Approach to the semiconductor cavity QED in high-Q regimes with q-deformed boson

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The high density Frenkel exciton which interacts with a single mode microcavity field is dealed with in the framework of the q-deformed boson. It is shown that the q-defomation of bosonic commutation relations is satisfied naturally by the exciton operators when the low density limit is deviated. An analytical expression of the physical spectrum for the exciton is given by using of the dressed states of the cavity field and the exciton. We also give the numerical study and compare the theoretical results with the experimental results

PACS number(s): 42.50 Fx, 71.35-y

I. INTRODUCTION

It's well known that the exciton system is a quasiparticle system. At the low density, excitons are approximately treated as bosons which obey Bose statistics [1–3]. But when the density of the excitons become higher, the excitons, which somewhat deviate ideal bosons, are no longer ideal bosons. There are two ways of dealing with this problem : one way is to put these deviations into the effective interaction between the hypothetical ideal bosons and the exciton operators are still presented by the bosonic operators [4,5]. Another way is the implement of the atomic operators model [6–8]. The question naturelly arises whether the exciton system is equivalent to the atomic system.

In this paper we show that the exciton system could be described by $q-$ deformed boson system which interpolates between Bose system and Fermi system [9] ; and the deformation parameter q is well defined by the total atomic particle number N,rather than it is phenomenological in the previous discussions .The concept of the q-deformed boson was even extensively applied due to the q-deformed boson realization of quantum group theory by different authors ten years ago [10–13]. Since then, many physicists make efforts to find its real physical realizations. For example, they give some phenomenological investigations to fit the deformed spectra of rotation and oscillation for molecules and nuclei [14–16]. In our opinion, those investigations can be regarded as merely phenomenological because a q-deformed structure is postulated in advance without giving it a microscopic mechanism. In this paper it will be shown that a physical and natural realization of the q-deformed boson is provided by the exciton operators, which was presented recently by Gardiner [17] for the description of Bose-Einstein condensation (BEC). In fact, the similar quasiparticles scheme for particle-number conservation has already been introduced by Girardeau and Arnowitt almost 40 years ago [18]. The relationship between Gardiner's phonons and these quasiparticles has been discussed in a recent comment [19] .

Following these ways, we find that the exciton system also could be described by the q-deformed boson. When the density of the exciton (the particles excited in upper state) is low enough, it return to the ideal boson case. Using this theory, we could give a good explanation on the semiconductor cavity QED in high-Q regimes.What will be investigated here is the case that the total atomic particle number N is very large but not infinite. That is, we shall consider the effects of order $o(1/N)$. And we shall focus on an algebraic method, a q-deformed boson algebra, of treating the effects of finite particle number. As it turns out, the commutation relations for the exciton operators will no longer obey the commutation relation of the Heisenberg-Weyl algebra but the q-deformed bosonic commutation relation

$$\left[b_q, b_q^\dagger\right]_q \equiv b_q b_q^\dagger - q b_q^\dagger b_q = 1, \tag{1}$$

where the deformation constant q depends on the total atomic particle number.

This paper is organized as follows. In section 2 we firstly deduce the q-deformed commutation relation for the exciton in the high-Q cavity in case of the large but finite lattice molecule number N . In section 3, only keeping the first order term of $\frac{1}{N}$,we model the Frenkel excitons in a micro-cavity as the dressed q-deformed boson system. In section 4, the quantum approach for angular momentum is used to obtain the eigen- values and eigen-function of the system under the one order approximation. The stationary physical spectrum of the system is calculated in the section 5. Finally we summarize our results with some comments.

II. Q-DEFORMED BOSONIC ALGEBRA FOR EXCITON

Gardiner's starting point [17]to introduce the exciton operators is to consider a system of the weakly interacting Bose gas. Without losing generality, we consider a thin molecular crystal film containing N identical two-level molecules interacting resonantly with a single mode quantum field. The intermolecular interaction is neglected. We assume that all molecules have equivalent mode positions, so they have the same coupling constant κ. By using Dick model [20], we could write the Hamiltonian under the rotating wave approximation as following:

$$H = \hbar\Omega(S_z + a^+a) + \hbar\kappa(aS_+ + a^+S_-), \tag{2}$$

where, a is annihilation operator of the quantum cavity field and

$$\begin{array}{rcl} S_Z &=& \sum_{n=1}^{N} s_z(n), \\ S_+ &=& \sum_{n=1}^{N} s_+(n), \\ S_- &=& \sum_{n=1}^{N} s_-(n), \end{array} \tag{3}$$

where, $s_z(n) = \frac{1}{2}(|e_n><e_n| - |g_n><g_n|)$, $s_+(n) = |e_n><g_n|$ and $s_-(n) = |g_n><e_n|$ are quasi spin operators of the nth molecule. $|e_n>$ and $|g_n>$ are the excited state and the ground state of n'th molecule.

