Entangled SU(2) and SU(1,1) coherent states

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Abstract

Entangled SU(2) and SU(1,1) coherent states are developed as superpositions of multiparticle SU(2) and SU(1,1) coherent states. In certain cases, these are coherent states with respect to generalized su(2) and su(1,1) generators, and multiparticle parity states arise as a special case. As a special example of entangled SU(2) coherent states, entangled binomial states are introduced and these entangled binomial states enable the contraction from entangled SU(2) coherent states to entangled harmonic oscillator coherent states. Entangled SU(2) coherent states are discussed in the context of pairs of qubits. We also introduce the entangled negative binomial states and entangled squeezed states as examples of entangled SU(1,1) coherent states. A method for generating the entangled SU(2) and SU(1,1) coherent states is discussed and degrees of entanglement calculated. Two types of SU(1,1) coherent states are discussed in each case: Perelomov coherent states and Barut-Girardello

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coherent states.

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I. INTRODUCTION

Qubits are the basic elements of quantum information technology, in the same way that bits are basic units of information in computers. Whereas bits are binary digits, qubits are spin-1/2, or two-level, quantum systems. The advantage of quantum computing over classical computing is the capacity for producing entangled qubits: the large state space available for entangled qubits enables certain problems, thought not to be computable on classical computers, to be solved on quantum computers [1].

A bit can be in an off, or '0', state or an on, or '1', state, but the qubit can be in a superposition of an off, or '$|0\rangle$', state and an on, or '$|1\rangle$', state. We can represent such a state (a general qubit) by

$$\begin{aligned} |\theta,\phi\rangle &= \exp\left[-\frac{\theta}{2}(\sigma_+ e^{-i\phi} - \sigma_- e^{i\phi})\right]|1\rangle \\ &= \cos\frac{\theta}{2}|1\rangle + e^{i\phi}\sin\frac{\theta}{2}|0\rangle, \end{aligned} \tag{1.1}$$

up to a global phase. Here $\sigma_\pm = \sigma_x \pm i\sigma_y$ for σ_x and σ_y are Pauli matrices. In fact such a state is an SU(2) coherent state, also known as an atomic coherent state [2,3]. Two qubits, prepared in a product state, could then be expressed as $|\theta_1,\phi_1\rangle \otimes |\theta_2,\phi_2\rangle$. However, quantum computation is based on the exploitation of entanglement, and such product states are of limited value. The simplest extension of this arbitrary two-qubit product state to a two-qubit entangled state is the unnormalized state

$$\begin{aligned} &\cos\theta|\theta_1,\phi_1\rangle \otimes |\theta_2,\phi_2\rangle \\ &+e^{i\phi}\sin\theta|\theta_1',\phi_1'\rangle \otimes |\theta_2',\phi_2'\rangle, \end{aligned} \tag{1.2}$$

which is a product state for θ a multiple of $\pi/2$. Of course the Bell states [4,5]

$$\begin{aligned} |\Phi\rangle_\pm &= \frac{1}{\sqrt{2}}\left(|0\rangle \otimes |0\rangle \pm |1\rangle \otimes |1\rangle\right), \\ |\Psi\rangle_\pm &= \frac{1}{\sqrt{2}}\left(|0\rangle \otimes |1\rangle \pm |1\rangle \otimes |0\rangle\right) \end{aligned} \tag{1.3}$$

are special cases of the general state (1.2).

The SU(2) coherent states form an overcomplete basis for the Hilbert space, and the qubit states correspond to spin-1/2 representations of SU(2). There are therefore subtleties concerning this entanglement of non-orthogonal states: such subtleties have been considered with respect to entangled coherent states (or superpositions of multimode coherent states) where the coherent states have been harmonic oscillator coherent states [6–22]. Our objective here is to introduce and analyze entangled SU(2) and SU(1,1) coherent states.

The entangled SU(1,1) coherent states are closely related to the SU(2) coherent states because the algebra su(1,1) and su(2) are so similar. However, there are two commonly considered coherent states for SU(1,1). One SU(1,1) coherent state is the analog of the harmonic oscillator coherent state achieved by displacing the vacuum state and the SU(2) coherent state obtained by "rotating" the lowest- or highest-weight state. The analogous SU(1,1) coherent state is obtained via an SU(1,1) transformation of lowest-weight state. This SU(1,1) coherent state is a member of Perelomov's category of generalized coherent state, and we refer to this state as a Perelomov SU(1,1) coherent state [3]. The second SU(1,1) coherent state, introduced by Barut and Girardello [23], is the analog of the harmonic oscillator coherent state being an eigenstate of the annihilation operator; an SU(2) coherent state of this type does not exist due to the SU(2) Hilbert space being finite. We treat entangled SU(1,1) coherent states of both the Perelomov and Barut-Girardello types. As a special case of the entangled Perelomov SU(1,1) coherent state, we obtain superpositions of squeezed vacuum states [24] and entangled squeezed states [12]. Squeezed states are significant in quantum limited measurements, quantum communications and exotic spectroscopy of atoms [25].

In association with the parity operator, this paper first develops parity coherent states and entangled SU(2) and SU(1,1) coherent states in Section II. Section III generalizes the parity coherent states and considers nonlinear SU(2) and SU(1,1) coherent states. In Section IV, we discuss how to represent entangled SU(2) and SU(1,1) coherent states in Fock space, and we have obtained the entangled binomial states, entangled negative binomial states and entangled squeezed states as well as the contraction of entangled SU(2) and SU(1,1) coherent states to entangled harmonic oscillator coherent states. Section V investigates how to generate the entangled coherent states in Hamiltonian systems. Section VI discusses the degree of entanglement for these entangled coherent states, and a conclusion is given in Section VII. The Appendix gives the most general entangled SU(2) and SU(1,1) coherent states with certain special cases, including the quantum Fourier transform state of a product multiparticle SU(2) coherent state acquired by Shor's algorithm [26].

II. ENTANGLED SU(2) AND SU(1,1) COHERENT STATES

A. Entangled coherent states of the harmonic oscillator

For the harmonic oscillator, a general unnormalized two-particle entangled coherent state can be expressed as [9]

$$\begin{aligned}&\cos\theta|\alpha_1\rangle\otimes|\alpha_2\rangle+e^{i\phi}\sin\theta|\alpha_1'\rangle\otimes|\alpha_2'\rangle\\&=[\cos\theta\exp(\vec{\alpha}\cdot\vec{a}^\dagger-\vec{\alpha}^*\cdot\vec{a})\\&+e^{i\phi}\sin\theta\exp(\vec{\alpha}'\cdot\vec{a}^\dagger-\vec{\alpha}'^*\cdot\vec{a})]|0,0\rangle,\end{aligned}\tag{2.1}$$

for $\vec{a}\equiv(a_1,a_2)$, $\vec{\alpha}\equiv(\alpha_1,\alpha_2)$ and $|0,0\rangle=|0\rangle_1\otimes|0\rangle_2$ the vacuum state. It is particularly helpful to concentrate on the balanced entangled coherent state

$$2^{-1/2}\left[|\alpha_1\rangle\otimes|\alpha_2\rangle+e^{i\phi}|-\alpha_1\rangle\otimes|-\alpha_2\rangle\right],\tag{2.2}$$

where the word "balanced" refers to each component in the superposition having coefficients of the same magnitude. For $\phi=\pi/2$ such a state can, in principle, be generated via a nonlinear interferometer [7,20] and is an eigenstate of the canonically transformed pair annihilation operator [14]

$$\vec{\tilde{a}}\equiv\Pi\vec{a}\tag{2.3}$$

with

$$\Pi = e^{i\pi\vec{a}^\dagger\cdot\vec{a}}, \ \Pi\vec{a}\Pi = -\vec{a}. \tag{2.4}$$

Thus, the entangled coherent state (2.1) is actually a pair coherent state [27] with respect to the canonically transformed vector annihilation operator [14], although this pair coherent state does not have the restriction that the number difference of the two modes is fixed.

