

# From time inversion to nonlinear QED\*

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In Minkowski flat space-time, it is perceived that time inversion is unitary rather than antiunitary, with energy being a time vector changing sign under time inversion. The Dirac equation, in the case of electromagnetic interaction, is not invariant under unitary time inversion, giving rise to a “Klein paradox”. To render unitary time inversion invariance, a nonlinear wave equation is constructed, in which the “Klein paradox” disappears. In the case of Coulomb interaction, the revised nonlinear equation can be linearized to give energy solutions for Hydrogen-like ions without singularity when nuclear number  $Z > 137$ , showing a reversed energy order pending for experimental tests such as Zeeman effects. In non-relativistic limit, this nonlinear equation reduces to nonlinear Schrödinger equation with soliton-like solutions. Moreover, particle conjugation and electron-proton scattering with a nonsingular current-potential interaction are discussed. Finally the explicit form of gauge function is found, the uniqueness of Lorentz gauge is proven and the Lagrangian density of quantum electrodynamics (QED) is revised as well. The implementation of unitary time inversion leads to the ultimate derivation of nonlinear QED.

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# 1. INTRODUCTION

Decades ago, Wigner [1] first pointed out the significance of time inversion and later made detailed discussions on antiunitary time inversion (Wigner [2]). His theory about time inversion is based on a classical motion picture (Wigner [3]): “Time inversion . . . replaces every velocity by the opposite velocity, so that the position of particles at  $+t$  becomes the same as it was, without time inversion at  $-t$ . . . .”

Also well known is Einstein’s relativity theory (Einstein [4]) that completely changes our perception of space and time, as best manifested by Minkowski’s “world-postulate” (see Minkowski’s paper in [4]). Minkowski unifies space and time by introducing time as an independent coordinate in addition to three space coordinates. Usually “Minkowski space-time” refers to four-dimensional flat space-time in special relativity, in which relativistic quantum mechanics and quantum electrodynamics (QED) are established.

On the development of quantum mechanics and relativity, there always exist heated debates on the physical meaning of the theories. Dirac however was indifferent in such debates. Instead he had been looking for mathematical possibilities to reconcile quantum mechanics and relativity. As a consequence, he founded relativistic quantum mechanics with a relativistic wave equation called “Dirac equation” (Dirac [5]). However, the Dirac equation was also not perfect. In the following year, Klein [6] found a paradox in the Dirac equation. Now seven decades has gone by, the “Klein paradox” has still not been resolved mathematically, though it can be explained away by physical reasoning.

This paper is organized as follows. Given that a correct concept in Newtonian classical mechanics may not be necessarily correct in Einsteinian relativistic mechanics, first I am going to show that the above mentioned classical motion picture is just one of those concepts, correct classically but not relativistically. Then I will proceed to clarify what is supposed to mean to make a time inversion in special relativity. Based on the general principles of quantum mechanics and special relativity, it can be shown that time inversion turns out to be a unitary transformation with energy being a time vector changing sign under time inversion in Minkowski space-time.

With unitary time inversion understood, the next step is to study the transformation of the Dirac equation. For clarity, let us name the Dirac equation in the case of electromagnetic interaction as Dirac EM-equation to distinguish it from the Dirac equation in other cases. What happens is: the Dirac EM-equation is not invariant under unitary time inversion. As a direct consequence of this non-invariance, the Dirac EM-equation has a non-symmetric positive-negative energy spectrum with a mathematical singularity that causes a “Klein paradox”. Here in this paper, I do not intend to argue too much about whether or not the “Klein paradox” has physical meaning. Rather I hold such a point of view that we would be better off from the outset without mathematical singularity.

To implement unitary time inversion, I come up with a revised equation that looks

similar to the Dirac EM-equation on the one hand, while differs from the Dirac EM-equation in several aspects on the other. The main difference is: the revised equation preserves the invariance under unitary time inversion in addition to the invariance under the other Lorentz transformations. Consequently, it gives a positive-negative symmetric energy spectrum without singular crossing point. In the case of Coulomb interaction, the energy bound states for Hydrogen-like ions are solved without singularity when nuclear number  $Z > 137$ , and the order of some energy levels found opposite to the conventional one, which entails further high precision tests from experiments like Zeeman effects.

Another difference is: the Dirac EM-equation, involving the external electromagnetic potential with minimal coupling, is linear on fermion field; while the revised wave equation, involving the electro-dynamical interaction potential, is nonlinear on fermion field. In non-relativistic limit, the Dirac EM-equation reduces to the linear Schrödinger equation, while the revised nonlinear wave equation reduces to the nonlinear Schrödinger equation with soliton-like solutions well known in nonlinear physics.

Typically, when the finite size of fermion is considered, the interaction potential can be generalized by a convolution between four-current and four-potential with an integral over the finite four-dimensional size of the fermion. Using such a nonsingular convolution can avoid the singularity and infinity troubles in calculating the self-energy of electron and the divergent integrals of Feynman Diagrams.

The gauge invariance is to be discussed at the end. To preserve Lorentz invariance including the invariance under unitary time inversion, it can be shown that gauge function is of particular form, not arbitrary, and the only gauge condition is Lorentz gauge. It is also straightforward to write a nonlinear Lagrangian that derives the Maxwell electromagnetic field equation and the revised nonlinear fermion field equation. With the basic principles of quantum field theory, we may then establish a nonlinear QED that would show many interesting applications in experiment. What I am actually up to, is to make necessary improvements on QED in its own framework, namely in the language of space-time, without further drastic changes as in superstring theory. So much has been said, let me now turn into more detailed discussions.

## 2. UNITARY TIME INVERSION

Usually under a coordinate transformation  $x' = ax$ , the transformation of wave function in quantum mechanics is defined by

$$A(a)\Psi(x) = \Psi(ax). \quad (2.1)$$

But antiunitary time inversion is defined by (see Bjorken and Drell [7])

$$T_*\Psi(t, \mathbf{x}) = \Psi^*(-t, \mathbf{x}). \quad (2.2)$$

where the extra complex conjugation  $*$  on wave function is used to comply with the assumption that under time inversion  $(t, \mathbf{x}) \rightarrow (-t, \mathbf{x})$ , energy and momentum transform as  $(E, \mathbf{p}) \rightarrow (E, -\mathbf{p})$ . In this case, the phase of a plane wave  $\phi = \mathbf{p} \cdot \mathbf{x} - Et$  (natural units are used in this paper), is not invariant under antiunitary time inversion. This also leads to another result that the imaginary unit  $i$  has to be supposed to change to  $-i$  under antiunitary time inversion, while constant  $i$  has nothing to do with time.

In quantum mechanics, the group velocity of a wave packet is expressed by  $d\mathbf{x}/dt = dE(\mathbf{p})/d\mathbf{p}$ . By making a positive non-relativistic energy assumption  $E(\mathbf{p}) = \mathbf{p}^2/2m$ , one can then get a classical correspondence  $d\mathbf{x}/dt = \mathbf{p}/m = \mathbf{v}$ , from which the classical motion picture comes. The point is: in the derivation of classical mechanics velocity from quantum mechanics group velocity, the positive energy assumption has been taken, which may not be correct under time inversion in special relativity. It would be logically odd to discuss symmetry under time inversion based on a truncated classical motion picture that has already broken the symmetry.

Now that quantum mechanics is more general than classical mechanics, it is better not to impose any properties for time inversion classically until we obtain certain results from quantum mechanics. As we understand, time inversion is nothing but a kind of coordinate transformation in space-time, and it is natural to adapt the general definition (2.1) for time inversion rather than to follow the truncated classical motion picture. Therefore putting the general principles of quantum mechanics in the first place, we define time inversion as:

$$\mathsf{T}\Psi(t, \mathbf{x}) = \Psi(-t, \mathbf{x}). \quad (2.3)$$

For a plane wave  $\Psi(t, \mathbf{x}) = C(\mathbf{p}, E) \exp[i(\mathbf{p} \cdot \mathbf{x} - Et)]$ , it is clear that the expectation value of space momentum operator  $\hat{\mathbf{p}} = -i\partial_{\mathbf{x}}$  remains the same and that of energy operator  $\hat{E} = i\partial_t$  changes sign under time inversion. Any wave function can be Fourier-transformed into a combination of plane waves. So it is easy to check these results hold for any kind of matter waves.

In 4-d space-time, the space-time interval squared  $\Delta X_{\mu}\Delta X^{\mu} = g^{\mu\nu}\Delta X_{\mu}\Delta X_{\nu}$  is Lorentz invariant, where  $g^{00} = -g^{ii} = 1 (i = 1, 2, 3)$  and  $g^{\mu\nu} = 0 (\mu \neq \nu)$ . Mathematically if contravariant four-coordinate transforms as  $X'^{\mu} = a^{\mu}_{\nu}X^{\nu}$ , then covariant four-coordinate transforms in the way  $X'_{\mu} = a^{\nu}_{\mu}X_{\nu}$ . Since  $\Delta X'_{\mu}\Delta X'^{\mu} = \Delta X_{\mu}\Delta X^{\mu}$ , we get  $a^{-1} = ga^{\sim}g$  where  $a^{\sim}$  is the transpose of  $a$ , leading to  $\det(a^{-1}) = \det(a) = \pm 1$  for proper and improper Lorentz transformations respectively. On the other hand, the phase of a plane wave  $\phi = -P_{\mu}X^{\mu}$  characterizing the physical correlation of space-time world-points, is also invariant under homogeneous Lorentz transformations, therefore four-momentum must transform in a covariant way  $P'_{\mu} = a^{\nu}_{\mu}P_{\nu}$ . In the case of time inversion  $(t, \mathbf{x}) \rightarrow (-t, \mathbf{x})$ , it is naturally found an energy-momentum transformation

$$(E, \mathbf{p}) \rightarrow (-E, \mathbf{p}). \quad (2.4)$$

Moreover, if a plane wave with a constant four-momentum  $P_\mu = (E, \mathbf{p})$  travels from space-time point 1 to 2, the phase difference between them  $\Delta\phi = \phi_2 - \phi_1$  describing the causality of this process, is an invariant under all inhomogeneous Lorentz transformations  $X'^\mu = a^\mu_\nu X^\nu + b^\mu$ . The phase differences between points 1 and 2 before and after space-time inversions can be written down as:  $\Delta\phi = \mathbf{p} \cdot (\mathbf{x}_2 - \mathbf{x}_1) - E(t_2 - t_1)$  and  $\Delta\phi' = \mathbf{p}' \cdot (\mathbf{x}'_2 - \mathbf{x}'_1) - E'(t'_2 - t'_1)$ . Since  $\Delta\phi' = \Delta\phi$ , we obtain  $\mathbf{p}' = -\mathbf{p}$  under space inversion  $\mathbf{x}'_2 - \mathbf{x}'_1 = -(\mathbf{x}_2 - \mathbf{x}_1)$ , and  $E' = -E$  under time inversion  $t'_2 - t'_1 = -(t_2 - t_1)$ . It is such a Lorentz invariant physical phase space that determines the unique energy-momentum transformation  $(E, \mathbf{p}) \rightarrow (-E, \mathbf{p})$  under time inversion.

