Universal Quantum Entanglement Concentration Gate

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We construct a Universal Quantum Entanglement Concentration Gate (QEC-Gate). Special times operations of QEC-Gate can transform a pure 2-level bipartite entangled state to nearly maximum entanglement. The transformation can attain any required fidelity with optimal probability by adjusting *concentration step*. We also generate QEC-Gate to the Schmidt decomposable multi-partite system.

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The entanglement nature is a key source to distinguish quantum and classical information theory. Some authors $[1-3]$ have been successful in identifying many fundamental properties of entanglement. Because many important quantum processes, such as teleportation [4], superdense code [5] and quantum computation [6], require maximally entangled states, the methods of entanglement enhancement and entanglement concentration was frequently discussed $[1-3, 7-9]$. Such work is closely related to the fundamental problems of state transformation $[10-18]$. Two main protocols, probabilistic [7,11] and approximate [16], has been applied in the optimal process. An unavoidable problem of these work is that the transformation greatly depends on the input states.

In this report, we construct a Universal Quantum Entanglement Concentration Gate (QEC-Gate), multi-operation of which can transform a 2-level bipartite partial pure entangled state to a nearly maximally entangled state with optimal probability. Its fidelity can be entirely controlled by the parameter *concentration step*. Any required fidelity can be approached by adjusting *concentration step* and enhancing operation times. Additionally, each operation of QEC-Gate can always process with the optimal probability.

In general, a 2-level bipartite entangled state can be represented as the following according to Schmidt decomposition,

$$|\psi\rangle = \cos\theta\,|11\rangle + \sin\theta\,|00\rangle\,, \tag{1}$$

where $0 \leq \theta \leq \frac{\pi}{4}$. Its entanglement, $E(|\psi\rangle) = -\cos^2\theta\log_2\cos^2\theta - \sin^2\theta\log_2\sin^2\theta$ can be compared directly by θ.

Suppose the pure entangled state is shared by two distant observers, Alice and Bob. Since any operation in quantum mechanics can be represented by a unitary evolution together with a measurement, a probe P with a Hilbert space, $|P_0\rangle \otimes |P_1\rangle$ ($|P_0\rangle$ and $|P_1\rangle$ are orthogonal), is introduced. Our gate operation is a local unitary evolution (Alice side) together with the postselection of the measurement results which can be generally represented as

$$(\hat{U}_{AP}\otimes I_B)\,|\psi\rangle\,|P_0\rangle = \sqrt{\gamma}\left|\tilde{\psi}\right\rangle|P_0\rangle + \sqrt{1-\gamma}\,|\phi\rangle_{AB}\,|P_1\rangle\,, \tag{2}$$

where γ is the probability of successful state transformation of $|\psi\rangle \rightarrow \left|\tilde{\psi}\right\rangle$, $|\phi\rangle_{AB}$ is an arbitrary state in the composite system of Alice and Bob. If the measurement of the probe results in $|P_0\rangle$, this transformation is successful.

The QEC-Gate $G(\xi,\eta)$ executes the unitary-reduction operation U_{AP} which is described as the following with two parameters ξ and η, $0<\xi<\eta<\frac{\pi}{4}$,

$$\hat{U}_{AP}\,|1\rangle_A\,|P_0\rangle = \sqrt{\gamma_0}\frac{\cos\eta}{\cos\xi}\,|1\rangle_A\,|P_0\rangle + \sqrt{1-\gamma_0}\frac{1}{\cos\xi}\,|1\rangle_A\,|P_1\rangle\,, \tag{3}$$

