A Remark on One-Dimensional Many-Body Problems with Point Interactions

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Abstract

The integrability of one dimensional quantum mechanical many-body problems with general contact interactions is extensively studied. It is shown that besides the pure (repulsive or attractive) δ-function interaction there is another singular point interactions which gives rise to a new one-parameter family of integrable quantum mechanical many-body systems. The bound states and scattering matrices are calculated for both bosonic and fermionic statistics.

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Quantum mechanical solvable models describing a particle moving in a local singular potential concentrated at one or a discrete number of points have been extensively discussed in the literature, see e.g. [1, 2, 3] and references therein. One dimensional problems with contact interactions at, say, the origin ($x = 0$) can be characterized by separated or nonseparated boundary conditions imposed on the (scalar) wave function φ at $x = 0$. The classification of one dimensional point interactions in terms of singular perturbations is given in [4]. In the present paper we are interested in many-body problems with pairwise interactions given by such singular potentials. The first model of this type with the pairwise interactions determined by δ-functions was suggested and investigated in [5]. Intensive studies of this model applied to statistical mechanics (particles having boson or fermion statistics) are given in [6, 7] (these also leads to the well known Yang-Baxter equations).

Nonseparated boundary conditions correspond to the cases where the perturbed operator is equal to the orthogonal sum of two self-adjoint operators in $L_2(-\infty, 0]$ and $L_2[0, \infty)$. The family of point interactions for the one dimensional Schrödinger operator $-\frac{d^2}{dx^2}$ can be described by unitary 2×2 matrices via von Neumann formulas for self-adjoint extensions of symmetric operators, since the second derivative operator restricted to the domain $C_0^\infty(\mathbf{R} \setminus \{0\})$ has deficiency indices $(2, 2)$. The boundary conditions describing the self-adjoint extensions have the following form

$$\left(\begin{array}{c}\varphi\\ \varphi'\end{array}\right)_{0^+} = e^{i\theta}\left(\begin{array}{cc}a & b\\ c & d\end{array}\right)\left(\begin{array}{c}\varphi\\ \varphi'\end{array}\right)_{0^-}, \tag{1}$$

where

$$ad - bc = 1, \qquad \theta, a, b, c, d \in I\!R. \tag{2}$$

$\varphi(x)$ is the scalar wave function of two spinless particles with relative coordinate x. (1) also describes two particles with spin s but without any spin coupling between the particles when they meet (i.e. for $x = 0$), in this case φ represents any one of the components of the wave function. The values $\theta = b = 0$, $a = d = 1$ in (1) correspond to the case of a positive (resp. negative) δ-function potential for $c > 0$ (resp. $c < 0$). For general

a, b, c and d, the properties of the corresponding Hamiltonian systems have been studied in detail, see e.g. [4, 8, 9].

The separated boundary conditions are described by

$$\varphi'(0_+) = h^+\varphi(0_+) \ , \quad \varphi'(0_-) = h^-\varphi(0_-), \tag{3}$$

where $h^\pm \in I\!R \cup \{\infty\}$. $h^+ = \infty$ or $h^- = \infty$ correspond to Dirichlet boundary conditions and $h^+ = 0$ or $h^- = 0$ correspond to Neumann boundary conditions. In this case it is impossible to express the perturbed operator as the orthogonal sum of two self-adjoint operators in $L_2(-\infty, 0]$ and $L_2[0, \infty)$.

In the following we study the integrability of one dimensional systems of N-identical particles with general contact interactions described by the boundary conditions (1) or (3) that are imposed on the relative coordinates of the particles. We first consider the case of two particles ($N = 2$) with coordinates x_1, x_2 and momenta k_1, k_2 respectively. Each particle has n-'spin' states designated by s_1 and s_2, $1 \leq s_i \leq n$. For $x_1 \neq x_2$, these two particles are free. The wave functions φ are symmetric (resp. antisymmetric) with respect to the interchange $(x_1, s_1) \leftrightarrow (x_2, s_2)$ for bosons (resp. fermions). In the region $x_1 < x_2$, from the Bethe ansatz the wave function is of the form,

$$\varphi = \alpha_{12} e^{i(k_1x_1+k_2x_2)} + \alpha_{21} e^{i(k_2x_1+k_1x_2)}, \tag{4}$$