Actually four-vector momentum  $P_\mu = (E, \mathbf{p})$  consists of time component  $E$  and space component  $\mathbf{p}$ . While momentum  $\mathbf{p}$  is usually known as a space vector changing sign under space inversion, energy  $E$  may be similarly regarded as a time vector changing sign under time inversion. The idea of uncertainty principles in quantum mechanics:  $\Delta\mathbf{p}\Delta\mathbf{x} \sim 1$  and  $\Delta E\Delta t \sim 1$ , is that the strict measurement of momentum and position as well as that of energy and time can not be taken simultaneously since they do not commute, but they do have close relationship with each other. The determination of momentum is tightly related to that of position not to that of time. Similarly the measurement of energy is directly connected to that of time not to that of position. This one-to-one correspondence reveals the following fact that 4-d space-time and energy-momentum, combined into a Lorentz invariant phase space, are the reciprocal systems relatively to each other

$$P_\mu = i\partial_\mu, \quad (2.5)$$

with a commutator relation  $[X^\mu, P_\mu] = -i$ .

The Fourier transformation used in quantum mechanics also shows a good example of this one-to-one correspondence. According to this correspondence, we prefer to call 4-d energy-momentum system, the “reciprocal world” of space-time. Generally speaking, this is only a question of using different representations or interchangeable languages. There is nothing special, in the sense of relativistic covariance, that producing an inversion in space-time world is equivalent to making a corresponding inversion in its energy-momentum reciprocal world by the hint of (2.5). This idea now enables us to understand why momentum switches sign under space inversion while energy reverses sign under time inversion.

The physical picture becomes clear if in a “local framework”, we do not impose absolute future and past concepts by setting a standard clock A but rather reverse time axis by setting another clock B running counterclockwise. If a plane wave is propagating toward the future with a positive (or negative) frequency by the standard clock A, then it can also be equivalently looked upon as propagating toward the past with a negative (or positive) frequency by the other clock B. On the principle of special relativity, it is arbitrary in setting clocks in locally flat space-time. Therefore this plane wave may have

either positive or negative energy depending on which clock we use.

By the correspondence between space-time and energy-momentum worlds, if we indeed want to introduce a time inversion concept in Lorentz group, then it is natural to introduce a corresponding “energy inversion” concept to map these two worlds completely. Once again we would like to emphasize, based on what we try to clarify here from the intrinsic structures of space-time world and energy-momentum reciprocal world, and from the general principles of special relativity and quantum mechanics, that in Minkowski flat space-time, energy is not a scalar any more, it is a “time vector” changing sign under time inversion.

Similar to space inversion, by definition (2.3) it is easy to prove: (a)  $T$  is linear; (b)  $T^2 = 1$ ; (c)  $T^* = T$ ; (d)  $T^\sim = T$ . So time inversion operator is unitary

$$T^{-1} = T = T^\dagger. \quad (2.6)$$

To find its eigenvalues let  $T\Psi = \lambda\Psi$ . From relation (b) we get  $\lambda = \pm 1$  representing even and odd “time parities” respectively.

For a free relativistic particle with an energy-momentum relation  $E^2 - \mathbf{p}^2 = m^2$ , the common eigenfunctions of momentum operator  $\hat{p} = -i\partial_{\mathbf{x}}$  and energy operator  $\hat{E} = i\partial_t$  are expressed by  $\Phi_{\pm}(t, \mathbf{x}) = C(\mathbf{p}, E)\phi_{\mathbf{p}}(\mathbf{x})\exp(\pm iEt)$ . Obviously they are not the eigenfunctions of time inversion operator  $T$  since  $T$  does not commute with  $\hat{E}$ :  $[T, \hat{E}] \neq 0$ . Now make linear combinations:  $\Psi_+ = (\Phi_+ + \Phi_-)/2$  and  $\Psi_- = (\Phi_+ - \Phi_-)/2i$ , which are the common eigenstates of  $\hat{p}$  and  $T$  but not those of  $\hat{E}$  any more. Generally any state wave function can be divided by  $\Psi = \Psi_+ + \Psi_-$ , where the even and odd time parity eigenstates are  $\Psi_{\pm} = (1 \pm T)\Psi/2$  respectively. And any operator can be expressed by  $\hat{W} = \hat{W}_+ + \hat{W}_-$ , where  $\hat{W}_{\pm} = (\hat{W} \pm T\hat{W}T)/2$  and  $T\hat{W}_{\pm}T = \pm\hat{W}_{\pm}$ . Here  $\hat{W}_+$  represents the even time parity operator such as momentum  $\hat{p} = -i\partial_{\mathbf{x}}$  and  $\hat{W}_-$  represents the odd time parity operator such as energy  $\hat{E} = i\partial_t$ .

### 3. DIRAC EQUATION REVISITED

At the end of 19th century, Zeeman [8] detected the splitting of spectral lines under the influence of external magnetic fields, revealing the existence of electron spin. To explain the Zeeman effect, the imagination of electron structure has been made: electron has a magnetic moment and thus has an intrinsic spin angular momentum instead of a classical point particle (Lorentz [9]). It has been clarified that electron spin relates to the extra degrees of freedom and can be well described by a two component complex variable called “spinor”. One is then faced with such a typical question: how to construct a minimal complete set of variables in the combination of space-time and intrinsic spin. In Dirac’s mind (Dirac [10]), spinors, like tensors, are geometrical objects, yielding covariant transformations with respect to Minkowski space-time, and specifically under

Lorentz transformations, the Dirac equation is to obey the covariance principle of special relativity. Along this line of thought, we may say the whole domain of space-time and intrinsic spin is Lorentz invariant, and name it “common space” for clarity.

A common variable is now defined by

$$\Omega = \gamma^\mu X_\mu, \quad (3.1)$$

where matrices  $\gamma^\mu (\mu = 0, 1, 2, 3)$ , in the representation of Bjorken and Drell [7], are ascribed to the spin geometrical factors. The common variable constructed in this way is a function of space-time coordinates and also a metrical quantity. The common variable interval squared becomes  $\Delta\Omega^2 = \Delta X_\mu \Delta X^\mu$ , leading to the anticommutation relations of Dirac matrices  $\{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu}$ . Moreover, the derivative with respect to common variable can be deduced as

$$\partial_\omega = \left(\frac{1}{\partial_\mu \Omega}\right) \partial_\mu = \gamma^\mu \partial_\mu, \quad (3.2)$$

and a momentum operator in common space is naturally introduced as

$$P_\omega = i\partial_\omega = \gamma^\mu P_\mu. \quad (3.3)$$

By (3.1) and (3.3) it is straightforward to prove a commutator relation

$$[\Omega, P_\omega] = -4i. \quad (3.4)$$

If noticing that the common momentum squared of a free particle with mass  $m$  is constant  $P_\omega^2 = E^2 - \mathbf{p}^2 = m^2$ , we find  $P_\omega$  has two eigenvalues  $\pm m$ . If choosing  $+m$  eigenvalue, we arrive at the Dirac equation of free spin one-half particle

$$\gamma^\mu P_\mu \Psi(x) = m\Psi(x), \quad (3.5)$$

The Dirac equation, on the principle of special relativity, is required to be invariant under Lorentz transformation  $L_\omega$  in common space, which is a direct product of spinor one  $L_s$  and coordinate one  $L_c$ :  $L_\omega = L_s L_c$ . Making  $L_\omega$  on both sides of (3.5) and noting that  $L_s$  commutes with space-time vectors and  $L_c$  commutes with spinor vectors, we have

$$L_s \gamma^\mu L_s^{-1} L_c P_\mu L_c^{-1} L_\omega \Psi(x) = m L_\omega \Psi(x). \quad (3.6)$$

Since momentum  $P_\mu$  is a covariant four-vector in space-time,

$$L_c P_\mu L_c^{-1} = a_\mu^\nu P_\nu, \quad (3.7)$$

matrices  $\gamma^\mu$  must be correspondingly contravariant in spinor space,

$$L_s \gamma^\mu L_s^{-1} = a_\nu^\mu \gamma^\nu. \quad (3.8)$$

The above discussions also hold for two special cases. First, there is a unitary space inversion in common space  $S_\omega = S_s S_c$ , where  $S_s = \gamma^0$  is a unitary space inversion in spinor space:

$$S_s \gamma^0 S_s^{-1} = \gamma^0, S_s \boldsymbol{\gamma} S_s^{-1} = -\boldsymbol{\gamma}, \quad (3.9)$$

and  $S_c$  is a unitary space inversion in space-time, defined as usual in quantum mechanics. Second, there is a unitary time inversion in common space  $T_\omega = T_s T_c$ , where  $T_s = \gamma^1 \gamma^2 \gamma^3$  is a unitary time inversion in spinor space:

$$T_s \gamma^0 T_s^{-1} = -\gamma^0, T_s \boldsymbol{\gamma} T_s^{-1} = \boldsymbol{\gamma}, \quad (3.10)$$

and  $T_c$  is a unitary time inversion in space-time, defined by (2.3).