Consider the second quantization of the above model. Let $b_e^\dagger$ and b_e denote the creation and annihilation operators for the atoms in the excited state and $b_g^\dagger$ and b_g for the creation and annihilation operators of the atoms in the ground state. The simplified Hamiltonian in second quantization reads

$$H = \hbar\Omega(b_e^\dagger b_e - b_g^\dagger b_g + a^+a) + \hbar\kappa[ab_e^\dagger b_g + H.\text{c.}]. \tag{4}$$

Note that the total atomic particle number $\mathbf{N} = b_e^\dagger b_e + b_g^\dagger b_g$ is conserved. For convenience we define $\eta = 1/N$ for large particle number.

In the thermodynamical limit $N \to \infty$, the Bogoliubov approximation [21,22]is usually applied, in which the ladder operators $b_g^\dagger, b_g$ of the ground state are replaced by a c-number $\sqrt{N_c}$, where N_c is the average number of the ininital condensated atoms. As a result Hamiltonian Eq.(4) becomes a two-coupling harmonic oscillator system

$$H_b = \hbar\Omega(b_e^\dagger b_e + a^+a) + \hbar\kappa\sqrt{N_c}[ab_e^\dagger + H.\text{c.}]. \tag{5}$$

However, this apporoximation destroyes a symmetry of the Hamiltonian Eq.(4), i.e., the conservation of the total particle number is violated because of $[N, H_b] \neq 0$.

To avoide this problem, the exciton operators are defined as:

$$b_q = \frac{1}{\sqrt{N}} b_g^\dagger b_e, \qquad b_q^\dagger = \frac{1}{\sqrt{N}} b_g b_e^\dagger. \tag{6}$$

according to Gardiner [17].These operators act invariantly on the subspace V^N spanned by bases $|N;n\rangle \equiv |N-n,n\rangle$ $(n = 0, 1, \ldots, N)$, where Fock sates $(m, n = 0, 1, 2, \ldots)$

$$|m,n\rangle = \frac{1}{\sqrt{m!n!}} b_e^{\dagger m} b_g^{\dagger n}|0\rangle$$

spann the Fock space H_{2b} of a two mode boson.

A straightforward calculation leads to the following commutation relation between the exciton operaotr and its Hermitian conjugate:

$$\left[b_q, b_q^\dagger\right] = 1 - \frac{2}{N} b_e^\dagger b_e = f(b_q^\dagger b_q; \eta), \tag{7}$$

with $f(x;\eta) = \sqrt{1 + 2(1-2x)\eta + \eta^2} - \eta$. Keeping only the lowest order of η for a very large total particle number, the commutator above becomes

$$\left[b_q, b_q^\dagger\right] = 1 - 2\eta b_q^\dagger b_q \tag{8a}$$

or

$$\left[b_q, b_q^\dagger\right]_q = 1, \tag{8b}$$

with $q = 1 - 2\eta$. This is exactly a typical q-deformed commutation relation. As $N \to \infty$ or $q \to 1$, the usual commutation relation of Heisenberg-Weyl algebra is regained.

In the above discussion about the phonon excitation, we have linearized commutator $h \equiv f(b^\dagger b;\eta)$ so that a q-deformed commutation rule was obtained. Essentially this linearization establishes a physical realization of the q-deformed algebra. However, if the total particle number N is not large enough, then h can not be approximated by a linear function. From the commutation relations between h and $b_q, b_q^\dagger$

$$\left[h, b_q^\dagger\right] = -\frac{2}{N} b_q^\dagger, \qquad [h, b_q] = \frac{2}{N} b_q, \tag{9}$$

we see that the algebra of exciton operators is a rescaling of algebra $su(2)$ with factor N.