The most general superposition of harmonic oscillator coherent states is [20]

$$\int \frac{d^{2N}\vec{\alpha}}{\pi^N} f(\vec{\alpha})|\vec{\alpha}\rangle \tag{2.5}$$

with $|\vec{\alpha}\rangle = |\alpha_1\rangle \otimes |\alpha_2\rangle \otimes \cdots \otimes |\alpha_N\rangle$ the $N-$particle coherent state. The superposition (2.5) is entangled when it cannot be expressed as a product state in any representation.

We can introduce an entangled SU(2) coherent state as a generalization of (1.2). However, we make this analysis more general than necessary for studying qubits. We wish to treat general irreducible representations j, where j is the angular momentum parameter and can be integer or half-odd integer. The single qubit case corresponds to $j = 1/2$. A pair of qubits can of course be treated as a single system: in this case we have one $j = 0$ state (the singlet state) and $j = 1$ states (the triplet states). The four states together constitute the Bell states (1.3).

B. Entangled SU(2) coherent states

The SU(2) coherent state can be expressed as [2,3,28]

$$\begin{aligned}
|j\ \gamma\rangle &\equiv R(\gamma)|jj\rangle \\
&= \exp\left[-\frac{1}{2}\theta\left(J_+e^{-i\phi} - J_-e^{i\phi}\right)\right]|jj\rangle \\
&= \left(1+|\gamma|^2\right)^{-j}\sum_{m=0}^{2j}\binom{2j}{m}^{1/2}\gamma^m|j\,j-m\rangle,
\end{aligned} \tag{2.6}$$

where $\gamma = \exp(i\phi)\tan(\theta/2)$, $R(\gamma)$ is the rotation operator, and J_- and J_+ are lowering and raising operators of the su(2) Lie algebra, respectively. The su(2) generators $J_\pm$ and J_z satisfy the su(2) commutation relations

$$[J_+, J_-] = 2J_z,\ [J_z, J_\pm] = \pm J_\pm. \tag{2.7}$$

We can define an su(2) parity operator as

$$\Pi = (-1)^{\mathcal{M}},\ \Pi^2 = 1,\ \Pi^\dagger = \Pi. \tag{2.8}$$

Here $\mathcal{M} = J_z + j$ is the 'number' operator such that

$$\mathcal{M}|j, m\rangle = (j + m)|j, m\rangle,\ 0 \leq j + m \leq 2j. \tag{2.9}$$

It is easy to see that

$$\Pi J_\pm \Pi = -J_\pm,\ \Pi J_z \Pi = J_z. \tag{2.10}$$

Using the above equation we have a new su(2) representation:

$$[\tilde{J}_+, \tilde{J}_-] = 2\tilde{J}_z,\ [\tilde{J}_z, \tilde{J}_\pm] = \pm\tilde{J}_\pm, \tag{2.11}$$

where

$$\tilde{J}_+ = J_+\Pi,\ \tilde{J}_- = \Pi J_-,\ \tilde{J}_z \equiv J_z. \tag{2.12}$$

We define a new SU(2) coherent state associated with the su(2) algebra (2.11) as

$$\begin{aligned} |j\ \gamma\rangle_\Pi &\equiv \tilde{R}(\gamma)|jj\rangle \\ &= \exp[-\frac{\theta}{2}(\tilde{J}_+ e^{-i\phi} - \tilde{J}_- e^{i\phi})]|jj\rangle. \end{aligned} \tag{2.13}$$

We call $|j\ \gamma\rangle_\Pi$ the parity SU(2) coherent state, because the parity operator plays a central role in its definition. This term follows from that of the parity (harmonic oscillator) coherent state [14]. The state $|j\ \gamma\rangle_\Pi$ is different from the SU(2) coherent state due to the nontrivial introduction of the su(2) parity operator Π.

The antinormally ordered rotation operator is

$$\tilde{R}(\gamma) = \exp(\gamma\tilde{J}_-)(1 + |\gamma|^2)^{-\tilde{J}_z} \exp(-\gamma^*\tilde{J}_+). \tag{2.14}$$

Using the above equation, we obtain

$$
\begin{aligned}
|j\ \gamma\rangle_{\Pi} = & \frac{1}{\sqrt{2}}[e^{-i\frac{\pi}{4}}|j\ \ -i(-1)^{2j}\gamma\rangle \\
& +e^{i\frac{\pi}{4}}|j\ i(-1)^{2j}\gamma\rangle].
\end{aligned}
\tag{2.15}
$$

The SU(2) parity coherent state is a superposition of two SU(2) coherent states with a phase difference π. It is similar to the parity harmonic oscillator coherent states [14,29].

The general entangled SU(2) coherent state, analogous to the entangled coherent state (2.5), is discussed in Appendix A. Here we introduce a specific SU(2) coherent state by employing the su(2) parity operator (2.8). Let us consider two independent su(2) Lie algebras. From these two algebras, we can define two new algebras

$$
\begin{aligned}
& [\tilde{J}^n_+, \tilde{J}^l_-] = 2\delta_{nl}\tilde{J}^n_z,\ [\tilde{J}^n_z, \tilde{J}^l_\pm] = \pm\delta_{nl}\tilde{J}^n_\pm, \\
& [\tilde{J}^n_-, \tilde{J}^l_-] = 0,
\end{aligned}
\tag{2.16}
$$

where

$$
\begin{aligned}
& \tilde{J}^n_+ = J^n_+\Pi,\ \tilde{J}^n_- = \Pi J^n_-,\ \tilde{J}^n_z \equiv J^n_z, \\
& \Pi = (-1)^{\ \mathcal{M}_1+\mathcal{M}_2},\ n, l = 1, 2.
\end{aligned}
\tag{2.17}
$$

It is easy to see that the parity operator Π satisfies $\Pi^2 = 1$ and $\Pi^\dagger = \Pi$. These two new su(2) representation are dependent on each other. The SU(2) coherent state of the two original su(2) algebra is

$$
|j\ \vec{\gamma}\rangle = R(\vec{\gamma})|jj\rangle_1 \otimes |jj\rangle_2,
\tag{2.18}
$$

where $\vec{\gamma} \equiv (\gamma_1, \gamma_2)$, $|j\ \vec{\gamma}\rangle \equiv |j\ \gamma_1\rangle_1 \otimes |j\ \gamma_2\rangle_2$, and $R(\vec{\gamma}) \equiv R_1(\gamma_1)R_2(\gamma_2)$. It is important that both Hilbert spaces concerned in the entanglement are restricted to the same irreducible representation j for the entangled coherent states as we introduce them to be well-defined for the $j = 1/2$ case. We consider entangled qubit states; hence the restriction to the same j is of particular value as well. However the SU(2) coherent state for the two new su(2) representations is obtained as

$$
\begin{aligned}
|j\ \vec{\gamma}\rangle_\Pi &= \tilde{R}(\vec{\gamma})|jj\rangle_1 \otimes |jj\rangle_2 \\
&= \frac{1}{\sqrt{2}}\left(e^{-i\frac{\pi}{4}}|j\ \ -i\vec{\gamma}\rangle + e^{i\frac{\pi}{4}}|j\ i\vec{\gamma}\rangle\right)
\end{aligned}
\tag{2.19}
$$

The state $|j\ \vec{\gamma}\rangle_\Pi$ is an entangled SU(2) coherent state with a two-particle parity symmetry.