When a spin one-half particle is put into an external Maxwell electromagnetic field, the Dirac EM-equation with minimal coupling can be written down in a general form (Dirac [5] [11]):

$$\gamma^\mu (P_\mu - e A_\mu) \Psi(x) = m \Psi(x), \quad (3.11)$$

where  $A_\mu = (\Phi, \mathbf{A})$  is the space-time four-potential of electromagnetic field, proposed by Minkowski [12]. Under time inversion, time component  $\Phi$  does not change, while space vector  $\mathbf{A}$  is supposed to change sign. The Minkowski four-potential constructed in this way is therefore “time-anticovariant” under time inversion:

$$(\Phi, \mathbf{A}) \rightarrow (\Phi, -\mathbf{A}). \quad (3.12)$$

If four-momentum  $P_\mu$  is also “time-anticovariant”, one may make spinor space yield a sort of transformation to keep the equation invariant as done in conventional theory. However from what we have shown in the last section, four-momentum is not “time-anticovariant”, but rather “time-covariant” (2.4), just as it is covariant under all the other Lorentz transformations. Here exists a nontrivial dilemma that the summation of “time-covariant” four-momentum and “time-anticovariant” Minkowski four-potential multiplied by charge and a negative sign is neither time-covariant nor time-anticovariant, keeping in no way the Dirac EM-equation invariant under time inversion.

This dilemma gives rise to an unsymmetrical positive-negative energy spectrum when we solve this equation. Consequently when the electric potential  $\Phi$  such as Coulomb potential is strong enough, the positive and negative energy spectra will be shifted to overlap, leading to a mathematical singularity difficulty - “Klein paradox” (Klein [6]). Particularly, for a Hydrogen-like ion there is no real bound energy solution but appears a singularity when the nuclear number of the ion is larger than 137.

At first, Dirac could hardly understand why his equation has negative energy solutions that seem “nonphysical”. To find an explanation, Dirac constructed a hole theory with a vacuum of fully filled negative energy states (Dirac [13]). The predicted positron was found a few years later (Anderson [14]). Despite Dirac’s profound prediction of



antiparticles, the hole theory has difficulties such as infinite density, infinite negative energy and vacuum fluctuation in the vacuum state. Based on the hole theory, the “Klein paradox” is explained in terms of vacuum polarization accompanying particle-antiparticle pair production and annihilation under strong fields. Along this line of work, one has been trying to explain away the “Klein paradox” rather than to avoid it. Eventually, one is led to such a conclusion: “‘Klein paradox’ is not a paradox”.

Just as in classical mechanics, energy in the Dirac hole theory is still viewed as a scalar: namely negative energy is always “lower” than positive energy. If energy in special relativity is a frame-dependent “time vector” rather than just a scalar, as we have shown earlier, then it makes little sense to insert negative energy into vacuum or explain the “Klein paradox” as vacuum polarization. There should exist a more fundamental solution for these problems. In the next section, we would like to provide a down-to-earth approach, to remove the “Klein paradox” by revising the Dirac EM-equation. It sounds a little radical. Indeed this can not be done without the understanding that time inversion is a unitary transformation and energy is a “time vector” changing sign under time inversion in Minkowski space-time.

## 4. NONLINEAR EQUATION RECONSTRUCTED

As shown in §2, energy is a time vector changing sign under time inversion. Therefore the whole energy spectrum we obtain from whatever relativistic wave equation we might construct, must be positive-negative symmetric. This is, on the other hand, a necessary check if the constructed equation is truly invariant under unitary time inversion or not. By looking into the Dirac EM-equation, we find that the non-symmetric energy spectrum is caused by a matrix  $\beta$  in front of electric potential  $\Phi$ . If  $\beta$  is removed there, the equation becomes invariant under unitary time inversion. But it is no longer invariant under the other proper Lorentz transformations, if  $\Phi$  and  $\mathbf{A}$  are still external potentials. After careful checking, we realize that interaction potentials have to be utilized in lieu of external potentials in a special way, to render the invariance under all Lorentz transformations. The following equation is what we get:

$$(\gamma^\mu P_\mu - e\Phi^{\mathbf{I}} + e\boldsymbol{\gamma} \cdot \mathbf{A}^{\mathbf{I}})\Psi(x) = m\Psi(x), \quad (4.1)$$

where  $\Phi^{\mathbf{I}}$  and  $\mathbf{A}^{\mathbf{I}}$  are the interaction potentials related to the external potentials  $\Phi$  and  $\mathbf{A}$ . We are going to derive their relations by considering Lorentz transformations, among which time inversion is defined in the unitary way we presented earlier.

As in §3, one can see this modified equation is invariant under unitary space inversion  $S_\omega = \gamma^0 S_c$  and time inversion  $T_\omega = \gamma^1 \gamma^2 \gamma^3 T_c$  where  $T_c$  is defined by (2.3), by assuming that scalar potential  $\Phi^{\mathbf{I}}$  does not change sign under space-time inversions while space vector potential  $\mathbf{A}^{\mathbf{I}}$  changes sign under space inversion but not under time inversion.

These assumptions can be directly verified after we derive the explicit forms of  $\Phi^{\mathbf{I}}$  and  $\mathbf{A}^{\mathbf{I}}$  in terms of  $\Phi$  and  $\mathbf{A}$ .

Below we are going to prove that the modified equation (4.1) does give a symmetric positive-negative energy spectrum, to enhance our confidence that this equation is indeed invariant under unitary time inversion. From the equation (4.1), the energy operator is expressed in the way

$$\hat{\mathbf{E}} = \rho_1 \boldsymbol{\sigma} \cdot (\mathbf{p} - e\mathbf{A}^{\mathbf{I}}) + \rho_3(m + e\Phi^{\mathbf{I}}), \quad (4.2)$$

where

$$\rho_1 = \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}, \rho_2 = \begin{pmatrix} 0 & -iI \\ iI & 0 \end{pmatrix}, \rho_3 = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}, \quad (4.3)$$

form a set of  $4 \times 4$  matrices analogous to  $2 \times 2$  Pauli matrices  $\sigma_i (i = 1, 2, 3)$ . Here  $I$  is a  $2 \times 2$  unit matrix. To diagonalize (4.2), take a look at the square of  $\hat{\mathbf{E}}$

$$\hat{\mathbf{E}}^2 = (\mathbf{p} - e\mathbf{A}^{\mathbf{I}})^2 - e\boldsymbol{\sigma} \cdot (\nabla \times \mathbf{A}^{\mathbf{I}} + \mathbf{A}^{\mathbf{I}} \times \nabla) + (m + e\Phi^{\mathbf{I}})^2 - e\rho_2 \boldsymbol{\sigma} \cdot [\nabla, \Phi^{\mathbf{I}}]. \quad (4.4)$$

Making a unitary transformation ( $U^{-1} = U^\dagger$ )

$$U = \frac{1}{\sqrt{2}} \begin{pmatrix} I & iI \\ iI & I \end{pmatrix}, \quad (4.5)$$

on both sides of (4.4) we can diagonalize it by using

$$U\rho_2U^{-1} = -\rho_3, \quad (4.6)$$

to the following form

$$\begin{aligned} \hat{\mathbf{E}}'^2 &= U\hat{\mathbf{E}}^2U^{-1} = \begin{pmatrix} \hat{\mathbf{E}}_u^2 & 0 \\ 0 & \hat{\mathbf{E}}_l^2 \end{pmatrix} \\ &= (\mathbf{p} - e\mathbf{A}^{\mathbf{I}})^2 - e\boldsymbol{\sigma} \cdot (\nabla \times \mathbf{A}^{\mathbf{I}} + \mathbf{A}^{\mathbf{I}} \times \nabla) + (m + e\Phi^{\mathbf{I}})^2 + e\rho_3 \boldsymbol{\sigma} \cdot [\nabla, \Phi^{\mathbf{I}}]. \end{aligned} \quad (4.7)$$

If supposing the following solutions

$$\hat{\mathbf{E}}_u^2 \chi_u = E_u^2 \chi_u, \quad \hat{\mathbf{E}}_l^2 \chi_l = E_l^2 \chi_l, \quad (4.8)$$

we can rewrite (4.7) as

$$\hat{\mathbf{E}}'^2 = \sum_{\chi} (|\chi_u \rangle E_u^2 \langle \chi_u| + |\chi_l \rangle E_l^2 \langle \chi_l|). \quad (4.9)$$

Transforming back to

$$\psi_u = U^{-1}\chi_u, \quad \psi_l = U^{-1}\chi_l, \quad (4.10)$$

we get

$$\hat{\mathbf{E}}^2 = \sum_{\psi} (|\psi_u \rangle E_u^2 \langle \psi_u| + |\psi_l \rangle E_l^2 \langle \psi_l|). \quad (4.11)$$

It is straightforward to check that the energy operator turns out to be

$$\hat{E} = \sum_{\psi, \lambda} (|\psi_u > \lambda E_u < \psi_u| + |\psi_l > \lambda E_l < \psi_l|), \quad (4.12)$$

with  $\lambda = \pm 1$ , which indeed gives a symmetric positive-negative energy spectrum.