where α_{12} and α_{21} are $n^2 \times 1$ column matrices. In the region $x_1 > x_2$,

$$\varphi = (P^{12}\alpha_{12}) e^{i(k_1x_2+k_2x_1)} + (P^{12}\alpha_{21}) e^{i(k_2x_2+k_1x_1)}, \tag{5}$$

where according to the symmetry or antisymmetry conditions, $P^{12} = p^{12}$ for bosons and $P^{12} = -p^{12}$ for fermions, p^{12} being the operator on the $n^2 \times 1$ column that interchanges $s_1 \leftrightarrow s_2$.

Let $k_{12} = (k_1 - k_2)/2$. In the center of mass coordinate $X = (x_1 + x_2)/2$ and the relative coordinate $x = x_2 - x_1$, we get, by substituting (4) and (5) into the boundary conditions at $x = 0$,

$$\left\{ \begin{array}{l} \alpha_{12} + \alpha_{21} = e^{i\theta} a P^{12}(\alpha_{12} + \alpha_{21}) + i e^{i\theta} b k_{12} P^{12}(\alpha_{12} - \alpha_{21}), \\ i k_{12}(\alpha_{21} - \alpha_{12}) = e^{i\theta} c P^{12}(\alpha_{12} + \alpha_{21}) + i e^{i\theta} d k_{12} P^{12}(\alpha_{12} - \alpha_{21}) \end{array} \right. \tag{6}$$

for boundary condition (1), and

$$\left\{\begin{array}{l} ik_{12}(\alpha_{21}-\alpha_{12}) = h_+(\alpha_{12}+\alpha_{21})\ , \\ ik_{12}P^{12}(\alpha_{12}-\alpha_{21}) = h_-P^{12}(\alpha_{12}+\alpha_{21}) \end{array}\right. \tag{7}$$

for boundary condition (3) respectively.

Eliminating the term $P^{12}\alpha_{12}$ from (6) we obtain the relation

$$\alpha_{21} = Y_{21}^{12}\alpha_{12}\ , \tag{8}$$

where

$$Y_{21}^{12} = \frac{2ie^{i\theta}k_{12}P^{12} + ik_{12}(a-d) + (k_{12})^2 b + c}{ik_{12}(a+d) + (k_{12})^2 b - c}. \tag{9}$$

We remark that the system (7) is contradictory unless

$$h_+ = -h_- \doteq h \in I\!R \cup \{\infty\}. \tag{10}$$

In this case it also leads to equation (8) with

$$Y_{21}^{12} = \frac{ik_{12}+h}{ik_{12}-h}\ . \tag{11}$$

For $N \geq 3$ and $x_1 < x_2 < ... < x_N$, the wave function is given by

$$\psi = \alpha_{12...N}e^{i(k_1x_1+k_2x_2+...+k_Nx_N)} + \alpha_{21...N}e^{i(k_2x_1+k_1x_2+...+k_Nx_N)} + (N!-2)\ \textit{other terms}. \tag{12}$$

The columns α have $n^N \times 1$ dimensions. The wave functions in the other regions are determined from (12) by the requirement of symmetry (for bosons) or antisymmetry (for fermions). Along any plane $x_i = x_{i+1}$, $i \in 1, 2, ..., N-1$, from similar considerations as above we have

$$\alpha_{l_1l_2...l_il_{i+1}...l_N} = Y_{l_{i+1}l_i}^{ii+1}\alpha_{l_1l_2...l_{i+1}l_i...l_N}, \tag{13}$$

where

$$Y_{l_{i+1}l_i}^{ii+1} = \frac{2ie^{i\theta}k_{l_il_{i+1}}P^{ii+1} + ik_{l_il_{i+1}}(a-d) + (k_{l_il_{i+1}})^2 b + c}{ik_{l_il_{i+1}}(a+d) + (k_{l_il_{i+1}})^2 b - c} \tag{14}$$

for nonseparated boundary condition and

$$Y_{l_{i+1}l_i}^{ii+1} = \frac{ik_{l_il_{i+1}}+h}{ik_{l_il_{i+1}}-h} \tag{15}$$

for separated boundary condition. Here $k_{l_i l_{i+1}} = (k_{l_i} - k_{l_{i+1}})/2$ play the role of spectral parameters. $P^{ii+1} = p^{ii+1}$ for bosons and $P^{ii+1} = -p^{ii+1}$ for fermions, with p^{ii+1} the operator on the $n^N \times 1$ column that interchanges $s_i \leftrightarrow s_{i+1}$.