C. Entangled SU(1,1) coherent states

The generators of su(1,1) Lie algebras, $K_\pm$ and K_z, satisfy the commutation relations

$$
[K_+, K_-] = -2K_z,\ \ [K_z, K_\pm] = \pm K_\pm. \tag{2.20}
$$

By analogy to the su(2) case, we can define the su(1,1) parity operator as

$$
\Pi = (-1)^{\mathcal{N}},\ \ \Pi^2 = 1,\ \ \Pi^\dagger = \Pi. \tag{2.21}
$$

where the 'number' operator $\mathcal{N}$ is given by

$$
\mathcal{N} = K_z - k,\ \ \mathcal{N}|k\ n\rangle = n|k\ n\rangle. \tag{2.22}
$$

Here $|k\ n\rangle$ $(n = 0, 1, 2, ...)$ is the complete orthonormal basis and $k \in \{1/2, 1, 3/2, 2, ...\}$ is the Bargmann index labeling the irreducible representation[$k(k-1)$ is the value of the Casimir operator].

Using the su(1,1) parity operator we can introduce a new su(1,1) algebra with generators

$$
\tilde{K}_+ = K_+\Pi,\ \ \tilde{K}_- = \Pi K_-,\ \tilde{K}_z \equiv K_z. \tag{2.23}
$$

There are two distinct SU(1,1) coherent states to consider.

1. Entangled Perelomov SU(1,1) coherent states

The Perelomov coherent state of the su(1,1) algebra is defined as [3]

$$
\begin{aligned}
|k\ \eta\rangle_P &= S(\xi)|k\ 0\rangle \\
&= \exp(\xi K_+ - \xi^* K_+)|k\ 0\rangle \\
&= (1-|\eta|^2)^k \sum_{n=0}^{\infty} \sqrt{\frac{\Gamma(2k+n)}{\Gamma(2k)n!}}\eta^n |k\ n\rangle,
\end{aligned}
\tag{2.24}
$$

where $\xi = r\exp(i\theta)$, $\eta = \exp(i\theta)\tanh r$, $\Gamma(x)$ is the Gamma function and $S(\xi)$ is the su(1,1) displacement operator.

We define a new SU(1,1) coherent state in association with the su(1,1) parity operator

$$
\begin{aligned}
|k\ \eta\rangle_{P\Pi} &= \tilde{S}(\xi)|k\ 0\rangle \\
&= \exp(\xi\tilde{K}_+ - \xi^*\tilde{K}_-)|k\ 0\rangle,
\end{aligned} \tag{2.25}
$$

It is found that the Perelomov SU(1,1) coherent state is a nonlinear coherent state with the nonlinear function $1/(\mathcal{N}+2k)$ [30]. Therefore, the parity Perelomov SU(1,1) coherent state is a nonlinear coherent state with the nonlinear function$(-1)^{\mathcal{N}}/(\mathcal{N}+2k)$.

The normally-ordered su(1,1) displacement operator is

$$
\begin{aligned}
\tilde{S}(\xi) &= \exp(\eta\tilde{K}_+)(1-|\eta|^2)^{\tilde{K}_z}\exp(-\eta^*\tilde{K}_-), \\
\eta &= \frac{\xi}{|\xi|}\tanh(|\xi|).
\end{aligned} \tag{2.26}
$$

Using the above equation, we obtain

$$
\begin{aligned}
&|k\ \eta\rangle_{P\Pi} \\
&= \frac{1}{\sqrt{2}}\left(e^{i\frac{\pi}{4}}|k\ \ -i\eta\rangle_P + e^{-i\frac{\pi}{4}}|k\ i\eta\rangle_P\right).
\end{aligned} \tag{2.27}
$$

The parity Perelomov SU(1,1) coherent states are superpositions of two Perelomov SU(1,1) coherent states.

The general entangled SU(1,1) coherent state is treated in Appendix A, but here we consider the two-particle case. From the two su(1,1) algebras for the two Hilbert space concerned with the entanglement, we define two new su(1,1) algebras as

$$
\begin{aligned}
&[\tilde{K}^n_+, \tilde{K}^l_-] = -2\delta_{nl}\tilde{K}^n_z, \\
&[\tilde{K}^n_z, \tilde{K}^l_\pm] = \pm\delta_{nl}\tilde{K}^l_\pm,\ [\tilde{K}^n_-, \tilde{K}^l_-] = 0.
\end{aligned} \tag{2.28}
$$

where

$$
\begin{aligned}
&\tilde{K}^n_+ = K^n_+\Pi,\ \tilde{K}^n_- = \Pi K^n_-,\ \tilde{K}^n_z \equiv K^n_z, \\
&\Pi = (-1)^{\mathcal{N}_1+\mathcal{N}_2},\ n, l = 1, 2.
\end{aligned} \tag{2.29}
$$

The su(1,1) parity operator Π satisfies $\Pi^2 = 1$ and $\Pi^\dagger = \Pi$.

The Perelomov SU(1,1) coherent state of the two new su(1,1) algebras is obtained as

$$\begin{aligned}|k\ \vec{\eta}\rangle_{P\Pi} &= \tilde{S}(\vec{\xi})|k\ \vec{0}\rangle \\ &= \frac{1}{\sqrt{2}}\left(e^{i\frac{\pi}{4}}|k\ -i\vec{\eta}\rangle_P + e^{-i\frac{\pi}{4}}|k\ i\vec{\eta}\rangle_P\right)\end{aligned} \tag{2.30}$$

Here $\tilde{S}(\vec{\xi}) \equiv \tilde{S}_1(\xi_1)\tilde{S}_2(\xi_2)$. The state $|k\ \vec{\eta}\rangle_{P\Pi}$ is the entangled Perelomov SU(1,1) coherent state. Again each Hilbert space is restricted to the same irrep k, similar to the restriction for the su(2) case.

2. Entangled Barut-Girardello SU(1,1) coherent states

There is another coherent state of the su(1,1) algebra known as the Barut-Girardello coherent state [23]. It is defined as the eigenstate of the lowering operator K_-

$$K_-|k\ \eta\rangle_{BG} = \eta|k\ \eta\rangle_{BG}, \tag{2.31}$$

and it can be expressed as [23]

$$|k\ \eta\rangle_{BG} = \sqrt{\frac{|\eta|^{2k-1}}{I_{2k-1}(2|\eta|)}}\sum_{n=0}^{\infty}\frac{\eta^n}{\sqrt{n!\Gamma(n+2k)}}|k\ n\rangle, \tag{2.32}$$

where $I_\nu(x)$ is the modified Bessel function of the first kind. The Perelomov coherent state is defined with respect to the displacement operator formalism, whereas the Barut-Girardello coherent state is defined with respect to the ladder operator formalism. Thus, we define the parity Barut-Girardello SU(1,1) coherent state as

$$\tilde{K}_-|k\ \eta\rangle_{BG\Pi} = (-1)^{\mathcal{N}}K_-|k\ \eta\rangle_{BG\Pi} = \eta|k\ \eta\rangle_{BG\Pi}. \tag{2.33}$$

The state $|k\ \eta\rangle_{BG\Pi}$ is a nonlinear coherent state with the nonlinear function $(-1)^{\mathcal{N}}$. From the general expression of a SU(1,1) nonlinear coherent state [30], we obtain the expression of the state $|k\ \eta\rangle_{BG\Pi}$ as

$$|k\ \eta\rangle_{BG\Pi} = \frac{1}{\sqrt{2}}\left(e^{i\frac{\pi}{4}}|k\ -i\eta\rangle_{BG} + e^{-i\frac{\pi}{4}}|k\ i\eta\rangle_{BG}\right). \tag{2.34}$$

Eigenstates of operators $\tilde{K}_-^n (n = 1, 2)$ are constructed as

$$|k\ \vec{\eta}\rangle_{BG\Pi} \sim \exp\left(\frac{\eta_1}{\mathcal{N}_1 + 2k - 1}\tilde{K}_+^1\right) \exp\left(\frac{\eta_2}{\mathcal{N}_2 + 2k - 1}\tilde{K}_+^2\right)|k\ \vec{0}\rangle = \frac{1}{\sqrt{2}}\left(e^{i\frac{\pi}{4}}|k\ \ -i\vec{\eta}\rangle_{BG} + e^{-i\frac{\pi}{4}}|k\ i\vec{\eta}\rangle_{BG}\right), \tag{2.35}$$

which is the entangled Barut-Girardello SU(1,1) coherent states. Here we have used the exponential form of the unnormalized Barut-Girardello coherent state [30]

$$|k\ \eta\rangle_{BG} \sim \exp\left(\frac{\eta}{(\mathcal{N} + 2k - 1)}K_+\right)|k\ 0\rangle. \tag{2.36}$$

The generalization of (2.19), (2.30) and (2.35) to multiparticle entangled SU(2) and SU(1,1) coherent states is treated in Appendix A.