To keep (4.1) invariant under the proper Lorentz transformations

$$\Psi' = L_p \Psi, \bar{\Psi}' = \bar{\Psi} L_p^{-1}, \quad (4.13)$$

where  $\bar{\Psi} = \Psi^\dagger \gamma^0$ , we need to let the following term:

$$\bar{\Psi}(\Phi^{\mathbf{I}} - \boldsymbol{\gamma} \cdot \mathbf{A}^{\mathbf{I}})\Psi \equiv \bar{\Psi} \Psi J^\mu A'_\mu, \quad (4.14)$$

be invariant, where

$$J_\mu = \bar{\Psi} \gamma_\mu \Psi, \quad (4.15)$$

and

$$A'_\mu = \left( \frac{\Phi^{\mathbf{I}}}{\bar{\Psi}^\dagger \Psi}, \frac{\mathbf{A}^{\mathbf{I}}}{\bar{\Psi} \Psi} \right). \quad (4.16)$$

Since  $\bar{\Psi} \Psi$  is a proper Lorentz scalar and  $J_\mu$  is a proper Lorentz four-current,  $A'_\mu$  must be a proper Lorentz four-vector as well. If we define the following correspondence between the interaction and external potentials:

$$\Phi^{\mathbf{I}} = (\Psi^\dagger \Psi) \Phi, \quad (4.17a)$$

$$\mathbf{A}^{\mathbf{I}} = (\bar{\Psi} \Psi) \mathbf{A}, \quad (4.17b)$$

then we see  $A'_\mu$  becomes the Minkowski four-potential  $A_\mu = (\Phi, \mathbf{A})$ .

Considering a four-current  $eJ_\mu$  as a source of four-potential  $A_\mu$ , we have a Poisson equation

$$\partial_\rho \partial^\rho A_\mu = eJ_\mu, \quad (4.18)$$

which is the Maxwell equation under a Lorentz gauge

$$\partial^\mu A_\mu = 0, \quad (4.19)$$

with a current continuity equation derivable from (4.1)

$$\partial^\mu J_\mu = 0. \quad (4.20)$$

This whole set of the Maxwell equations is invariant under the proper Lorentz transformations as well as unitary space-time inversions, although the Minkowski four-potential and four-current are “both” time-anticovariant.

Take a look at an example: if an electron feels the external force driven by a proton, from the correspondence (4.17) and the Maxwell equation (4.18) we get the following potential expressions

$$\Phi^{\mathbf{I}}(x) = \Psi_e^\dagger(x)\Psi_e(x) \int d^4x' [e_p \Psi_p^\dagger(x')\Psi_p(x')]G(x, x'), \quad (4.21a)$$

$$\mathbf{A}^{\mathbf{I}}(x) = \bar{\Psi}_e(x)\Psi_e(x) \int d^4x' [e_p \bar{\Psi}_p(x')\gamma\Psi_p(x')]G(x, x'), \quad (4.21b)$$

where Green's function  $G(x, x')$  satisfies

$$\partial_\rho \partial^\rho G(x, x') = \delta^{(4)}(x - x'). \quad (4.22)$$

Hence  $\Phi^{\mathbf{I}}$  and  $\mathbf{A}^{\mathbf{I}}$ , which depend on not only the wave function of driving source  $\Psi_p(x')$  but also that of testing body  $\Psi_e(x)$ , are the interaction potentials between the driving source and testing body, gaining different meaning from the background potentials  $\Phi$  and  $\mathbf{A}$  driven only by external sources. Though the Minkowski four-potential has been successfully used in the Maxwell equation (4.18), the situation in the wave equation (4.1) is different: the interaction potentials  $\Phi^{\mathbf{I}}$  and  $\mathbf{A}^{\mathbf{I}}$  between the spin one-half particles and the external electromagnetic fields, should be taken into account, even if  $\Phi^{\mathbf{I}}$  and  $\mathbf{A}^{\mathbf{I}}$  do not yet form a covariant four-vector. While a time-covariant interaction four-potential will be derived in §8. Particularly in the linear approximation, the equation (4.21a) gives a point-like static Coulomb potential  $e_e \Phi^{\mathbf{I}} = -\alpha/r$  as we expect.

From the relations (4.14) through (4.17), the equation (4.1) can also be alternatively written down as

$$\gamma^\mu P_\mu \Psi(x) = (m + eJ^\mu A_\mu)\Psi(x). \quad (4.23)$$

Here  $eJ^\mu A_\mu$  happens to be the well-known electro-dynamical interaction potential. So unlike the Dirac EM-equation, the constructed equation is a nonlinear wave equation, which, in most cases, does not have exact explicit analytical solutions. This result is actually a consequence of the Maxwell equation, in which four-potential transforms in the same way as four-current under the Lorentz transformations, leading to the appearance of a Lorentz invariant electro-dynamical interaction potential in (4.23).

Originally the Dirac EM-equation was set up to deal with the interaction between a point-like electron and an external electromagnetic field, while the internal size of electron was ignored. In contrast, the nonlinear equation (4.23) may become valid for extended electron, when the interaction potential is expressed by a convolution between four-current and four-potential

$$V(x) = eJ^\mu * A_\mu = e \int d^4x' J^\mu(x')A_\mu(x - x'), \quad (4.24)$$

with an integral over the finite 4-d size of extended electron. Using such a nonsingular convolution can avoid the infinity troubles like the infinite self-energy of electron and

divergent integrals in Feynman diagrams, caused by the interaction of singular point particles.

If a fermion also participates in other types of interactions, more Lorentz invariant interaction terms should be added into  $V(x)$ . We may just as well generalize (4.23) by the following form:

$$i\gamma^\mu \partial_\mu \Psi(x) = [m + V(x)]\Psi(x), \quad (4.25)$$

where  $V(x)$  representing the sum of interaction terms must be invariant under all the Lorentz transformations. This equation is basically a nonlinear equation since the interaction  $V(x)$  depends on fermion fields. To completely solve the problem, one needs to establish more equations for intermediate bosons by gauge theory. Of course, this is only true in principle. When more complicated interactions are involved, solving a complete set of nonlinear equations are very impractical and much unnecessary, and further realizable techniques are required.

## 5. ENERGY ORDER OF HYDROGEN-LIKE IONS

Though (4.1) is a nonlinear equation without exact analytical solutions, we can still make a fair approximation that the interaction potentials  $\Phi^{\mathbf{I}}$  and  $\mathbf{A}^{\mathbf{I}}$  are slowly-changing functions of particle wavefunctions on account of the small scale of particle itself compared with the long-range electromagnetic interaction. Using this approximation and noticing the differences between upper larger components and lower smaller components of wave function, we obtain the non-relativistic Hamiltonian of four major terms after taking  $\mathbf{A}^{\mathbf{I}} = \mathbf{B}^{\mathbf{I}} \times \mathbf{r}/2$  which is valid in the atomic range where  $\mathbf{B}^{\mathbf{I}}$  is approximately constant:

$$H_{NR} = \frac{\mathbf{p}^2}{2m} + e\Phi^{\mathbf{I}} - \frac{e}{2m}\mathbf{k} \cdot (\nabla \times \mathbf{A}^{\mathbf{I}}) - \frac{e}{2m^2 r^2}(\mathbf{r} \cdot \nabla \Phi^{\mathbf{I}})(\mathbf{s} \cdot \mathbf{l}), \quad (5.1)$$

here  $\mathbf{k} = \mathbf{l} + \boldsymbol{\sigma}$ , while  $\mathbf{l} = \mathbf{r} \times \mathbf{p}$  is the orbital angular momentum,  $\mathbf{s} = \boldsymbol{\sigma}/2$  is the spin, and the total angular momentum is  $\mathbf{j} = \mathbf{l} + \mathbf{s}$ . In (5.1), the first term is the kinetic energy; the second is the electric potential; the third shows the magnetic Zeeman effect; the fourth represents the spin-orbit coupling. As we can see, the first three terms are formally the same as the conventional results although  $\Phi^{\mathbf{I}}$  and  $\mathbf{A}^{\mathbf{I}}$  refer to the interaction potentials not the pure external potentials. The fourth spin-orbit coupling term has a different sign from convention, showing a reversed order of splitting energy levels.

With the non-relativistic Hamiltonian (5.1), we can write a wave equation  $i\partial_t \psi = H_{NR}\psi$ , which is kind of “nonlinear Schrödinger equation” since the interaction potentials  $\Phi^{\mathbf{I}}$  and  $\mathbf{A}^{\mathbf{I}}$  depend on the wave functions. The nonlinear Schrödinger equation has certain soliton-like solutions, and has been applied in many aspects of nonlinear physics. However in the derivation of the above non-relativistic Hamiltonian, the positive-energy assumption has been taken. This implies: a wave equation with a non-relativistic Hamil-

tonian is not invariant under unitary time inversion. In other words, the results obtained by applying time inversion in a truncated non-relativistic wave equation like the Schrödinger equation may not be correct.

As an example, we can solve the nonlinear relativistic wave equation (4.1) for the single-electron model of Hydrogen-like ions under the linear approximation that there is a point-like Coulomb potential  $e\Phi^{\text{I}} = -Z\alpha/r$ , but no vector magnetic potential  $e\mathbf{A}^{\text{I}} = 0$ . In this model, (4.1) reduces to

$$(\gamma^\mu P_\mu + \frac{Z\alpha}{r})\Psi(x) = m\Psi(x), \quad (5.2)$$

which differs from the Dirac equation by a matrix  $\beta$  on the Coulomb term. A more general form than (5.2) for scalar central potential ( $C/r$ ) has been investigated in detail (Greiner [15] [16]). Following the standard procedures of solving this type of equations, we arrive at the bound state energy solutions of (5.2):

$$E_{nj} = \pm m \left\{ 1 - \frac{(Z\alpha)^2}{[n - (j + 1/2) + \sqrt{(j + 1/2)^2 + (Z\alpha)^2}]^2} \right\}^{1/2}, \quad (5.3)$$

which gives a symmetric positive-negative energy spectrum and does not have any difficulty when  $Z\alpha > 1$ .