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$$\hat{U}_{AP}\left|0\right\rangle_A \left|P_0\right\rangle = \sqrt{\gamma_0}\frac{\sin\eta}{\sin\xi}\left|0\right\rangle_A \left|P_0\right\rangle, \quad (4)$$

where $\gamma_0 = \frac{\sin^2\xi}{\sin^2\eta}$ is exactly the optimal probability to transform entangled state $\cos\xi\left|11\right\rangle_{AB} + \sin\xi\left|00\right\rangle_{AB}$ to $\cos\eta\left|11\right\rangle_{AB} + \sin\eta\left|00\right\rangle_{AB}$ [11]. In fact, U_{AP} can be implemented by qubit A controlling P rotation. It can be expressed as a generalized Toffoli gate $\Lambda_1(U)$ [19] that applies U to qubit P if and only if A on $\left|1\right\rangle$, where $U = \begin{pmatrix} \delta & -\sqrt{1-\delta^2} \\ \sqrt{1-\delta^2} & \delta \end{pmatrix}$ with $\delta = \frac{\tan\xi}{\tan\eta}$.

We set $\theta = \theta_0 \neq 0, \frac{\pi}{4}$ in Eq.(1) as the partial entangled initial input state (it is reasonable because if the initial input state's Schmidt decomposition is represented on another base, an unitary operation $V_A \otimes V_B$ is needed to transform it to the form of Eq.(1)). One operation of $G(\xi,\eta)$ on the input can be represented as

$$\begin{aligned} &\left(\hat{U}_{AP}\otimes I_B\right)\left(\cos\theta_0\left|11\right\rangle_{AB} + \sin\theta_0\left|00\right\rangle_{AB}\right)\left|P_0\right\rangle \quad (5) \\ &= \sqrt{\gamma_0}\left(\cos\theta_0\frac{\cos\eta}{\cos\xi}\left|11\right\rangle_{AB} + \sin\theta_0\frac{\sin\eta}{\sin\xi}\left|00\right\rangle_{AB}\right)\left|P_0\right\rangle \\ &\quad + \sqrt{1-\gamma_0}\frac{\cos\theta_0}{\cos\xi}\left|11\right\rangle_{AB}\left|P_1\right\rangle. \end{aligned}$$

The normalization of output state in Eq.(5) can yield the transformation probability $\gamma_1 = \frac{\sin^2\theta_0}{\sin^2\theta_1}$. This probability is also optimal according to Vidal Theorem [11].

Denote the output entangled state as $\cos\theta_1\left|11\right\rangle_{AB} + \sin\theta_1\left|00\right\rangle_{AB}$, we get

$$\tan\theta_1 = \frac{1}{\delta}\tan\theta_0. \quad (6)$$

Define *Concentration Step* $\Delta(\xi,\eta) = \frac{1}{\delta} = \frac{\tan\eta}{\tan\xi}$. Notice that if ξ and η are set properly, which means that $\Delta(\xi,\eta) - 1$ has a small value, we can promise the enhancement of entanglement. A natural application is to operate $G(\xi,\eta)$ on the initial input state more than one times. For each operation, the entanglement get an increase. Surely this situation can not continue infinitely and the converting operation indicates the approach of maximum. Another obvious property of QEC-Gate is that the enhancement of entanglement is discrete. So in general cases, the final concentrated state is not exactly the maximum entangled state. Here we can determine the optimal operation times T, the final output state $\left|\Phi_{final}\right\rangle$, total probability Γ and fidelity F.

$$T = \left[-\frac{\ln\tan\theta_0}{\ln\Delta(\xi,\eta)}\right], \quad (7)$$

where $[x]$ is the Gauss Functor which gives the integer part of real number x. Notice that this result is based on the limit of $\theta_T \in \left(0, \frac{\pi}{4}\right]$. If we permit $\theta_T > \frac{\pi}{4}$, when $\theta_i \leq \frac{\pi}{4} < \theta_{i+1}$, we set $T = i$ if $\left|\theta_i - \frac{\pi}{4}\right| \leq \left|\theta_{i+1} - \frac{\pi}{4}\right|$, otherwise, $T = i+1$. This choice can promise the optimal output.