III. NONLINEAR SU(2) AND SU(1,1) COHERENT STATES

A. SU(2) case

The su(2) parity operator $\Pi = (-1)^{\mathcal{M}}$ is a special case of the unitary operator $U(\mathcal{M}) = \exp[-i\vartheta(\mathcal{M})])$. Here $\vartheta(\mathcal{M})$ is a nonsingular function of $\mathcal{M}$. It is easy to check that the operators

$$\bar{J}_+ = J_+ U(\mathcal{M}),\ \bar{J}_- = U^\dagger(\mathcal{M})J_-,\ \bar{J}_z \equiv J_z \tag{3.1}$$

satisfy the su(2) commutation relations (2.7). Then we can define the SU(2) coherent state corresponding to this su(2) algebra as

$$|\vartheta;\ j\ \gamma\ \rangle \equiv \bar{R}(\gamma)|jj\rangle = \exp\left[-\frac{1}{2}\theta\left(\bar{J}_+ e^{-i\phi} - \bar{J}_- e^{i\phi}\right)\right]|jj\rangle. \tag{3.2}$$

By use of (2.14), the state $|\vartheta;\ j\ \gamma\rangle$ is obtained as

$$|\vartheta;\ j\ \gamma\ \rangle = \left(1+|\gamma|^2\right)^{-j} \sum_{m=0}^{2j} \binom{2j}{m}^{1/2} \gamma^m$$
$$\times \exp\left[i\sum_{n=1}^{m} \vartheta(2j-n)\right] |j\, j-m\rangle, \tag{3.3}$$

In the derivation of the above equation, we have used the relation

$$[f(\mathcal{M})J_-]^m = (J_-)^m \prod_{n=1}^{m} f(\mathcal{M}-n) \tag{3.4}$$

If we choose $\vartheta(\mathcal{M}) = \pi\mathcal{M}$, equation (3.3) reduces to (2.15) as we expected. We refer to the state $|\vartheta;\ j\ \gamma\ \rangle$ as a nonlinear SU(2) coherent state if $\vartheta(\mathcal{M})$ is a nonlinear function of $\mathcal{M}$.

B. SU(1,1) case

By analogy to the su(2) case (3.1), we define

$$\bar{K}_+ = K_+ V(\mathcal{N}),\ \bar{K}_- = V^\dagger(\mathcal{N})K_-,\ \bar{K}_z \equiv K_z. \tag{3.5}$$

where $V(\mathcal{N}) = \exp[-i\varphi(\mathcal{N})]$. These operators satisfy the su(1,1) commutation relations (2.20). Then we can define the Perelomov SU(1,1) coherent state corresponding to this su(1,1) algebra as

$$|\varphi;\ k\ \eta\rangle_P = \bar{S}(\xi)|k\ 0\rangle$$
$$= \exp(\xi\bar{K}_+ - \xi^*\bar{K}_-)|k\ 0\rangle. \tag{3.6}$$

As is mentioned in the last section, the Perelomov SU(1,1) coherent state $|k\ \eta\rangle_P$ is a nonlinear coherent state with the nonlinear function $1/(\mathcal{N}+2k)$. Therefore, the state $|\varphi;\ k\ \eta\rangle_P$ is a nonlinear coherent state with the nonlinear function $\exp[i\varphi(\mathcal{N})]/(\mathcal{N}+2k)$. From the general expression for an su(1,1) nonlinear coherent [30], we obtain

$$|\varphi;\ k\ \eta\rangle_P = (1-|\eta|^2)^k \sum_{n=0}^{\infty} \sqrt{\frac{\Gamma(2k+n)}{\Gamma(2k)n!}}\eta^n$$
$$\times \exp\left[-i\prod_{m=0}^{n-1} \varphi(m)\right] |k\ n\rangle. \tag{3.7}$$

In the derivation of the above equation, we have used the relation

$$[K_+ f(\mathcal{N})]^m = (K_+)^m \prod_{n=1}^{m} f(\mathcal{N}+n-1). \tag{3.8}$$

The nonlinear Barut-Girardello coherent state is defined as

$$\begin{aligned}\bar{K}_-|\varphi;\ k\ \eta\rangle_{BG} &= \exp[i\varphi(\mathcal{N})]K_-|\varphi;\ k\ \eta\rangle_{BG} \\ &= \eta|\varphi;\ k\ \eta\rangle_{BG}.\end{aligned} \tag{3.9}$$

The state $|\varphi;\ k\ \eta\rangle_{BG}$ is also a nonlinear coherent state with the nonlinear function $\exp[i\varphi(\mathcal{N})]$. The expansion of this state yields

$$\begin{aligned}|\varphi;\ k\ \eta\rangle_{BG} = \sqrt{\frac{|\eta|^{2k-1}}{I_{2k-1}(2|\eta|)}} \sum_{n=0}^{\infty} \frac{1}{\sqrt{\Gamma(2k+n)n!}}\eta^n \\ \times \exp[-i\prod_{m=0}^{n-1}\varphi(m)]|k\ n\rangle.\end{aligned} \tag{3.10}$$

If we choose $\varphi(\mathcal{N})$ =$\pi\mathcal{N}$, equations (3.7) and (3.10) reduce to (2.27) and (2.34), respectively, as we expected.

IV. REPRESENTATION OF ENTANGLED SU(2) AND SU(1,1) COHERENT STATES IN FOCK SPACE

A. Entangled binomial states

It is well known that the operators

$$J_+ = a^\dagger\sqrt{M-N},\ J_- = \sqrt{M-N}a,\ \ J_z = N - M/2 \tag{4.1}$$

generate the su(2) algebra via the Holstein-Primakoff representation [31] in the spin $M/2$ representation. Here $N = a^\dagger a$ is the number operator. The vacuum $|0\rangle$ is the lowest weight state

$$J_-|0\rangle = 0,\ \ J_z|0\rangle = -M/2|0\rangle. \tag{4.2}$$

Then we define the SU(2) coherent state in the displacement operator form

$$\begin{aligned}|M\ \eta\rangle &= \exp(\zeta J_+ - \zeta^* J_-)|0\rangle \\ &= \exp(\zeta a^\dagger \sqrt{M-N} - \zeta^* \sqrt{M-N}a)|0\rangle \end{aligned} \tag{4.3}$$

where we choose $\zeta = \exp(i\theta)\arctan(|\eta|/\sqrt{1-|\eta|^2})$, $\eta = |\eta|\exp(i\theta)$. The normally ordered form of the displacement operator is

$$\begin{aligned}&\exp(\zeta J_+ - \zeta^* J_-) \\ &= \exp(\tilde{\eta} J_+)(1+|\tilde{\eta}|^2)^{J_z}\exp(-\tilde{\eta}^* J_-)\end{aligned} \tag{4.4}$$

where $\tilde{\eta} = \zeta/|\zeta|\tan|\zeta| = \eta/\sqrt{1-|\eta|^2}$. By use of (4.4), we obtain

$$|M\ \eta\rangle = (1-|\eta|^2)^{M/2}\sum_{n=0}^{M}\binom{M}{n}^{1/2}\left(\frac{\eta}{\sqrt{1-|\eta|^2}}\right)^n |n\rangle. \tag{4.5}$$

This is the binomial state [32–37]. Therefore, the binomial state is a special type of SU(2) coherent state via the Holstein-Primakoff representation. From (2.8) and (4.1) we see that the parity operator is $(-1)^N$. By analogy to the discussion in Section II, we can obtain the entangled binomial states of the type (2.19). Now we show how to generate entangled binomial states in a particular Hamiltonian system.