If  $Z\alpha \ll 1$ , we can expand the positive  $E_{nj}$  in powers of  $Z\alpha$

$$E_{nj} = m \left[ 1 - \frac{(Z\alpha)^2}{2n^2} + \frac{(Z\alpha)^4}{2n^3} \left( \frac{1}{j + 1/2} - \frac{1}{4n} \right) \right] + O[(Z\alpha)^6]. \quad (5.4)$$

Where the first term is the rest mass of electron, the second term is the classical quantum mechanics result, and the third relativistic term is different from the conventional one by a crucial sign change. Although the fine structure splitting energy happens to be the same as usual

$$\Delta E_n(j_1 \rightarrow j_2) = \frac{m(Z\alpha)^4}{2n^3} \left| \frac{1}{j_1 + 1/2} - \frac{1}{j_2 + 1/2} \right|, \quad (5.5)$$

the energy order for quantum number  $j$  with fixed  $n$  is changed. Our conclusion is: the smaller the total angular momentum  $j$ , the larger the positive energy  $E_{nj}$  in a certain level  $n$  of Hydrogen-like ions. For example, the energy level  $2P_{1/2}$  is higher than the energy level  $2P_{3/2}$  in the doublet  $2P_{1/2} - 2P_{3/2}$  of Hydrogen-like ions, contrary to the conventional order. The common practice in experiment is: first detect a spectrum of energy levels, then arrange these energy levels in the order derived from the Dirac equation. Up till now, the energy order of the doublet of Hydrogen-like ions, has not been precisely verified in experiment as far as we know. Let us examine if it is possible to double check it by experiment.

The fine structures of the lines  $H_\alpha(\lambda 6563, n = 3 \rightarrow 2)$  and  $H_\beta(\lambda 4861, n = 4 \rightarrow 2)$  in the Balmer series of Hydrogen have been carefully investigated (Lewis and Spedding [17]; Spedding *et al.* [18]), of which no more than two strong components can be resolved

within experimental accuracy. This type of experiments, however, can only show the energy difference between two levels, i.e.,  $2P_{1/2} - 2P_{3/2}$  of Hydrogen atom. It can not, by itself, indicate the energy order: which line corresponds to which transition. The fine structure of the line  $\lambda 4686(n = 4 \rightarrow 3)$  of singly ionized Helium  $\text{He}^+$  has also been studied in detail (Paschen [19]). However, if the new energy order is assigned to those energy levels, the fit to the experimental data is not worse in consideration of experimental accuracies. On the other hand, the Paschen-Back [20] effect by applying strong magnetic fields shows a symmetric triplet optical spectrum, and does not make any difference if the order of energy levels is reversed.

On the other hand, the Zeeman effects of multi-electron atoms and ions have been vastly investigated (White [21]; Kuhn [22]), which show both “normal order”, the bigger  $j$  the higher energy level, and “abnormal order”, the smaller  $j$  the higher energy level. When a number of electrons are involved in multi-electron atoms or ions, more complicated effects must be taken into account, for example, electrostatic screening, orbit penetration and Fermi-Dirac statistics properties of multi-electron systems, and more complete equations like the self-consistent Hartree-Fock equations need to be set up. They are beyond the scope that the single-electron equation can describe.

In principle, only the Zeeman effects for Hydrogen-like ions under weak magnetic fields can directly and clearly give us the answer of the energy order of Hydrogen-like ions, since the asymmetric spectral lines for different transitions may be presented (White [21]; Kuhn [22]). The principal fine structure doublet splitting ( $2P_{1/2} - 2P_{3/2}$ ) of Hydrogen atom is less than half wave number, hence it is very difficult to detect more splitting levels under weak magnetic fields in such a narrow band with good resolution. For this reason, the Zeeman effect of Hydrogen atom has not been observed (White [21]), and the energy order of the doublet  $2P_{1/2} - 2P_{3/2}$  in Hydrogen is still an unsolved puzzle in experiment. If high resolution (say, one tenth of wave number) in analyzing spectral lines is realized, the detection of the Hydrogen Zeeman effect will be easier. For the other Hydrogen-like ions such as  $\text{He}^+$ ,  $\text{Li}^{++}$ ,  $\text{Be}^{+++}$ , etc., having bigger fine structure splitting energies ( $\propto Z^4$ ) as seen in (5.5), the Zeeman effects under weak magnetic fields seem detectable, even if most of the simple transitions ( $n = 2 \rightarrow 1$ , or  $3 \rightarrow 2$ ) drop into ultraviolet or X-ray range, not seen in the range of visible light spectrum. By modern techniques in analyzing laser optical spectrum, these experiments are believed realizable.

From (5.3), the positive ground state energy for  $n = 1$  and  $j = 1/2$  is

$$E_0 = \frac{m}{\sqrt{1 + (Z\alpha)^2}}, \quad (5.6)$$

which approaches zero when  $Z \rightarrow \infty$ . While the conventional ground state energy of a Hydrogen-like ion has a singularity at  $Z\alpha \sim 1$ :

$$E_{con} = m\sqrt{1 - (Z\alpha)^2}, \quad (5.7)$$

which has no real meaningful solution when  $Z\alpha > 1$ . The percentage difference between (5.6) and (5.7) is

$$1 - \frac{E_{con}}{E_0} = 1 - \sqrt{1 - (Z\alpha)^4}, \quad (5.8)$$

which is 1% when  $Z \sim 50$ , 5% when  $Z \sim 75$  and 15% when  $Z \sim 100$ . One can see that the errors are significant only for high  $Z$  Hydrogen-like ions which are unstable on the other hand. This causes real difficulty in doing experiments. An alternative way is to measure the ground states of high  $Z$  atoms instead of ions, by assuming the outer shell electrons have little screening effects on the inner ground state electrons, which may be approximately described by single-electron model. So a possible experiment is to use laser beam to pump out the electrons of high  $Z$  atoms. One may find out the largest ionization energies and compare them with the theoretical results (5.6) and (5.7).

In summary, the energy solutions (5.3) differ from conventional results in a subtle way. The differences may not be noticeable if no special attention is paid to it in experiment. In fact, the fine structure splitting is exactly the same as usual. Only the energy order in a certain level  $n$  of Hydrogen-like ions is reversed. This is why we emphasize the importance of a thorough experimental investigation of the Zeeman effects of Hydrogen-like ions. Ironically no such experiment can be found by far.

## 6. PARTICLE CONJUGATION

After a simple check, we find that charge conjugation defined by  $C\Psi = i\gamma^2\Psi^*$ , does not turn (4.23) into the other one with opposite charge. So we need to try something else. Now making a complex conjugation  $*$  on both sides of (4.23), we get

$$-\gamma^{\mu*}P_\mu\Psi^*(x) = (m + eJ^\mu A_\mu)\Psi^*(x). \quad (6.1)$$

Considering  $\gamma^{2*} = -\gamma^2$  and  $\gamma^{i*} = \gamma^i (i = 0, 1, 3)$ , we may pick a unitary transformation

$$O_s = i\gamma^0\gamma^1\gamma^3, \quad (6.2)$$

which is a complex conjugation operator in spinor space:

$$O_s\gamma^{\mu*}O_s^{-1} = \gamma^\mu. \quad (6.3)$$

Making transformation  $-O_s$  on both sides of (6.1), we get

$$\gamma^\mu P_\mu(O_\omega\Psi(x)) = (-m - eJ^\mu A_\mu)(O_\omega\Psi(x)), \quad (6.4)$$

where  $O_\omega$  is a complex conjugation in common space, defined by

$$O_\omega\Psi(x) = i\gamma^0\gamma^1\gamma^3\Psi^*(x). \quad (6.5)$$



Comparing (6.4) with (4.23), we find that  $O_\omega$  changes a particle with charge  $e$  and mass  $m$  into another one with opposite charge  $-e$  and opposite mass  $-m$ . So we like to call  $O_\omega$  “particle conjugation”.

In §3, we have mentioned that common momentum has two eigenvalues  $\pm m$ . So we may just as well write another equation:

$$\gamma^\mu P_\mu \Psi_{-m}(x) = -m \Psi_{-m}(x), \quad (6.6)$$

which gives the same type of solutions with the same energy spectrum as the Dirac equation (3.5). Adding an interaction term, we get an equation similar to (4.23) but with “opposite” mass:

$$\gamma^\mu P_\mu \Psi_{-m}(x) = (-m + eJ^\mu A_\mu) \Psi_{-m}(x), \quad (6.7)$$

which can be transformed under “particle conjugation” into

$$\gamma^\mu P_\mu (O_\omega \Psi_{-m}(x)) = (m - eJ^\mu A_\mu) (O_\omega \Psi_{-m}(x)). \quad (6.8)$$

Clearly (6.8) and (4.23) show opposite charge but same mass.

Negative charge has been found in experiment long ago, but no negative mass. In present-day cosmology, the existence of negative mass is highly controversial. Somehow it involves the precise definition of mass. It would be naive to draw any definitive conclusion just from this single-electron model. Hence we consider this “particle conjugation” as a speculative thought, though we do feel equations (4.23), (6.4), (6.7) and (6.8) are all legitimate forms in certain sense.

Due to our different definitions of unitary time inversion and “particle conjugation”, the conventional CP or CPT theorem does not hold in our theory. It is possible to find more conventional theorems that may not be derivable in our nonlinear theory. We may either derive similar ones as replacements, or find completely new theorems.

## 7. PERTURBATION TREATMENT

As long as the interaction potential  $V(x)$  is weak enough in (4.25), we may follow Feynman’s perturbation treatment (Feynman [23]), to derive the lowest-order differential cross section:

$$\frac{d\sigma}{d\Omega} = \frac{|\mathbf{p}_f|^3}{|\mathbf{p}_i|} \frac{m^2}{E_f E_i} |\bar{u}(p_f, s_f^0) u(p_i, s_i^0)|^2 |V(q)|^2, \quad (7.1)$$

where subscripts  $i$  and  $f$  represent the incoming and outgoing waves respectively,  $\bar{u}$  and  $u$  are the spinors,  $q = p_f - p_i$  is the four-momentum transfer and

$$V(q) = \int d^4x V(x) e^{iq \cdot x}, \quad (7.2)$$

is the Fourier amplitude of the interaction potential  $V(x)$ .