III. THEORETICAL MODEL

Based on the above analysis about the algebraic structure of exciton operator, we consider the case of the low density of atoms in excited sate for the Hamiltonian (2).

Since the second quantization forms of S_+ and S_- are $S_+ = b_e^\dagger b_g, S_- = b_g^\dagger b_e$, it is straighforward to prove that the collective operator $\frac{S_+}{\sqrt{N}}$ and $\frac{S_-}{\sqrt{N}}$ are approximately considered as the simple bosonic operators as $N \to \infty$. These collective operators are called exciton operators. But in case of the high density of molecules in excited state with finite N, many molecules are in the excited

states, the bosonic approximation could no longer work well. The Hamiltonian (2) is rewritten as the effective Hamiltonian in the form of q-deformed boson:

$$H = \hbar\Omega(a^+a + b_q^+b_q) + \hbar g(a^+b_q + b_q^+a) \tag{10}$$

with $g = \sqrt{N}k$, b_q and b_q^+ satisfy q-deformed relation:

$$[b_q, b_q^+] = b_q b_q^+ - q b_q^+ b_q = 1, \tag{11}$$

where,

$$q = 1 - \frac{2}{N} \tag{12}$$

Here, the deformation parameter q is determined by the lattice molecule number. So q is no longer phenomenological.

Up to the first order approximation, $b_q^+(b_q)$ could be expressed as following

$$b_q^+ = b^+ + \frac{b^+b^+b}{2N}, \tag{13}$$

$$b_q = b + \frac{b^+bb}{2N}. \tag{14}$$

in terms of the general bosonic operators $b^+(b)$ .So the Hamitonian H is rewritten in form of perturbation

$$H = H_0 + H' \tag{15}$$

where

$$H_0 = \hbar\Omega(a^+a + b^+b) + \hbar g(a^+b + b^+a), \tag{16}$$

$$H' = \frac{\hbar}{2N}(2\Omega b^+b^+bb + gb^+b^+ab + a^+b^+bb). \tag{17}$$

It is clearly that H' is equivalent to the attractive exciton-exciton collisions due to the bi-exciton effect and decreased exciton-photon coupling constants due to the phase phase filling effect [24] .

IV. APPROXIMATE ANALYTICAL SOLUTIONS

To solve the Schroedinger equation governed by eq.(15),we implement the quantum angular momentum theory [23]

If we define the angular momentum operators

$$J_z = \frac{1}{2}(a^+a - b^+b),$$
$$J_+ = a^+b, J_- = ab^+. \tag{18}$$

then

$$J_x = \frac{1}{2}(a^+b + ab^+),$$
$$J_y = \frac{1}{2i}(a^+b - ab^+). \tag{19}$$

We rewrite the Hamitonian (5)

$$H_0 = \hbar\Omega\hat{N} + 2\hbar g J_x = \hbar\Omega\hat{N} + 2\hbar g e^{-i\frac{\pi}{2}J_y} J_z e^{i\frac{\pi}{2}J_y}. \tag{20}$$

In terms of a $SO(3)$ rotation $\hbar(\Omega\hat{N} + 2gJ_x)$ by $e^{i\frac{\pi}{2}J_y}$. Note that the excitation number operator $\hat{N} = a^+a + b^+b$ is a constant under the a $SO(3)$ rotation and

$$J^2 = J_x^2 + J_y^2 + J_z^2 = \frac{\hat{N}}{2}(\frac{\hat{N}}{2} - 1) \tag{21}$$

is the total angular momentum operator. In terms of the eigen states of J^2 and J_z

$$|jm> = \frac{(a^+)^{j+m}(b^+)^{jm}}{\sqrt{(j+m)!(j-m)!}}|0>, \tag{22}$$

where the eigen values of the J^2 and J_z are

$$j = \frac{\mathcal{N}}{2}, m = -\frac{\mathcal{N}}{2}, \cdots, \frac{\mathcal{N}}{2} \tag{23}$$

the eigen functions and the eigen values of H_0 are constructed as

$$\psi_{jm}^0 = e^{-i\frac{\pi}{2}J_y}|jm>, E_{jm}^{(0)} = \hbar\Omega\mathcal{N} + 2\hbar g m. \tag{24}$$