For consistency Y must satisfy the Yang-Baxter equation with spectral parameter [6, 10], i.e.,

$$Y_{ij}^{m,m+1} Y_{kj}^{m+1,m+2} Y_{ki}^{m,m+1} = Y_{ki}^{m+1,m+2} Y_{kj}^{m,m+1} Y_{ij}^{m+1,m+2},$$

or

$$Y_{ij}^{mr} Y_{kj}^{rs} Y_{ki}^{mr} = Y_{ki}^{rs} Y_{kj}^{mr} Y_{ij}^{rs} \tag{16}$$

if m, r, s are all unequal, and

$$Y_{ij}^{mr} Y_{ji}^{mr} = 1, \qquad Y_{ij}^{mr} Y_{kl}^{sq} = Y_{kl}^{sq} Y_{ij}^{mr} \tag{17}$$

if m, r, s, q are all unequal.

The operators Y given by (14) satisfy the relation (17) for all θ, a, b, c, d. However the relations (16) are satisfied only when $\theta = 0$, $a = d$ and $b = 0$, that is, according to the constraint (2), $\theta = 0$, $a = d = \pm 1$, $b = 0$, c arbitrary. The case $a = d = 1$, $\theta = b = 0$ corresponds to the usual δ-function interactions, which has been investigated in [6, 7]. The case $a = d = -1$, $\theta = b = 0$, which we shall refer to as 'anti-δ' interaction, is related to another singular interactions between any pair of particles (for $a = d = -1$ and $\theta = b = c = 0$ see [4, 8]). Associated with the separated boundary condition, the operators Y given by (15) satisfy both the relations (17) and (16) for arbitrary h.

We have thus found that with respect to N-particle (either boson or fermion) problems, altogether there are three integrable one parameter families with contact interactions of type δ, anti-δ and separated one, described respectively by one of the following conditions on the wave function along the plane $x_i = x_j$ for any pair of particles with coordinates x_i and x_j,

$$\varphi(0_+) = +\varphi(0_-), \quad \varphi'(0_+) = c\varphi(0_-) + \varphi'(0_-)\ ,\ c \in I\!R\ ; \tag{18}$$

$$\varphi(0_+) = -\varphi(0_-), \quad \varphi'(0_+) = c\varphi(0_-) - \varphi'(0_-)\ ,\ c \in I\!R\ ; \tag{19}$$

$$\varphi'(0_+) = h\varphi(0_+), \quad \varphi'(0_-) = -h\varphi(0_-)\ ,\ h \in I\!R \cup \{\infty\}\ . \tag{20}$$

The wave functions are given by (12) with the α's determined by (13) and initial conditions. The operators Y in (13) are given respectively by

$$Y^{ii+1}_{l_{i+1}l_i} = \frac{i(k_{l_i} - k_{l_{i+1}})P^{ii+1} + c}{i(k_{l_i} - k_{l_{i+1}}) - c}\ ; \tag{21}$$

$$Y^{ii+1}_{l_{i+1}l_i} = -\frac{i(k_{l_i} - k_{l_{i+1}})P^{ii+1} + c}{i(k_{l_i} - k_{l_{i+1}}) + c}\ ; \tag{22}$$

and

$$Y^{ii+1}_{l_{i+1}l_i} = \frac{i(k_{l_i} - k_{l_{i+1}}) + 2h}{i(k_{l_i} - k_{l_{i+1}}) - 2h}\ . \tag{23}$$

Nevertheless, from (21) and (22) we see that if we simultaneously change $c \to -c$ and $P^{ii+1} \to -P^{ii+1}$, these two formulas are interchanged. There is a sort of duality between bosons (resp. fermions) with δ-interaction of strength c and fermions (resp. bosons) with anti-δ interaction of strength $-c$. It can be checked that under the "kink type" gauge transformation $\mathcal{U} = \prod_{i>j}\ \mathrm{sgn}(x_i - x_j)$, the N-boson (resp. fermion) δ-type contact interaction goes over to the N-fermion (resp. boson) anti-δ interaction. Therefore these two situations are in fact unitarily equivalent under a gauge transformation $\mathcal{U}$ that is non-smooth and does not factorize through one particle Hilbert spaces.