In the case of Coulomb interaction between the incident electron beam and nucleon target of nuclear number  $Z$ :  $V(x) = -Z\alpha/|\mathbf{x}|$ , energy is conserved:  $E_f = E_i$ . We have  $V(q) = -Z\alpha/\mathbf{q}^2$  where  $\mathbf{q}^2 = (\mathbf{p}_f - \mathbf{p}_i)^2 = -4\mathbf{p}^2(\sin \frac{\theta}{2})^2$ . For any unpolarized incident electron beam, the cross section is a sum over final spin states and an average over initial spin states. So we get

$$\frac{d\bar{\sigma}}{d\Omega} = \frac{Z^2\alpha^2m^2}{\mathbf{q}^4}\Sigma_1. \quad (7.3)$$

where the spin sum

$$\Sigma_1 = \frac{1}{2} \sum_{s_f^0, s_i^0} |\bar{u}(p_f, s_f^0)u(p_i, s_i^0)|^2 = \frac{1}{2}(1 + \frac{p_i \cdot p_f}{m^2}), \quad (7.4)$$

is different from the conventional result (Bjorken and Drell [7]) since our Coulomb term as in (5.2) does not have a matrix  $\beta$ . Ignoring a constant factor 1/16, finally we have

$$\frac{d\bar{\sigma}}{d\Omega} = \frac{Z^2\alpha^2m^2}{\mathbf{p}^4(\sin \frac{\theta}{2})^4} [1 + \frac{\mathbf{p}^2}{m^2}(\sin \frac{\theta}{2})^2]. \quad (7.5)$$

In the non-relativistic case  $|\mathbf{p}| \ll m$ , it reduces to the Rutherford formula:

$$\frac{d\bar{\sigma}}{d\Omega} = \frac{Z^2\alpha^2m^2}{\mathbf{p}^4(\sin \frac{\theta}{2})^4}. \quad (7.6)$$

And in the relativistic limit  $|\mathbf{p}| \gg m$ , it turns out to be

$$\frac{d\bar{\sigma}}{d\Omega} = \frac{Z^2\alpha^2}{\mathbf{p}^2(\sin \frac{\theta}{2})^2}, \quad (7.7)$$

which differs from the Mott [24] formula by a factor  $[\cot(\theta/2)]^2$ .

Usually the Coulomb interaction is applicable only in low energy scattering, not in high energy scattering due to the recoil of nuclear target. In the electron-proton elastic scattering experiments (McAllister and Hofstadter [25]), the incident electron energy 188 Mev is in the same order as the proton mass 938 Mev. Considering the recoil effect of hydrogen, we would better utilize a current-potential interaction with a nonsingular convolution (4.24):

$$V(x) = eJ_e^\mu(x) * A_\mu^p(x). \quad (7.8)$$

Its Fourier amplitude becomes

$$V(q) = e \int d^4x' J_e^\mu(x') e^{iq \cdot x'} \int d^4x A_\mu^p(x - x') e^{iq \cdot (x - x')} = eJ_e^\mu(q)A_\mu^p(q). \quad (7.9)$$

From the Maxwell equation (4.18), we may express the four-potential driven by proton as follows:

$$A_\mu^p(x) = G(x) * e_p J_\mu^p(x) = \int d^4y G(x - y) e_p J_\mu^p(y), \quad (7.10)$$

where  $G(x)$  is Green's function defined by (4.22). In analogy to (7.9), we get

$$A_\mu^p(q) = e_p G(q) J_\mu^p(q), \quad (7.11)$$

where  $G(q) = -(q^2 + i\epsilon)^{-1}$ . Finally we write down

$$\begin{aligned} V(q) &= -e^2 J_e^\mu(q) G(q) J_\mu^p(q) \\ &= -\alpha \sqrt{\frac{m^2}{E_f E_i}} \sqrt{\frac{M^2}{E_f^p E_i^p}} \bar{u}(p_f, s_f) \gamma^\mu u(p_i, s_i) \frac{-1}{q^2 + i\epsilon} \bar{u}(P_f, S_f) \gamma_\mu u(P_i, S_i). \end{aligned} \quad (7.12)$$

In the relativistic elastic scattering as shown in Fig. 1, the four-momentum transfer squared is  $q^2 = (p' - p)^2 = -4E_f E_i (\sin \frac{\theta}{2})^2$ , and the finite mass recoil factor is  $E_f/E_i = [1 + (2E/M)(\sin \frac{\theta}{2})^2]^{-1}$  where  $E = E_i$ . For any unpolarized electron beam, the electron spin is random over space-time. So when we calculate the interaction between electron and proton, their spins are chosen randomly. The incoming electron spin  $s_i$  in calculating the Feynman diagrams is not related to the incident electron spin  $s_i^0$  in the past. Same is true for the outgoing electrons. To calculate the total differential cross section per solid angle, we sum over all spin states independently:

$$\frac{d\bar{\sigma}}{d\Omega} = \frac{\alpha^2 m^4 E_f}{q^4 E_i^3} \Sigma_1 \Sigma_2, \quad (7.13)$$

where  $\Sigma_1$ , an extra term that the conventional theory does not have, is given by (7.4):  $\Sigma_1 = (E_f E_i/m^2)(\sin \frac{\theta}{2})^2$  when  $E \gg m$ , and  $\Sigma_2$  is the same as in the conventional theory (Bjorken and Drell [7]):

$$\begin{aligned} \Sigma_2 &= \frac{1}{4} \sum_{s_f, s_i} \sum_{S_f, S_i} |\bar{u}(p_f, s_f) \gamma^\mu u(p_i, s_i) \bar{u}(P_f, S_f) \gamma_\mu u(P_i, S_i)|^2 \\ &= \frac{E_f E_i}{m^2} (\cos \frac{\theta}{2})^2 [1 - \frac{q^2}{2M^2} (\tan \frac{\theta}{2})^2]. \end{aligned} \quad (7.14)$$

Combining all these results we obtain

$$\frac{d\bar{\sigma}}{d\Omega} = \frac{\alpha^2}{E^2} (\cot \frac{\theta}{2})^2 \left[ \frac{1 - \frac{q^2}{2M^2} (\tan \frac{\theta}{2})^2}{1 + \frac{2E}{M} (\sin \frac{\theta}{2})^2} \right], \quad (7.15)$$

which differs from the conventional result by a factor  $[\sin(\theta/2)]^{-2}$ .

In comparison with the experimental data (McAllister and Hofstadter [25]) as shown in Fig. 2, it is hard to tell which one is better. More advanced research need to be made. Generally two form factors need to be considered due to the finite size and anomalous magnetic moment of proton (Cahn and Goldhaber [26]). The modification due to the internal structure of proton is significant in high energy scattering processes: the higher the incident electron energy, the larger the modification. In the extreme relativistic limit  $|\mathbf{p}| \gg M$ , the result (7.15) is not valid, and in the certain range  $|\mathbf{p}| \sim M$ , it is not accurate, but in the low relativistic limit  $m \ll |\mathbf{p}| \ll M$ , it may be good enough. The lowest-order scattering is quite preliminary. So we would not draw any definitive conclusions as yet.

## 8. GAUGE INVARIANCE

From the correspondence (4.17) and the following relation

$$\bar{\Psi}(\Phi^{\mathbf{I}} - \boldsymbol{\gamma} \cdot \mathbf{A}^{\mathbf{I}})\Psi = \bar{\Psi}J^{\mu}A_{\mu}\Psi = \bar{\Psi}\gamma^{\mu}(\bar{\Psi}\Psi)A_{\mu}\Psi, \quad (8.1)$$

we can also write (4.1) or (4.23) as follows:

$$\gamma^{\mu}(P_{\mu} - eA_{\mu}^{\mathbf{T}})\Psi = m\Psi, \quad (8.2)$$

by introducing an interaction four-potential

$$A_{\mu}^{\mathbf{T}} = (\bar{\Psi}\Psi)A_{\mu}. \quad (8.3)$$

Here  $A_{\mu}^{\mathbf{T}}$  is “time-covariant”, since  $A_{\mu}$  is “time-anticovariant” and  $\bar{\Psi}\Psi$  changes sign under unitary time inversion. The equation (8.2) implies a formal similarity to the Dirac equation (3.5) if we define an “effective” time-covariant four-momentum

$$P_{\mu}^{eff} = P_{\mu} - eA_{\mu}^{\mathbf{T}}, \quad (8.4)$$

or an “effective” time-covariant four-derivative

$$\partial_{\mu}^{eff} = \partial_{\mu} + ieA_{\mu}^{\mathbf{T}}. \quad (8.5)$$

In classical electrodynamics, observable physical quantities can be expressed in terms of electromagnetic field strengths:  $\mathbf{E} = -\nabla\Phi - \partial_t\mathbf{A}$  and  $\mathbf{B} = \nabla \times \mathbf{A}$  which seem invariant under gauge transformation for an arbitrary scalar function  $f(x)$  (Bjorken and Drell [27]; Aithison and Hey [28]):

$$A_{\mu} \rightarrow A_{\mu} + \partial_{\mu}f. \quad (8.6)$$

However there are several restrictions that need to be taken into account seriously, if we want to preserve both gauge invariance and Lorentz invariance of the Maxwell field equation and relativistic fermion field equation.

First of all, as we have pointed out, Minkowski four-potential  $A_{\mu}$  is time-anticovariant (3.12) but four-derivative  $\partial_{\mu}$  is apparently time-covariant under time inversion:

$$(\partial_t, \partial_{\mathbf{x}}) \rightarrow (-\partial_t, \partial_{\mathbf{x}}). \quad (8.7)$$

So gauge function  $f(x)$  must change sign under time inversion to keep the transformed four-potential  $A_{\mu} + \partial_{\mu}f$  time-anticovariant as same as  $A_{\mu}$ . Second, gauge function  $f(x)$  needs to be a proper Lorentz invariant to ensure the transformed four-potential  $A_{\mu} + \partial_{\mu}f$  covariant under the proper Lorentz transformations. Third, by making a local phase transformation on fermion field

$$\Psi \rightarrow \Psi \exp[-i\alpha(x)], \quad (8.8)$$

and a local gauge transformation (8.6) on Minkowski field simultaneously in (8.2), we arrive at another constraint

$$\partial_\mu \alpha - e(\bar{\Psi}\Psi)\partial_\mu f = 0. \quad (8.9)$$

The general solutions satisfying the above three conditions are

$$f(x) = \sum C_n (\bar{\Psi}\Psi)^n, \quad (n = \pm 1, \pm 3, \pm 5, \dots), \quad (8.10a)$$

$$\alpha(x) = e\left(\sum \frac{n}{n+1} C_n (\bar{\Psi}\Psi)^{n+1} - C_{-1} \log |\bar{\Psi}\Psi|\right), \quad (n = 1, \pm 3, \pm 5, \dots), \quad (8.10b)$$

where  $C_n$  are arbitrary real constants to cover the whole phase space. Note:  $\bar{\Psi}\Psi \neq 0$ , otherwise, there is no interaction term in the equation (8.2).