Up to the first order approximation, the eigen values of H are obtained as

$$E_{jm} = E_{jm}^{(0)} + < mj||e^{i\frac{\pi}{2}J_y} H' e^{-i\frac{\pi}{2}J_y}|jm>, \tag{25}$$

whith their corresponding eigen functions

$$\psi_{jk} = \psi_{jk}^{(0)} + \sum_{n\neq k} \frac{H'_{nk}}{E_{jk}^{(0)} - E_{jn}^{(0)}}\psi_{jn}^{(0)}. \tag{26}$$

where,n and k present the subscript (jm') and (jm). We calculate the matrix elements of the pretubation Hamiltonian H':

$$\begin{aligned}
&< m'j||e^{i\frac{\pi}{2}J_y} H' e^{-i\frac{\pi}{2}J_y}|jm> \\
&= \frac{\hbar}{4N}\Omega\sqrt{(j+m)(j+m-1)} \\
&\times \sqrt{(j-m+1)(j-m+2)}\delta_{m-2,m'} \\
&+ \frac{\hbar}{4N}\Omega\sqrt{(j+m+1)(j+m+2)} \\
&\times \sqrt{(j-m)(j-m-1)}\delta_{m+2,m'} \\
&+ \frac{\hbar}{4N}(2\Omega - g)\sqrt{(j-m)(j+m+1)} \\
&\times (j-m-1)\delta_{m+1,m'} \\
&+ \frac{\hbar}{4N}(2\Omega + g)\sqrt{(j-m)(j+m+1)} \\
&\times (j+m)\delta_{m+1,m'} \\
&+ \frac{\hbar}{4N}(2\Omega + g)\sqrt{(j+m)(j-m+1)} \\
&\times (j+m-1)\delta_{m-1,m'}
\end{aligned}$$

$$+\frac{\hbar}{N}(2\Omega-g)\sqrt{(j+m)(j-m+1)}\times(j-m)\delta_{m-1,m'}+\frac{\hbar}{4N}(\Omega+g)(j+m)(j+m-1)\delta_{m,m'}+\frac{\hbar}{4N}(\Omega-g)(j-m)(j-m-1)\delta_{m,m'}+\frac{\hbar}{N}\Omega(j^2-m^2)\delta_{m,m'} \quad (27)$$

so the eigen values of H are

$$E_{jm}=\hbar\Omega\mathcal{N}+2m\hbar g+\frac{\hbar}{N}\Omega(j^2-m^2)+\frac{\hbar}{4N}(\Omega+g)(j+m)(j+m-1)+\frac{\hbar}{4N}(\Omega-g)(j-m)(j-m-1) \quad (28)$$

In general, we could obtain all eigen functions of H under one order approximation by using eq.(24), eq.(26), and eq.(27). So the time evolution operator of the system is written as:

$$U(t)=exp(-i\frac{H}{\hbar}t)=\sum_{j=0}\sum_{m=-j}^{j}exp(-i\frac{E_{jm}}{\hbar}t)|\psi_{jm}><\psi_{jm}| \quad (29)$$

V. FLUORESCENCE SPECTRUM OF THE EXCITON

We firstly give an analytic expression for the physical spectrum of the q-deformed exciton in terms of the Fock state of the quantum field and the exciton. The standard definition of the physical spectrum is [25]

$$S(\omega)=2\gamma\int_0^t \mathrm{d}t_1\int_0^t \mathrm{d}t_2 e^{-(\gamma-i\omega)(t-t_2)}e^{-(\gamma+i\omega)(t-t_1)}G(t_1,t_2) \quad (30)$$

where γ is the half-bandwith of spectrometer which is being used to measure the spectrum, and t is time length of the excitation in the cavity. $G(t_1,t_2)$ is dipole correlation function, and

$$G(t_1,t_2)=<i|U^+(t_2)b_q^+U(t_2)U^+(t_1)b_qU(t_1)|i>, \quad (31)$$

where, $|i>$ is any initial state of the system. By using the bosonic approximation and substituting eq.(29) into eq.(31), the dipole correlation function is rewritten as following:

$$\begin{aligned}G(t_1,t_2)&=<i|U^+(t_2)b_q^+U(t_2)U^+(t_1)b_qU(t_1)|i>\\&=<i|U^+(t_2)(b^+ +\frac{b^+b^+b}{2N})U(t_2)U^+(t_1)(b+\frac{b^+bb}{2N}))U(t_1)|i>\\&=\sum_{j,k,l,m,n,}<i|\psi_{jl}><\psi_{jl}|(b^+ +\frac{b^+b^+b}{2N})|\psi_{km}>\\&\quad\times<\psi_{km}|(b+\frac{b^+bb}{2N}))|\psi_{jn}><\psi_{jn}|i>e^{i\omega_{lm}t_2}e^{-i\omega_{nm}t_1}\end{aligned} \quad (32)$$

where $\omega_{lm}=\frac{E_{jl}-E_{km}}{\hbar}$ and $\omega_{nm}=\frac{E_{jn}-E_{km}}{\hbar}$. It's evident that j is determined only by the initial state $|i>$. So we have

$$S(\omega)=\sum_{l,k,m}\frac{2\gamma}{\gamma^2+(\omega-\omega_{lm})^2}|<i|\psi_{jl}>|^2|<\psi_{jl}|(b^+ +\frac{b^+b^+b}{2N})|\psi_{km}>|^2 \quad (33)$$

Noting that we have passed the transient terms and slowly variation terms. This equation gives the stationary physical spectrum in terms of the system eigenvalues and eigenstaes. If $<m'j'|b^+|jm>\neq 0$, then we have $j'=j+\frac{1}{2}$ and $m'=m-\frac{1}{2}$. So the equation (25) is rewritten as:

$$S(\omega)=\sum_{l,m}\frac{2\gamma}{\gamma^2+(\omega-\omega_{lm})^2}|<i|\psi_{jl}>|^2|<\psi_{jl}|(b^+ +\frac{b^+b^+b}{2N})|\psi_{(j-\frac{1}{2})m}>|^2 \quad (34)$$

The eigenvalues determine the position of the spectral component and $|<i|\psi_{jl}>|^2|<\psi_{jl}|(b^+ +\frac{b^+b^+b}{2N})|\psi_{(j-\frac{1}{2})m}>|^2$ determine the intensity of the spectral lines.

In terms of the experimental condition of the reference [26], the bare excitons could be prepared by resonant femtosecond pulse pumping. If we prepare the initial state being in $\mathcal{N}=1$, then the eq.(34) shows that the emission spectrum of the $\mathcal{N}=1$ to the $\mathcal{N}=0$ transition has double peaks structure which is exactly equal to that of the two-level atomic system. When the pumping power is increased , the emission spectrum is quiet different from the case of the two-level atomic system. For example, if the system initially is in Fock state $\mathcal{N}=2$, then we have:

$$S(\omega)=\sum_{l,m}\frac{2\gamma}{\gamma^2+(\omega-\omega_{lm})^2}|<i|\psi_{1l}>|^2|<\psi_{1l}|(b^+ +\frac{b^+b^+b}{2N})|\psi_{\frac{1}{2}m}>|^2 \quad (35)$$

From this equation we know when $\mathcal{N}=2$ there are six peaks in the emission spectrum which are expected. Although there are three different initial state, they have similar spectrum shape. As is shown in Fig.1, this sextet structure is contrast to the triplet structure in the emission spectrum from strong pumped two-level system [27].

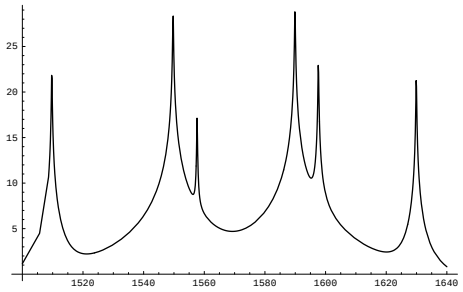

FIG. 1. Ω=1562 mev, N=100, g=20 mev, γ=0.1 mev

When the molecular number of the system is increased, Such as there are 10000 molecular in the system, the other conditions are the same as that of the Fig.1, then the coupling between the molecular and the cavity field is weak. There are two peaks in the emission spectrum (Fig. 2). In this case Bose approximation is good.