The entangled coherent states can be created using an ideal Kerr nonlinearity with three nonlinear media elements [20]. We show that the entangled binomial state can also be generated in this system. By an appropriate arrangement of the three nonlinear elements, the effective Kerr transformation with two input fields, 1 and 2, is given by [20]

$$S_{12}(\chi) = \exp(-i\chi a_1^\dagger a_1 a_2^\dagger a_2). \tag{4.6}$$

We assume that the initial state is a product of binomial states $|M\ \vec{\eta}\rangle = |M\ \eta_1\rangle_1 \otimes |M\ \eta_2\rangle_2$. For $\chi = \pi$, the resulting output state is

$$\begin{aligned}&\frac{1}{2}(|M\ \eta_1\rangle_1 + |M\ -\eta_1\rangle_1)\otimes|M\ \eta_2\rangle_2 \\ &+(|M\ \eta_1\rangle_1 - |M\ -\eta_1\rangle_1)\otimes|M\ -\eta_2\rangle_2\end{aligned} \tag{4.7}$$

This state contains both even and odd binomial states [38].

B. Entangled negative binomial states

The generators of the su(1,1) algebra via the Holstein-Primakoff realization of the discrete irreducible representation with Bargmann index $M/2$ are

$$K_+ = a^\dagger\sqrt{M+N},\ K_- = \sqrt{M+N}a,\ \ K_z = N + M/2, \tag{4.8}$$

and the vacuum $|0\rangle$ is the lowest-weight state:

$$K_-|0\rangle = 0,\ \ K_z|0\rangle = M/2|0\rangle. \tag{4.9}$$

The corresponding SU(1,1) coherent state is

$$|M\ \eta\rangle^- = \exp(\xi K_+ - \xi^* K_-)|0\rangle. \tag{4.10}$$

Here $\xi = \exp(i\theta)\,\text{arctanh}|\eta|$. Using (2.26), we obtain

$$|M\ \eta\rangle^- = (1-|\eta|^2)^{M/2}\sum_{n=0}^{\infty}\binom{M+n-1}{n}^{1/2}\eta^n|n\rangle. \tag{4.11}$$

This is the negative binomial state [39–42]. Therefore, the negative binomial state is a special type of SU(1,1) Perelomov coherent state. The parity operator is $(-1)^N$ and the entangled negative binomial states of the type (2.30) can be obtained. We do not write these entangled states explicitly here.

Under the transformation $S_{12}(\pi)$, the input negative binomial state $|M\ \vec{\eta}\rangle^-$ will be transformed into the entangled negative binomial states

$$\begin{aligned}&\frac{1}{2}(|M\ \eta_1\rangle_1^- + |M\ \ -\eta_1\rangle_1^-)\otimes|M\ \eta_2\rangle_2^- \\ &+(|M\ \eta_1\rangle_1^- - |M\ \ -\eta_1\rangle_1^-)\otimes|M\ \ -\eta_2\rangle_2^-.\end{aligned} \tag{4.12}$$

This state contains both even and odd negative binomial states [43,44]

C. Contraction of su(2) and su(1,1) algebra

Let us note the fact that the binomial distribution tends to the Poisson distribution in a certain limit. Let $M\to\infty, |\eta|\to 0$ in such a way that the product $|\eta|^2 M = |\alpha|^2$ is fixed. In this limit, the binomial distribution of the binomial state tends to the Poisson distribution $\exp(-|\alpha|^2)|\alpha|^{2n}/n!$, and the binomial state tends to the ordinary coherent state

$$|M\ \eta\rangle \to \exp(-|\alpha|^2/2)\sum_{n=0}^{\infty}\frac{\alpha^n}{\sqrt{n!}}|n\rangle. \tag{4.13}$$

Here we have used the relation

$$(1-|\eta|^2)^M \to \exp(-|\alpha|^2). \tag{4.14}$$

This limit can also be visualized as a contraction of the su(2) algebra into the Heisenberg-Weyl algebra [45]

$$|\eta|J_+ \to |\alpha|a^\dagger, |\eta|J_- \to |\alpha|a. \tag{4.15}$$

Thus, (4.5) tends to the coherent state

$$|M\ \eta\rangle \to \exp[\alpha a^\dagger - \alpha^* a]|0\rangle. \tag{4.16}$$

In this limit the entangled binomial state (4.7) reduces to the entangled coherent state [20]

$$\begin{aligned}&\frac{1}{2}[(|\alpha_1\rangle_1 + |-\alpha_1\rangle_1)\otimes|\alpha_2\rangle_2 \\ &+(|\alpha_1\rangle_1 - |-\alpha_1\rangle_1)\otimes|-\alpha_2\rangle_2\end{aligned} \tag{4.17}$$

which can be employed as qubits in quantum computation [46,47].

In the same limit described above, the negative binomial states reduce to coherent states, the su(1,1) algebra contracts into the Heisenberg-Weyl algebra [45], and the entangled negative binomial states (4.12) reduce to the entangled harmonic oscillator coherent states (4.17).

D. Entangled squeezed states

The amplitude-squared su(1,1) algebra is realized by

$$K_+ = \frac{1}{2}a^{+2}, K_- = \frac{1}{2}a^2, K_z = \frac{1}{2}\left(N+\frac{1}{2}\right). \tag{4.18}$$

The representation on the usual Fock space is completely reducible and decomposes into a direct sum of the even Fock space (S_0) and odd Fock space (S_1),

$$S_i = \text{span}\{|n\rangle_i \equiv |2n+i\rangle | n = 0, 1, 2, ...\}, \; i = 0, 1. \tag{4.19}$$

Representations on S_i can be written as

$$\begin{aligned} K_+|n\rangle_i &= \sqrt{(n+1)(n+i+1/2)}|n+1\rangle_i, \\ K_-|n\rangle_i &= \sqrt{(n)(n+i-1/2)}|n-1\rangle_i, \\ K_0|n\rangle_i &= (n+i/2+1/4)|n\rangle_i. \end{aligned} \tag{4.20}$$

The Bargmann index $k = 1/4$ $(3/4)$ for even (odd) Fock space. We see that the Perelomov SU(1,1) coherent states in even/odd Fock space are squeezed vacuum states and squeezed first Fock states

$$\begin{aligned} |\eta\rangle_V &= \exp\left(\frac{\xi}{2}a^{+2} - \frac{\xi^*}{2}a^2\right)|0\rangle, \\ |\eta\rangle_F &= \exp\left(\frac{\xi}{2}a^{+2} - \frac{\xi^*}{2}a^2\right)|1\rangle, \end{aligned} \tag{4.21}$$

respectively. From (2.21) and (4.18), the corresponding parity operator is $(-1)^{N/2}$ in even Fock space and $(-1)^{(N-1)/2}$ in odd Fock space. We consider two modes a_1 and a_2. Then from (2.30) and through the amplitude-squared su(1,1) realization, we obtain the entangled squeezed vacuum states and entangled squeezed first Fock states as

$$\begin{aligned} |\vec{\eta}\rangle_V &= \frac{1}{\sqrt{2}}\left(e^{i\frac{\pi}{4}}|-i\vec{\eta}\rangle_V + e^{-i\frac{\pi}{4}}|i\vec{\eta}\rangle_V\right), \\ |\vec{\eta}\rangle_F &= \frac{1}{\sqrt{2}}\left(e^{i\frac{\pi}{4}}|-i\vec{\eta}\rangle_F + e^{-i\frac{\pi}{4}}|i\vec{\eta}\rangle_F\right). \end{aligned} \tag{4.22}$$

The state (4.22) is a special case of the entangled squeezed coherent state [12] for zero amplitude and reduces to the superposition of two squeezed vacuum states [24].