Furthermore, the Lorentz gauge (4.19) leads to another constraint on the choice of gauge function  $f(x)$ :

$$\partial_\mu \partial^\mu f(x) = 0. \quad (8.11)$$

Using (4.23) or more generally (4.25) with (4.24), we can prove step by step that this is true if the gauge function takes the form of (8.10a). First we have an expansion

$$\partial_\mu \partial^\mu (\bar{\Psi}\Psi) = \bar{\Psi}(\partial_\mu \partial^\mu \Psi) + (\partial_\mu \partial^\mu \bar{\Psi})\Psi + 2\partial_\mu \bar{\Psi} \partial^\mu \Psi. \quad (8.12)$$

From (4.25) it comes out

$$\gamma^\mu \partial_\mu \Psi = -i(m + V)\Psi. \quad (8.13)$$

Making a hermitian conjugation on both sides and then multiplying it by  $\gamma^0$  from right, we get by the relations  $\gamma^{\mu\dagger}\gamma^0 = \gamma^0\gamma^\mu$  and  $\bar{\Psi} = \Psi^\dagger\gamma^0$

$$\partial_\mu \bar{\Psi} \gamma^\mu = i(m + V)\bar{\Psi}. \quad (8.14)$$

From (8.13) it is derived

$$\partial_\mu \partial^\mu \Psi = \gamma^\mu \partial_\mu \gamma^\nu \partial_\nu \Psi = -i\gamma^\mu (\partial_\mu V)\Psi - (m + V)^2 \Psi. \quad (8.15)$$

Repeating the procedures in deriving (8.14) leads to

$$\partial_\mu \partial^\mu \bar{\Psi} = i\bar{\Psi} \gamma^\mu \partial_\mu V - (m + V)^2 \bar{\Psi}. \quad (8.16)$$

Furthermore we have

$$2\partial_\mu \bar{\Psi} \partial^\mu \Psi = 2\partial_\mu \bar{\Psi} g^{\mu\nu} \partial_\nu \Psi = \partial_\mu \bar{\Psi} (\gamma^\mu \gamma^\nu + \gamma^\nu \gamma^\mu) \partial_\nu \Psi. \quad (8.17)$$

Combining (8.13) and (8.14), we get

$$\partial_\mu \bar{\Psi} \gamma^\mu \gamma^\nu \partial_\nu \Psi = (m + V)^2 \bar{\Psi} \Psi. \quad (8.18)$$

Then we need to find the second term in (8.17). Now let us make a transformation on all fermion fields (note  $\gamma^0 \gamma^{\mu\dagger} \gamma^0 = \gamma^\mu$ )

$$\Psi' = \gamma^\mu \Psi, \quad \bar{\Psi}' = \bar{\Psi} \gamma^\mu, \quad (8.19)$$

then (8.13) becomes

$$\gamma^\nu \partial_\nu (\gamma^\mu \Psi) = -i(m + V') \gamma^\mu \Psi. \quad (8.20)$$

Here  $V'$  is the interaction after the transformation, which for example can be expressed by its Fourier amplitude (7.13) in the electron-proton scattering

$$\begin{aligned} V'(x) = & -\alpha \sqrt{\frac{m^2}{E_f E_i}} \sqrt{\frac{M^2}{E_f^p E_i^p}} \int \frac{d^4 q}{(2\pi)^4} \frac{-1}{q^2 + i\epsilon} e^{-iq \cdot x} \\ & \times \bar{u}(p_f, s_f) \gamma^\mu \gamma^\rho \gamma^\mu u(p_i, s_i) \bar{u}(P_f, S_f) \gamma^\mu \gamma_\rho \gamma^\mu u(P_i, S_i). \end{aligned} \quad (8.21)$$

Given  $\gamma^\mu \gamma^\mu = \pm 1$ , it is clear to see (8.21) does not change when  $\gamma^\mu$  and  $\gamma^\rho$  switch twice in both electron and proton (or any fermion) currents. So the interaction will not change after the transformation (8.19):

$$V'(x) = V(x). \quad (8.22)$$

Combining (8.20), (8.22) and (8.14), we have

$$\partial_\mu \bar{\Psi} \gamma^\nu \gamma^\mu \partial_\nu \Psi = -i(m + V) \partial_\mu \bar{\Psi} \gamma^\mu \Psi = (m + V)^2 \bar{\Psi} \Psi. \quad (8.23)$$

With (8.18) and (8.23), then (8.17) becomes

$$2\partial_\mu \bar{\Psi} \partial^\mu \Psi = 2(m + V)^2 \bar{\Psi} \Psi. \quad (8.24)$$

Finally inserting (8.15), (8.16) and (8.24) into (8.12) we arrive at

$$\partial_\mu \partial^\mu (\bar{\Psi} \Psi) = 0, \quad (8.25)$$

leading to the constraint (8.11) if the gauge function is given by (8.10a). From the above proof, we can see that the gauge condition is quite stringent. It appears that the Lorentz gauge is inherent in our theory. Note: the Coulomb gauge, as a special case of the Lorentz gauge for static fields, is included.

In summary, with the explicit solutions (8.10) for  $f(x)$  and  $\alpha(x)$ , the local gauge transformations in our nonlinear QED look like:

$$\Psi \rightarrow \Psi \exp[-i\alpha(x)], \quad (8.26a)$$

$$A_\mu \rightarrow A_\mu + \partial_\mu f(x), \quad (8.26b)$$

$$\text{or } A_\mu^T \rightarrow A_\mu^T + \frac{1}{e} \partial_\mu \alpha(x), \quad (8.26c)$$

which ensure both gauge invariance and Lorentz invariance of the nonlinear equation (8.2) and the Maxwell equation with the Lorentz gauge. Note: our nonlinear QED involving higher-order fermion fields, is different from the one involving higher-order electromagnetic fields, and is also different from the one involving self-interaction potential in the Dirac EM-equation, mentioned in some literatures (see reference [29]).

We may also construct a nonlinear QED Lagrangian density

$$L(x) = \bar{\Psi}\gamma^\mu(i\partial_\mu - \frac{1}{2}eA_\mu^T)\Psi - m\bar{\Psi}\Psi - \frac{1}{8}F_{\mu\nu}F^{\mu\nu}\bar{\Psi}\Psi, \quad (8.27)$$

here  $F_{\mu\nu} = \partial_\nu A_\mu - \partial_\mu A_\nu$  is the second-rank antisymmetric electromagnetic field tensor. The Hamiltonian density changes sign under unitary time inversion, so does the Lagrangian density. All Euler-Lagrange field equations can be obtained by Hamilton's principle. Taking an infinitesimal arbitrary variation on the electromagnetic field  $\delta A_\mu$ , we get

$$\partial^\nu[F_{\mu\nu}(\bar{\Psi}\Psi)] = e(\bar{\Psi}\gamma_\mu\Psi)(\bar{\Psi}\Psi), \quad (8.28)$$

then keeping the fermion field unchanged  $\partial^\mu(\bar{\Psi}\Psi) \rightarrow 0$ , we eliminate  $F_{\mu\nu}\partial^\nu(\bar{\Psi}\Psi)$  from (8.28) and cancel  $\bar{\Psi}\Psi$  on both sides to obtain the Maxwell equation  $\partial^\nu F_{\mu\nu} = eJ_\mu$  that reduces to (4.18) under the Lorentz gauge (4.19). Similarly by taking an infinitesimal change on the fermion field  $\delta\bar{\Psi}$ , we get

$$i\gamma^\mu\partial_\mu\Psi - \frac{1}{2}e\gamma^\mu A_\mu^T\Psi - \frac{1}{2}e(\bar{\Psi}\gamma^\mu\Psi)A_\mu\Psi - m\Psi - \frac{1}{8}F_{\mu\nu}F^{\mu\nu}\Psi = 0, \quad (8.29)$$

then keeping the electromagnetic field unchanged  $\partial_\nu A_\mu \rightarrow 0$ , we eliminate the last term in (8.29) to obtain the nonlinear equation (8.2) by the relation (8.1).

Though the gauge transformation (8.26) does not keep the Lagrangian density (8.27) invariant, it does keep invariant the integral of the Lagrangian density since the extra term in the integral makes no contribution:  $\int d^4x J^\mu\partial_\mu\alpha = \int d^4x\partial_\mu(J^\mu\alpha) = 0$ . With the definition (8.3), the Maxwell equation (4.18) and the Lorentz gauge (4.19) or (8.25), we get

$$\partial_\rho\partial^\rho A_\mu^T = (\bar{\Psi}\Psi)J_\mu^{ext} + 2\partial_\rho(\bar{\Psi}\Psi)\partial^\rho A_\mu. \quad (8.30)$$

Suppose in the free field case ( $J_\mu^{ext} = 0$ ), the fluctuations of fermion field and photon field are small, namely

$$\partial_\rho(\bar{\Psi}\Psi) = \epsilon_\rho(x)(\bar{\Psi}\Psi), \quad (8.31a)$$

$$\partial^\rho A_\mu = \kappa^\rho(x)A_\mu, \quad (8.31b)$$

where  $\epsilon_\rho(x)$  and  $\kappa^\rho(x)$  are small fluctuation functions. Then we end up with

$$\partial_\rho\partial^\rho A_\mu^T + \mu^2(x)A_\mu^T = 0, \quad (8.32a)$$

$$\mu^2(x) = -2\epsilon_\rho(x)\kappa^\rho(x). \quad (8.32b)$$

If  $\mu^2(x) > 0$ , it is a Klein-Gordon equation for vector boson with a "fluctuation mass"  $\mu(x)$ , i.e, a harmonic oscillator with a fluctuation energy-momentum. If  $\mu^2(x) < 0$ , it shows the decay and oscillation of the interaction field. Both cases are complicated: the time-covariant intermediate vector boson field  $A_\mu^T$ , namely the coupled interaction field between the Dirac field  $\Psi$  and the Minkowski field  $A_\mu$ , fluctuates in space-time.