q

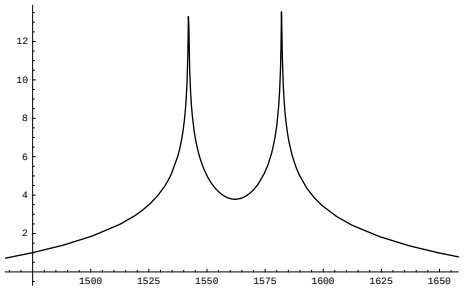

FIG. 2. Ω=1562 mev, N=10000, g=20 mev, γ=0.1 mev

VI. CONCLUSION

It has been shown that the higher density Frenkel exciton coupling to a single mode high-Q microcavity field can be described by the quantum dynamics for the dressed q-deformed boson. Keeping the first order term of $\frac{1}{N}$,the high density Frenkel exciton naturally obyes the q-deformed commutation relation. Based on this observation the quantum theory of angular momentum is used to obtain the eigen- values and eigen-function of the exciton system under the one order approximation. Comparing with the usual approach for Frenkel exciton dynamics our Hamiltonian is Hermitian and closed in form. An analytical expression for the stationaryphysical spectrum for the exciton is obtained by using of the dressed states of the cavity field and the exciton.

Acknowledgment

This work is supported by the NSF of China and "9-5 Pandeng " project.

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[1] H. Haken, Quantum field theory of solids (North-Holland Publishing Company, 1976)
[2] R. S. Knox, Theory of excitons (Academic Press, New York and London, 1963)
[3] A. S. Davydov, Theory of molecular excitons(Plenum Press, New York, 1971)
[4] E. Hanamura, J. Phys. Soc. Jp. 37, 1545(1974), 1553(1974)
[5] M. L. Steyn-Ross and C. W. Gardiner, Phys. Rev. A27, 310(1983)
[6] Y. Manabe, T. Tokihiro and E. Hanamura, Phys. Rev. B48, 2773(1993)
[7] T. Tokihiro, Y. Manabe and E. Hanamura, Phys. Rev, B47, 2019(1993)
[8] T. Tokihiro, Y. Manabe and E. Hanamura, Phys. Rev, B51, 7655(1995)
[9] M. Arik and D. Coon, J. Math. Phys. 17, 524 (1976)
[10] L.C. Biedenharn, J. Phys. A 22, L873 (1989)
[11] C.P. Sun and H.C. Fu, J. Phys. A 22, L983 (1989)
[12] A.J. Macfarlan, J. Phys. A 22, 4851 (1989)
[13] For a review article, see C.P. Sun and M.L. Ge, *The q deformed boson realization of quantum algebra and its application to Yang-Baxter equation*, in *Quantum Group and Quantum Integrable Systems* (World Scientific, ed. M.L. Ge) 133 (1992)
[14] D. Bonatsos, E.N. Argyres, S.B. Prenska, P.P. Reychev, R.R. Roussev and Y.F. Smirnov, Phys. Lett B 251, 477 (1990): J. Phys. G. 16, 137 (1990)
[15] Z. Chang, H.Y. Guo and H. Yan, Phys. Lett. A 192, 156 (1991); Z.Chang, H. Yan, Phys. Lett. A, 254, 154 (1991)

[16] J.M. Li and C.P. Sun Phys. Lett. A 180, 50 (1993)
[17] C.W. Gardiner, Phys. Rev. A 56, 1417 (1997)
[18] M. Girardeau and R. Arnowitt, Phys. Rev. 113, 755 (1959)
[19] M.D. Girardeau, Phys. Rev. A 58, 775 (1998)
[20] R. Dicke, Phys. Rev. 93, 99(1954).
[21] N.N. Bogoliubov, J. Phys. (USSR) 11, 23 (1947)
[22] A.L. Fetter and J.D. Walecka, *Quantum Theory of Many-Particle System* (McGraw-Hill, 1971)
[23] L.C.Beidenharn and J.D.Louck, *Angular Momentum in Quantum Physics,* (Addison-Wesley,London,1981), pp.205-264
[24] Jun-ichi Inoue, Tobias Brandes, Akira Shimizu, cond-mat/9905085
[25] J. H. Eberly and K. Wodkiewicz, J. Opt. Soc. Am. 67, 1252(1977)
[26] Y.Yamamoto,J.Jacobson,S.Pau, H.Cao and G.Bjork, in *Quantum Coherence and Decoherence*, ed.by K.Fujikawa and Y.A.Ono,, pp.331-334,(Amsterdam: Elsevier Sciense Press,1996)
[27] B. R. Mollow, Phys. Rev. 188, 1969(1969)