. Lett. **80**, 5239 (1998).

[3] P. Horodecki, Phys. Lett. A **232**, 333 (1997).

[4] C. H. Bennett, D. P. DiVincenzo, T. Mor, P. W. Shor, J. A. Smolin, and B. M. Terhal, Phys. Rev. Lett. **82**, 5385 (1999).

[5] M. Horodecki, P. Horodecki, and R. Horodecki, Phys. Lett. A **223**, 1 (1996).

[6] K. Kraus, *States, Effects ad Operations: Fundamental Notions of Quantum Theory*, (Wiley, New York 1991).

[7] S. L. Woronowicz, Rep. Math. Phys. **10**, 165 (1976).

[8] C. Bennett, G. Brassard, C. Crepeau, R. Jozsa, A. Peres, and W. K. Wooters, Phys. Rev. Lett. **70**, 1895 (1993).

[9] R. A. Horn and C. R. Johnson, *Matrix Ananlysis*, (Cambridge University Press, 1985)

[10] L. P. Hughston, R. Jozsa and W. K. Wootters, Phys. Lett. A **183**, 14 (1983).

[11] B. Schumacher, Phys. Rev. A **54**, 2614 (1996).

[12] R. F. Werner, Phys. Rev. A **40**, 4277 (1989).

[13] S. X. Yu and C. P. Sun, Phys. Rev. A (accepted).

[14] W. K. Wootters, Phys. Rev. Lett. **80**, 2245 (1998).

[15] M. Horodecki and P. Horodecki, Phys. Rev. A **59**, 4026 (1999).

[16] N. J. Cerf, C. Adami, and R. M. Gingrich, Phys. Rev. A **60**, 898 (1999).

[17] B. M. Terhal, LANL e-print quant-ph/9810091.

[18] V. Vedral, M. B. Plenio, M. A. Rippin, and P. L. Knight Phys. Rev. Lett. **78**, 2275 (1997).