By (8.27) and (8.1), the interaction Hamiltonian density can be written as

$$H^I(x) = \frac{1}{2}e\bar{\Psi}\gamma^\mu A_\mu^T\Psi = \frac{1}{2}e\bar{\Psi}J^\mu A_\mu\Psi. \quad (8.33)$$

If the fluctuation is small, we may simply take zeroth-order approximation by setting  $\mu(x) \rightarrow 0$ . In this limit, we may treat  $A_\mu^T$  as a time-covariant massless photon and linearize our nonlinear QED to fit experimental data just as accurately as conventional linear QED. Furthermore, we may investigate the nonlinear aspects of the theory, by considering  $A_\mu^T$  as a massive intermediate boson, or considering  $J^\mu A_\mu$  as a nonsingular convolution shown in (4.24) to avoid singularities, and rewrite (8.27) as:

$$L(x) = \bar{\Psi}(i\gamma^\mu\partial_\mu - \frac{1}{2}eJ^\mu * A_\mu)\Psi - m\bar{\Psi}\Psi - \frac{1}{8}F_{\mu\nu}F^{\mu\nu}\bar{\Psi}\Psi. \quad (8.34)$$

With the basic field quantization techniques, we can then deal with various problems in nonlinear QED starting from this Lagrangian density.

Orthodox QED, on the basis of the Dirac EM-equation, is just a linear theory. In this linear QED, electron is point-like and the self-energy of electron blows up when its size reduces to zero. Also at each vortex in Feynman diagrams, there is a singular term that causes ultraviolet divergences and requires renormalization. In contrast, a nonlinear equation of the type (4.25) with (4.24) gives certain “soliton-like” solutions, not “point-like” solutions. So the self-energy of electron would not go to infinity in our nonlinear QED. It appears at first sight that this theory is not renormalizable for there is a fourth-power term of fermion field in the Lagrangian. However the interaction potential  $eJ^\mu * A_\mu$  in (8.34) is considered as a nonsingular convolution, each vortex in Feynman diagrams is smeared, not singular any more as pictured in Fig. 1. Consequently, only two powers are left in the perturbation expansion with the other two being integrated out in each vortex. Hence this theory is free of ultraviolet divergences and is also renormalizable. These are just some general observations. Due to the scope limit of this paper, I leave these fundamental issues open for future research.

So much has been discussed, it is however not my intention to “solve” this nonlinear QED in such single paper. Rather my emphasis is on its “derivation” by the implementation of unitary time inversion. A complete understanding of nonlinear QED entails much more studies both theoretically and experimentally. It has been recognized for a long time that nonlinearity may result in fundamentally new phenomena (see reference [29] for a history overview). There is a typical problem posed for nonlinear theories: since the linear superposition principle of quantum mechanics is not valid in nonlinear field equations, it becomes quite problematic to make linear perturbation expansions. A great many efforts have been made in the past several decades, to seek non-perturbative techniques in solving nonlinear quantum field theories. Many interesting and important issues are still far from being settled.



## 9. REMARKS

The theory established in this paper is strictly limited to the Minkowski flat space-time, where the Lorentz group is the fundamental symmetry group. Naturally we would ponder on the possibility of extending our theory into curved space-time. In a globally curved space-time, the Lorentz group is only locally preserved. A conventional treatment on the Dirac fields in such a space-time is to use a covariant derivative with spin affine connection in the Dirac equation, as outlined in my dissertation (Jin [30]). As far as the electromagnetic interaction is concerned, a logical step is to include an electro-dynamical interaction potential rather than external potential in the Dirac equation, similar to what I have shown here in this paper. After all, a curved space-time with Lorentzian signature reduces locally to Minkowski space-time as a limit. Specifically in a static space-time with a time-independent metric, the time component of spin affine connection vanishes, we can separate time from space and anticipate a positive-negative symmetric energy spectrum (Jin [31] [32]). However in a general space-time with a time-dependent metric, time and space are not separable, time inversion is not applicable, and it is no longer possible to obtain a positive-negative symmetric energy spectrum.

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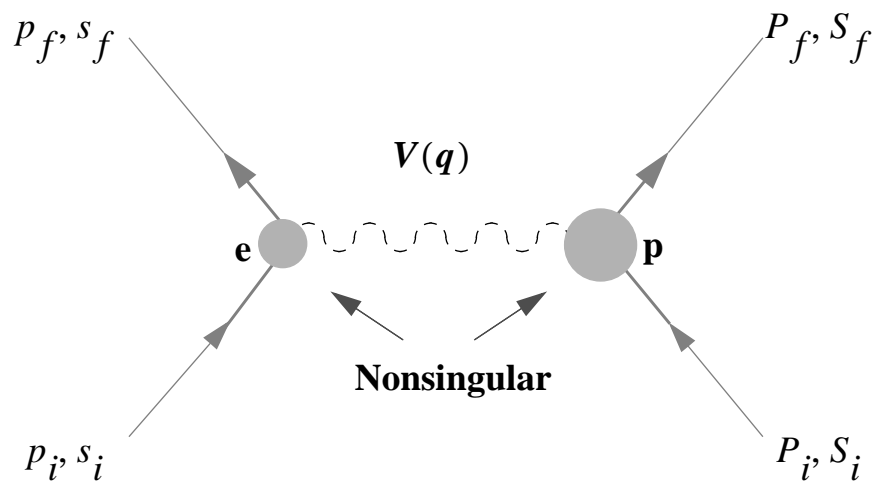


FIG. 1: Revised Feynman diagram of e-p scattering

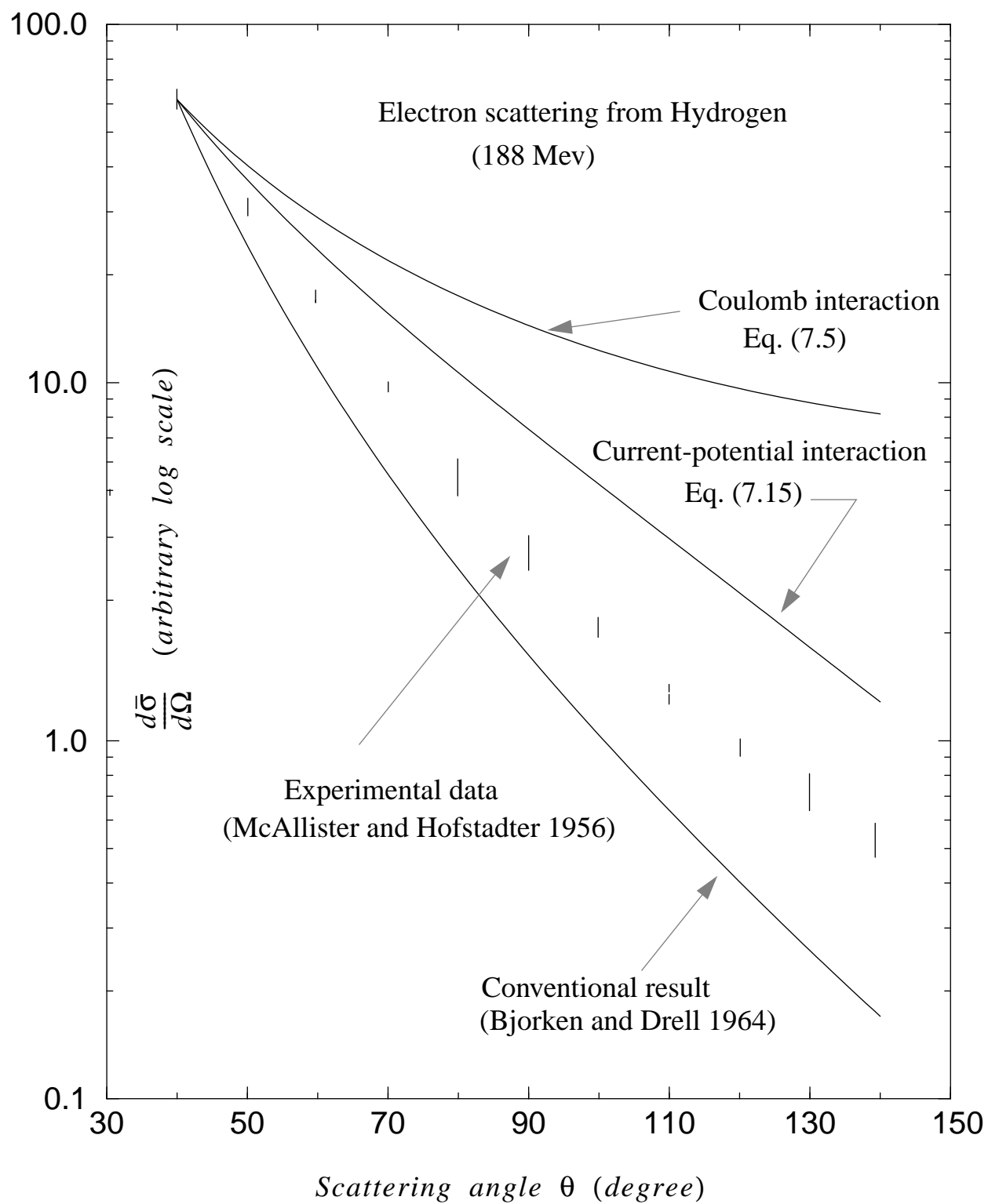


FIG. 2: Differential cross sections of e-p scattering