$$\begin{aligned} &\left|\Phi_{final}\right\rangle \quad (8) \\ &= G^T\left(\cos\theta_0\left|11\right\rangle_{AB} + \sin\theta_0\left|00\right\rangle_{AB}\right) \\ &= A\left(\cos\theta_0\left(\frac{\cos\eta}{\cos\xi}\right)^T\left|11\right\rangle_{AB} + \sin\theta_0\left(\frac{\sin\eta}{\sin\xi}\right)^T\left|00\right\rangle_{AB}\right), \end{aligned}$$

where A is a normalizing coefficient which is

$$A(\theta_0) = \left(\left(\cos^2\theta_0 + \Delta^{2T}\sin^2\theta_0\right)\left(\frac{1+\tan^2\xi}{1+\tan^2\eta}\right)^T\right)^{-\frac{1}{2}}. \quad (9)$$

The probability and fidelity are

$$\Gamma = \prod\gamma_i = \frac{\sin^2\theta_0}{\sin^2\theta_T} \to 2\sin^2\theta_0, \quad (10)$$

$$F = \frac{1}{2}\left(\frac{1+\tan\theta_0\Delta^T(\xi,\eta)}{\sqrt{1+\tan^2\theta_0\Delta^{2T}(\xi,\eta)}}\right)^2. \quad (11)$$

From the results above, we can see that the total probability is only related to the initial input state. The probability of each transformation approaches 1 when $\Delta(\xi,\eta) - 1$ is small enough. But with the increasing operation times, Γ remains approximately unchanged.

Because operation times T is not a continuous variable, the fidelity represented by Eq.(10) changes periodically with θ_0. The period is determined by *concentration step* Δ, that is $F(\theta_0) = F(\theta_0\arctan\Delta)$. For some special $\theta_0 \in \left\{\theta_0^{(k)} = \arctan\frac{1}{\Delta^k}, k = 1, 2, ...\right\}$, F can exactly attain 1. But the average fidelity, or even the minimum, is more important. To examine the relationship of F and *concentration step*, we firstly calculate the minimum of $F(\theta_0)$. In this situation, $\frac{\pi}{4} - \theta_T = \theta_{T+1} - \frac{\pi}{4}$, so we get

$$F_{\min}(\Delta) = \min F(\theta_0) = \frac{1}{2}\left(\frac{1+\sqrt{\Delta}}{\sqrt{1+\Delta}}\right)^2. \quad (12)$$

The differential coefficient shows that $F_{\min}(\Delta)$ is a monotonous decreasing function of Δ. We can increase the fidelity by reducing $\Delta(\xi, \eta)$. When $\Delta \rightarrow 1$, $F_{\min}(\Delta) \rightarrow 1$. Obviously, it is realized at the price of increasing operation times.

We (three of us) [18] have shown that two different entangled states can be concentrated by same local operations and classical communication if and only if they share the same marginal density operator on one side. Although the design of QEC-Gate makes it possible to attain any required fidelity, it doesn't contradict with the previous result. For different initial states, the final states and the operation times, which determine the practical operation of QEC-Machine, may be different. Actually, the discrete strategy which is adopted to simulate the continuous transformation space can create an identical unit to approximately measure the space. The finer the separation is, the better the measure is. This is exactly the reason why a universal gate can be constructed.

For the complicated situation of n-partite entanglement, the QEC-Gate can not be directly generated. But it is still efficient in transforming some special entanglements, such as the Schmidt decomposable entangled states $\left\{\cos\theta\,|11...1\rangle + \sin\theta\,|00...0\rangle, 0 < \theta < \frac{\pi}{4}\right\}$. Since the QEC-Gate operates only on the composite system of one of the n parties and the probe, the result is the same as that of 2-partite state transformation.

In summery, we have constructed a Universal Quantum Entanglement Concentration Gate, multi-operation of which can transform a partial entanglement to a nearly maximum entanglement with optimal probability. Any required fidelity can be approached by decreasing *concentration step.*

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