The integrable system related to the case (23) is not unitarily equivalent to either the δ or anti-δ cases. In fact their spectra are different (see the bound states below). In the following we study further the one dimensional integrable N-particle systems associated with (23).

When $h < 0$, there exist bound states. For $N = 2$, the space part of the orthogonal basis (labeled by $\pm$) in the doubly degenerate bound state subspace has the form, in the relative coordinate $x = x_2 - x_1$,

$$\psi_{2,\pm} = (\theta(x) \pm \theta(-x))e^{h|x|}. \tag{24}$$

The eigenvalue corresponding to the bound states (24) is $-h^2$. By generalization we get the $2^{N(N-1)/2}$ bound states for N-particle system

$$\psi_{N,\underline{\epsilon}} = \alpha_{\underline{\epsilon}} \prod_{k>l} (\theta(x_k - x_l) + \epsilon_{kl}\theta(x_l - x_k))e^{h\sum_{i>j}|x_i - x_j|}, \tag{25}$$

where $\alpha_{\underline{\epsilon}}$ is the spin wave function and $\underline{\epsilon} \equiv \{\epsilon_{kl} \ : \ k > l\}$; $\epsilon_{kl} = \pm$, labels the $2^{N(N-1)/2}$-fold degeneracy.

It can be checked that $\psi_{N,\underline{\epsilon}}$ satisfies the boundary condition (20) at $x_i = x_j$ for any $i \neq j \in 1, ..., N$. The spin wave function α here satisfies $P^{ij}\alpha = \epsilon_{ij}\alpha$ for any $i \neq j$, that is, $p^{ij}\alpha = \epsilon_{ij}\alpha$ for bosons and $p^{ij}\alpha = -\epsilon_{ij}\alpha$ for fermions. $\psi_{N,\underline{\epsilon}}$ is of the form (12) in each region. For instance comparing $\psi_{N,\underline{\epsilon}}$ with (12) in the region $x_1 < x_2 ... < x_N$, we get

$$k_1 = ih(N-1), \ k_2 = k_1 - 2ih, \ k_3 = k_2 - 2ih, ..., k_N = -k_1. \tag{26}$$

The energy of the bound state $\psi_{N,\underline{\epsilon}}$ is

$$E = -\frac{h^2}{3} N(N^2 - 1) \ . \tag{27}$$

The scattering matrix can readily be discussed. For real $k_1 < k_2 < ...k_N$, in each coordinate region such as $x_1 < x_2 < ...x_N$, the following term in (12) is an outgoing wave

$$\psi_{out} = \alpha_{12...N} e^{k_1 x_1 + ... + k_N x_N}. \tag{28}$$

An incoming wave with the same exponential as (28) is given by

$$\psi_{in} = [P^{1N} P^{2(N-1)} ...] \alpha_{N(N-1)...1} e^{k_N x_N + ... + k_1 x_1} \tag{29}$$

in the region $x_N < x_{N-1} < ... < x_1$. The scattering matrix is defined by $\psi_{out} = S\psi_{in}$. From (13) we have

$$\alpha_{12...N} = [Y_{21}^{12} Y_{31}^{23} ... Y_{N1}^{(N-1)N}] \alpha_{2...N1} = ...$$
$$= [Y_{21}^{12} Y_{31}^{23} ... Y_{N1}^{(N-1)N}][Y_{32}^{12} Y_{42}^{23} ... Y_{N2}^{(N-2)(N-1)}] ... [Y_{N(N-1)}^{12}] \alpha_{N(N-1)...1} \equiv S' \alpha_{N(N-1)...1},$$

where $Y_{l_{i+1} l_i}^{i i+1}$ is given by (23). Therefore

$$S = S' P^{N1} P^{(N-1)2} ... P^{1N} = S' [P^{12}][P^{23} P^{12}][P^{34} P^{23} P^{12}] ... [P^{(N-1)N} ... P^{12}].$$

Defining

$$X_{ij} = Y_{ij}^{ij} P^{ij} \tag{30}$$

we obtain

$$S = [X_{21} X_{31} ... X_{N1}][X_{32} X_{42} ... X_{N2}] ... [X_{N(N-1)}]. \tag{31}$$