In Fock space, we have obtained entangled binomial states, entangled negative binomial states and entangled squeezed states. They are special cases of entangled SU(2) coherent states or entangled SU(1,1) coherent states.

V. GENERATION OF THE ENTANGLED COHERENT STATES

A superposition of two distinct su(2) coherent states can be generated by the Hamiltonian system for the nonlinear rotator [28,48]

$$H_j = \omega J_z + \frac{\lambda}{2j} J_z^2 \ (\hbar = 1) \tag{5.1}$$

where ω is the linear precession frequency and λ is a positive constant. Let the initial state be the SU(2) coherent state $|j, \gamma\rangle$. At time $t_j = \pi j/\lambda$, the coherent state has evolved into the superposition state

$$\begin{aligned} &\exp(-iH_j t_j)|j \ \gamma\rangle \\ &= \frac{1}{\sqrt{2}} \left(e^{-i\frac{\pi}{4}} |j \ \bar{\gamma}\rangle + e^{i\frac{\pi}{4}} (-1)^j |j \ -\bar{\gamma}\rangle \right). \end{aligned} \tag{5.2}$$

Here $\bar{\gamma} = \exp(i\omega\pi j/\lambda)\gamma$. A superposition of two coherent states, separated by a phase π, has been generated from the initial SU(2) coherent state. The phase difference of the coefficients in (5.2) depends on whether j is odd or even. Let $\bar{\gamma} \to -i(-1)^{2j}\gamma$ and j be even; the above state is just the parity SU(2) coherent state $|j \ \gamma\rangle_\Pi$.

Gerry [49] and Bužek [50] have studied the dynamics of the nonlinear oscillator with the Hamiltonian H written as

$$H = \omega a^\dagger a + \frac{\lambda}{2} a^{\dagger 2} a^2 = \omega K_z + \lambda K_+ K_- \tag{5.3}$$

up to constant terms. Here K_z and $K_\pm$ are generators of the amplitude-squared su(1,1) algebra (4.18). At time $t = \pi/2\lambda$, the initial coherent state has evolved into the superposition state

$$\frac{1}{\sqrt{2}} \left[e^{-i\frac{\pi}{4}} |k \ \bar{\eta}\rangle_P + e^{i\frac{\pi}{4}} |k \ -\bar{\eta}\rangle_P \right], \tag{5.4}$$

$$\frac{1}{\sqrt{2}} \left[e^{-i\frac{\pi}{4}} |k \ \bar{\eta}\rangle_{BG} + e^{i\frac{\pi}{4}} |k \ -\bar{\eta}\rangle_{BG} \right], \tag{5.5}$$

where $\bar{\eta} = \exp\{-i\pi[\omega + (2k-1)\lambda]/(2\lambda)\}$. Here we choose the initial state to be the SU(1,1) coherent state $|k \ \eta\rangle_P$ and $|k \ \eta\rangle_{BG}$, respectively. Let $\bar{\eta} \to i\eta$; the above superposition states are then just the parity coherent states $|k \, \eta\rangle_{P\Pi}$ and $|k \ \eta\rangle_{BG\Pi}$.

Now we consider how to generate the entangled SU(2) and SU(1,1) coherent states. By analogy with the generation of entangled coherent states of the harmonic oscillator [14], the two Hamiltonians are given by

$$H_2 = \chi_1 \mathcal{M}_1^2 + \chi_2 \mathcal{M}_2^2 + \chi_3 \mathcal{M}_1 \mathcal{M}_2, \tag{5.6}$$

$$H_{11} = \lambda_1 \mathcal{N}_1^2 + \lambda_2 \mathcal{N}_2^2 + \lambda_3 \mathcal{N}_1 \mathcal{N}_2. \tag{5.7}$$

We assume that the initial state of the Hamiltonian system H_2 is $|j\ -i\vec{\gamma}\rangle$ and the time $t = \pi/(2\chi_1), \chi_2 = \chi_1, \chi_3 = 2\chi_1$. Then the initial state evolves into the entangled SU(2) coherent state $|j\ \vec{\gamma}\rangle_{\Pi}$ (j is an integer). Similarly, we assume that the initial state of the Hamiltonian system H_{11} is $|k\ i\vec{\eta}\rangle_P$ and the time $t = \pi/(2\lambda_1)$, $\lambda_2 = \lambda_1$, $\lambda_3 = 2\lambda_1$. The initial state evolves into the entangled SU(1,1) coherent state $|k\ \vec{\eta}\rangle_{P\Pi.}$ If we choose the initial state as an SU(1,1) coherent state $|k\ i\vec{\eta}\rangle_{BG}$, the resulting state will be the entangled SU(1,1) coherent states $|k\ \vec{\eta}\rangle_{BG\Pi.}$

It is interesting if we choose the parameters $\chi_1 = \chi_2 = 0, t = \pi/\chi_3$. The initial state $|j\ \vec{\gamma}\rangle$ will evolve into the following entangled SU(2) coherent state

$$\begin{aligned}
&\frac{1}{2}\{[|j\ \gamma_1\rangle_1 + (-1)^{2j}|j\ -\gamma_1\rangle_1] \otimes |j\ \gamma_2\rangle_2 + \\
&[(-1)^{2j}|j\ \gamma_1\rangle_1 - |j\ -\gamma_1\rangle_1] \otimes |j\ -\gamma_2\rangle_2\} \\
&= \frac{1}{2}\{|j\ \gamma_1\rangle_1 \otimes [|j\ \gamma_2\rangle_2 + (-1)^{2j}|j\ \text{-}\gamma_2\rangle_2] + \\
&|j\ -\gamma_1\rangle_1 \otimes [(-1)^{2j}|j\ \gamma_2\rangle_2 - |j\ -\gamma_2\rangle_2]\}.
\end{aligned} \tag{5.8}$$

The entangled coherent state (5.8) includes the SU(2) 'cat' states [28,51]. We assume that the initial state of the Hamiltonian system H_{11} is $|k\ \vec{\eta}\rangle$. Here the subscripts P and BG have been omitted since the discussions are the same for the Perelomov coherent states and the Barut-Girardello coherent states. At time $t = \pi/\lambda_3$ and $\lambda_1 = \lambda_2 = 0$, Resultant state is

$$\begin{aligned}
&\frac{1}{2}[(|k\ \eta_1\rangle_1 + |k\ -\eta_1\rangle_1) \otimes |k\ \eta_2\rangle_2 + \\
&(|k\ \eta_1\rangle_1 - |k\ -\eta_1\rangle_1) \otimes |k\ -\eta_2\rangle_2],
\end{aligned} \tag{5.9}$$

which includes the SU(1,1) 'cat' states [51].

VI. DEGREE OF ENTANGLEMENT

A. Bell inequality

A standard example of entanglement of two-particle nonorthogonal states is given by [11]

$$|\Psi\rangle = \mu|\alpha\rangle_1 \otimes |\beta\rangle_2 + \nu|\gamma\rangle_1 \otimes |\delta\rangle_2, \tag{6.1}$$

where $|\alpha\rangle_1$ and $|\gamma\rangle_1$ are nonorthogonal states of system 1 and similarly for $|\beta\rangle_2$ and $|\delta\rangle_2$ of system 2. We also assume that the states $|\alpha\rangle_1$ and $|\gamma\rangle_1$ are linearly independent as are $|\beta\rangle_2$ and $|\delta\rangle_2$. The entangled SU(2) and SU(1,1) coherent states obtained in the previous section are special cases of these entangled nonorthogonal states $|\Psi\rangle$. The state $|\Psi\rangle$ is a pure state whose density matrix is $\rho_{12} = |\Psi\rangle\langle\Psi|$. Then the reduced density matrices $\rho_1 = \mathrm{Tr}_2(\rho_{12})$ and $\rho_2 = \mathrm{Tr}_1(\rho_{12})$ for each subsystem can be obtained, and the two eigenvalues of ρ_1 are given by [11]

$$\begin{aligned} \lambda_\pm &= \frac{1}{2} \pm \frac{1}{2}\sqrt{1 - 4|\mu\nu\mathcal{A}_1\mathcal{A}_2|^2}, \\ \mathcal{A}_1 &= \sqrt{1 - |{}_1\langle\alpha|\gamma\rangle_1|^2},\ \mathcal{A}_2 = \sqrt{1 - |{}_2\langle\delta|\beta\rangle_2|^2}. \end{aligned} \tag{6.2}$$

The two eigenvalues of ρ_2 are identical to those of ρ_1. The corresponding eigenvectors of ρ_1 are $|\pm\rangle_1$, and the corresponding orthonormal eigenvectors of ρ_2 are denoted by $|\pm\rangle_2$. The general theory of the Schmidt decomposition [52–54] implies that the state $|\Psi\rangle$ can be expressed in the Schmidt form

$$|\Psi\rangle = c_-|-\rangle_1|-\rangle_2 + c_+|+\rangle_1|+\rangle_2, \tag{6.3}$$

where $|c_\pm|^2 = \lambda_\pm$, $|c_+|^2 + |c_-|^2 = 1$. The two-system entangled state(6.3) violates a Bell inequality. More specifically, we choose Hermitian operator $\hat{\Theta}_i$ for each subsystem such that the eigenvalues are ± 1. The general form for such an operator is

$$\begin{aligned} \hat{\Theta}_i(\lambda_i, \varphi_i) &= \cos\lambda_i(|+\rangle_i\langle+| - |-\rangle_i\langle-|) \\ &\quad + \sin\lambda_i(e^{i\varphi_i}|+\rangle_i\langle-| + e^{-i\varphi_i}|-\rangle_i\langle+|). \end{aligned} \tag{6.4}$$

The Bell operator is defined as [55]

$$\hat{B} = \hat{\Theta}_1\hat{\Theta}_2 + \hat{\Theta}_1\hat{\Theta}_2' + \hat{\Theta}_1'\hat{\Theta}_2 - \hat{\Theta}_1'\hat{\Theta}_2', \tag{6.5}$$

for $\hat{\Theta}_i \equiv \hat{\Theta}_i(\lambda_i, \varphi_i)$, $\hat{\Theta}_i' \equiv \hat{\Theta}_i(\lambda_i', \varphi_i')$.

For the choices

$$\begin{aligned} \lambda_1 &= 0,\ \ \lambda_1' = \pi/2, \\ \lambda_2 &= -\ \lambda_2' = \cos^{-1}[1 + |2c_+c_-|^2]^{-1/2}, \\ \varphi_1 + \varphi_2 &= \varphi_1' + \varphi_2' = \varphi_+ - \varphi_-, \end{aligned} \tag{6.6}$$

where $\varphi_\pm$ are the phases of $c_\pm$ in (6.3), the expectation value of the Bell operator for the state $|\Psi\rangle$ (6.1) is [11]

$$B = \langle\Psi|\hat{B}|\Psi\rangle = 2\sqrt{1 + 4\lambda_+\lambda_-} > 2. \tag{6.7}$$

The degree of violation depends on the values of $\lambda_\pm$, but a violation always occurs provided that the state is entangled.

Several entangled coherent states are obtained in the previous sections. Here we only consider two examples. The first example is the entangled SU(2) coherent state $|j\ \vec{\gamma}\rangle_\Pi$ (2.19). For simplicity $\gamma_0 = \gamma_1 = \gamma_2$ is introduced. The eigenvalues of the reduced density matrices ρ_1 and ρ_2 are

$$\lambda_\pm = \frac{1}{2} \pm \frac{1}{2}\sqrt{1 - \left[1 - \left(\frac{1 - |\gamma_0|^2}{1 + |\gamma_0|^2}\right)^{4j}\right]^2}. \tag{6.8}$$

Then the expectation value of the Bell operator for the entangled coherent state $|j\ \vec{\gamma}\rangle_\Pi$ is given by

$$B = 2\sqrt{1 + \left[1 - \left(\frac{1 - |\gamma_0|^2}{1 + |\gamma_0|^2}\right)^{4j}\right]^2}. \tag{6.9}$$

The second example is the entangled Perelomov SU(1,1) coherent state $|k\,\vec{\eta}\rangle_{P\Pi}$. The corresponding eigenvalues and expectation value of the Bell operator are obtained as

$$\lambda_{\pm} = \frac{1}{2} \pm \frac{1}{2}\sqrt{1 - \left[1 - \left(\frac{1-|\eta_0|^2}{1+|\eta_0|^2}\right)^{4k}\right]^2}, \tag{6.10}$$

$$B = 2\sqrt{1 + \left[1 - \left(\frac{1-|\eta_0|^2}{1+|\eta_0|^2}\right)^{4k}\right]^2}. \tag{6.11}$$

Here we have chosen $\eta_0 = \eta_1 = \eta_2$.

B. Entropy

The entropy S of a quantum state described by density operator ρ is defined by [56–58]

$$S = -k_B \mathrm{Tr}(\rho \ln \rho), \tag{6.12}$$

where k_B is the Boltzmann constant. The entropy defined above is zero for a pure state and positive for a mixed state. If we consider two systems, the entropy the system 1(2) is determined via the reduced density operator

$$S_{1(2)} = -k_B \mathrm{Tr}_{1(2)}(\rho_{1(2)} \ln \rho_{1(2)}). \tag{6.13}$$

In 1970 Araki and Lieb proved the following inequality [56]:

$$|S_1 - S_2| \leq S \leq S_1 + S_2. \tag{6.14}$$

One consequence of this inequality is that, if the total system is in a pure state, $S_1 = S_2$. From an information theory point of view, the entropy can be regarded as the amount of uncertainty contained within the density operator. We can use the index of correlation I_c as the amount of information lost in the tracing procedure [58]

$$I_c = S_1 + S_2 - S. \tag{6.15}$$

For the pure state $|\Psi\rangle$, the index of correlation is obtained as

$$I_c = 2S_1 = -2k_B(\lambda_+ \ln \lambda_+ + \lambda_- \ln \lambda_-). \tag{6.16}$$

If $I_c = 0$, the two subsystems are in a pure state and disentangled. The combination of (6.8-6.11) and (6.16) gives the index of correlation of entangled SU(2) and SU(1,1) coherent states.

C. Discussion

We now discuss two limit cases for certain parameters. When $|\gamma_0|(|\eta_0|) \to 0$ or ∞, $B = 2$, i.e., the entangled coherent states become product states and disentangled. When $|\gamma_0|(|\eta_0|) = 1$, $B = 2\sqrt{2}$, and the entangled coherent states are maximally entangled states. For this case, the two states $|j\ \gamma_0\rangle$ and $|j\ -\gamma_0\rangle$ are orthogonal and the states $|k\ \eta_0\rangle$ and $|k\ -\eta_0\rangle$ become orthogonal.

When $|\gamma_0|(|\eta_0|) \to 0$ or ∞, $\lambda_+ = 1$ and $\lambda_- = 0$, then the index of correlation is zero as we expect. The entangled states become product states. When $|\gamma_0|(|\eta_0|) = 1$, $\lambda_\pm = 1/2$, then the index of correlation is $2k_B \ln 2$, which results from the orthogonality of the states $|j\ \pm\gamma_0\rangle(|\gamma_0| = 1)$ or $|k\ \pm\eta_0\rangle(|\eta_0| = 1)$. An index of correlation is maximum for $I_c = 2k_B \ln 2$ [58].