The scattering matrix S is unitary and symmetric due to the time reversal invariance of the interactions. $< s_1' s_2' ... s_N' | S | s_1 s_2 ... s_N >$ stands for the S matrix element of the process from the state $(k_1 s_1, k_2 s_2, ..., k_N s_N)$ to the state $(k_1 s_1', k_2 s_2', ..., k_N s_N')$. The momenta (26) are imaginary for bound states. The scattering of clusters (bound states) can be discussed in a similar way as in [7]. For instance for the scattering of a bound state of two particles $(x_1 < x_2)$ on a bound state of three particles $(x_3 < x_4 < x_5)$, the scattering matrix is $S = [X_{32} X_{42} X_{52}][X_{31} X_{41} X_{51}]$.

We have extensively investigated the integrability of one dimensional quantum mechanical many-body problems with general contact interactions. Besides the repulsive or attractive δ and anti-δ function interactions, there is another integrable one parameter families associated with separated boundary conditions. From our calculations it is clear that these are all the integrable systems for one dimensional quantum identical many-particle models (of fermionic or bosonic statistics) with contact interactions. Here the possible contact coupling of the spins of two particles are not taken into account. A further study along this direction would possibly give rise to more interesting integrable quantum many-body systems.

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References

[1] S. Albeverio, F. Gesztesy, R. Høegh-Krohn and H. Holden, *Solvable Models in Quantum Mechanics*, New York: Springer, 1988.

[2] M. Gaudin, *La fonction d'onde de Bethe*, Masson, 1983.

[3] S. Albeverio and R. Kurasov, *Singular perturbations of differential operators and solvable Schrödinger type operators*, Cambridge Univ. Press., to appear in 1999.

[4] P. Kurasov, *Distribution theory for discontinuous test functions and differential operators with generalized coefficients*, J. Math. Analy. Appl. **201**(1996)297-323.

[5] J.B. McGuire, Study of exactly soluble one–dimensional N–body problems, *J.Math.Phys.*, **5** (1964), 622–636.

J.B. McGuire, Interacting fermions in one dimension.I. Repulsive potential, *J.Math.Phys.*, **6** (1965), 432–439.

J.B. McGuire, Interacting fermions in one dimension.II. Attractive potential, *J.Math.Phys.*, **7** (1966), 123–132.

J.B. McGuire and C.A. Hurst, The scattering of three impenetrable particles in one dimension, *J.Math.Phys.*, **13** (1972), 1595–1607.

J.B. McGuire and C.A.Hurst, Three interacting particles in one dimension: an algebraic approach, *J.Math.Phys.*, **29** (1988), 155–168.

[6] C.N. Yang, *Some exact results for the many-body problem in one dimension with repulsive delta-function interaction*, Phys. Rev. Lett. **19**(1967)1312-1315.

C.N. Yang, *S matrix for the one-dimensional N-body problem with repulsive δ-function interaction*, Phys. Rev. **168**(1968)1920-1923.

[7] C.H. Gu and C.N. Yang, *A one-dimensional N Fermion problem with factorized S matrix*, Commun. Math. Phys. **122** (1989)105-116.

[8] P. Chernoff and R. Hughes, *A new class of point interactions in one dimension*, J. Func. Anal. **111**(1993)97-117.

[9] S. Albeverio, Z. Brzeźniak and L Dąbrowski, *Time-dependent propagator with point interaction*, J. Phys. A**27**(1994)4933-4943.

[10] Z.Q. Ma, *Yang-Baxter Equation and Quantum Enveloping Algebras*, World Scientific, 1993.

V. Chari and A. Pressley, *A Guide to Quantum Groups*, Cambridge University Press, 1994.

C. Kassel, *Quantum Groups*, Springer-Verlag, New-York, 1995.

S. Majid, *Foundations of Quantum Group Theory*, Cambridge University Press, 1995.

K. Schmüdgen, *Quantum Groups and Their Representations*, Springer, 1997.