The maximum entanglement of SU(2) coherent states $|j\ \vec{\gamma}\rangle_{\Pi}$ for $\vec{\gamma} = (\gamma_0, \gamma_0)$ and $|\gamma_0| = 1$ can be understood as follows. From the expression for $\gamma = \exp(i\phi)\tan(\theta/2)$, we can see that $|\gamma| = 1$ corresponds to the set of all points on the equator ($\theta = \pi/2$) of the Poincaré sphere, which is used to represent the pure states of an SU(2) system. The states corresponding to $|j\ \gamma_0 = 1\rangle$ and $|j\,\gamma_0 = -1\rangle$ are antipodal; that is, one state is represented by a point at the intersection of the equator and the longitudinal line $\phi = 0$, and the other state is represented by the point at the intersection of the equator and the longitudinal line $\phi = \pi$. We can thus employ the notation of Eq. (1.1), but with an appropriate SU(2) rotation (and allow for arbitrary j) by replacing $|j\ \gamma_0 = 1\rangle$ by $|1\rangle$ and $|j\ \gamma_0 = -1\rangle$ by $|0\rangle$. For $j = 1/2$, we recover the qubit case (1.1). The entanglement state for $j = 1/2$ is just $2^{-1/2}\left[\exp(i\pi/4)|11\rangle + \exp(-i\pi/4)|00\rangle\right]$. This state is a form of Bell state (1.3) and the reason for B and I_c being maximal is evident. A similar analysis can be employed for the entangled SU(1,1) coherent states represented by points on the Lobachevsky plane.

VII. CONCLUSIONS

We have introduced entangled SU(2) and SU(1,1) coherent states. The general form for these entangled coherent states, which also incorporates entangled harmonic oscillator coherent states in the formalism, is given in the Appendix, but the main concern here is with two-particle coherent states. The two-particle coherent states present a diverse range of interesting phenomena, as we have shown. Aside from the mathematical elegance of entangled SU(2) and SU(1,1) coherent states, we have also applied these states to current topics of research, namely quantum information (specifically qubits) and nonclassical states of light (specifically squeezed vacuum states).

We have explored several aspects of coherent states. Aside from defining these states, we have employed the entangled binomial states to establish a rigorous contraction from entangled SU(2) coherent states to entangled harmonic oscillator coherent states. Two measures of entanglement, the Bell operator approach and the index of correlation, have been used to quantify the degree of entanglement. As the entanglement is generally between non-orthogonal states, the degree of entanglement ranges from no entanglement (product state) to being maximally entangled for various parametric choices.

The generation of entangled coherent states have been treated here by a Hamiltonian evolution which is a generalization of J_z^2 and K_z^2 nonlinear evolution for multiparticle systems. However, such evolutions are extremely sensitive to environmental-induced decoherence. Other methods for generating entangled coherent states could be considered, but the nonlinear evolution considered here illustrated one possible approach to producing these entangled states.

These results can be generalized in various ways which would be of interest. One generalization is to entangled generalized coherent states, with generalized coherent states of the type treated by Perelomov [3]. Another intriguing generalization is to entangled SU(2) and SU(1,1) coherent states for the respective Hilbert spaces, not restricted to the same irreducible representations. For example one could consider entanglement between spin-1/2 states (qubits) and spin-1 states (qutrits). Or one could consider entanglement of SU(1,1) coherent states with $k = 1/4$ (single mode squeezed vacuum states) and SU(1,1) coherent states with $k = 3/4$ (two-mode squeezed vacuum states). These ideas are topics for future research.

APPENDIX A: MOST GENERAL FORM OF ENTANGLED SU(2) AND SU(1,1) COHERENT STATES

The general form of entangled SU(2) and SU(1,1) coherent states can be written as

$$\int d\mu(\vec{\xi}) f(\vec{\xi}) |l\ \vec{\xi}\rangle, \tag{A1}$$

where $l = j$ and $\vec{\xi} = \vec{\gamma}$ for SU(2) coherent states, and $l = k$ and $\vec{\xi} = \vec{\eta}$ for either Perelomov or Barut-Girardello SU(1,1) coherent states. We can also obtain the entangled coherent state (2.5) via the same expression by replacing $\vec{\xi}$ by $\vec{\alpha}$ and ignoring the irrep index l. The measure $d\mu(\vec{\xi})$ is, for each case,

$$\begin{aligned} d\mu(\vec{\gamma}) &= \prod_n \frac{2j+1}{\pi} \frac{d^2\vec{\gamma}_n}{(1+|\gamma_n|^2)^2}, \\ d\mu_P(\vec{\eta}) &= \prod_n \frac{2k-1}{\pi} \frac{d^2\vec{\eta}_n}{(1-|\eta_n|^2)^2}, \\ d\mu_{BG}(\vec{\eta}) &= \prod_n \frac{2}{\pi} I^2_{2k-1}(2|\eta_n|)\ d^2\vec{\eta}_n, \\ d\mu(\vec{\alpha}) &= \prod_n \frac{d^2\vec{\alpha}}{\pi} \end{aligned} \tag{A2}$$

for SU(2) coherent states, Perelomov SU(1,1) coherent states and Barut-Girardello coherent states, and harmonic oscillator coherent states, respectively.

If we choose $l = j$ and

$$f(\vec{\xi}) = \frac{1}{\sqrt{2}} \left[\exp\left(-i\frac{\pi}{4}\right) \delta(\vec{\xi} + i\vec{\gamma}) + \exp\left(i\frac{\pi}{4}\right) \delta(\vec{\xi} - i\vec{\gamma}) \right], \tag{A3}$$

the general state (A1) will reduce to the entangled SU(2) coherent state (2.19). All examples of coherent states considered in this paper can be constructed accordingly.

As an interesting example, we consider the multiple qubit entanglement. Shor [26] introduced the quantum Fourier transforms in order to apply quantum computation to factorize a number a, with $0 \leq a \leq q$. This number a can be expressed in qubits as

$$|a\rangle \equiv |\vec{\varepsilon}\rangle = |\varepsilon_{1,}\ \varepsilon_{2,}\ \cdots, \varepsilon_N\rangle,\ \varepsilon_i \in \{0,1\}, \tag{A4}$$

where $\vec{\varepsilon}$ is the binary notation of a. Here N is the lowest integer greater than or equal to $\log_2 q$. The state $|a\rangle$ is a product multiparticle SU(2) coherent state for $j = 1/2$. The quantum Fourier transform of the state $|a\rangle$ is

$$q^{-1/2}\sum_{c=0}^{q-1}\exp(i2\pi ac/q)|c\rangle. \tag{A5}$$

The state $|c\rangle$ is also a product state for all c such that $0 \leq c \leq q-1$. We can express the transformed state as the entangled SU(2) coherent state (A1) such that

$$\begin{aligned} & q^{-1/2}\sum_{\vec{\varepsilon}_i\in\{0,1\}}\exp(i2\pi ac/q)|\vec{\varepsilon}\rangle \\ &= q^{-1/2}\sum_{\vec{\gamma}_i\in\{0,\infty\}}\exp(i2\pi ac/q)|j\ \vec{\gamma}\rangle \end{aligned} \tag{A6}$$

where $c \equiv \vec{\varepsilon}$. The above state can also be written in the form of (A1) with

$$f(\vec{\xi}) = q^{-1/2}\sum_{\vec{\gamma}_i\in\{0,\infty\}}\exp(i2\pi ac/q)\delta(\vec{\xi}-\vec{\gamma}_i). \tag{A7}$$

Hence the Fourier transform state (A5) can be treated within the formalism of entangled SU(2) coherent states.

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