Bound states of neutral particles in external electric fields *

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Neutral fermions of spin $\frac{1}{2}$ with magnetic moment can interact with electromagnetic fields through nonminimal coupling. The Dirac–Pauli equation for such a fermion coupled to a spherically symmetric or central electric field can be reduced to two simultaneous ordinary differential equations by separation of variables in spherical coordinates. For a wide variety of central electric fields, bound-state solutions of critical energy values can be found analytically. The degeneracy of these energy levels turns out to be numerably infinite. This reveals the possibility of condensing infinitely many fermions into a single energy level. For radially constant and radially linear electric fields, the system of ordinary differential equations can be completely solved, and all bound-state solutions are obtained in closed forms. The radially constant field supports scattering solutions as well. For radially linear fields, more energy levels (in addition to the critical one) are infinitely degenerate. The simultaneous presence of central magnetic and electric fields is discussed.

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I. INTRODUCTION

In relativistic quantum theory, a charged fermion of spin $\frac{1}{2}$ moving in a background electromagnetic field is described by the Dirac equation with minimal coupling to the vector potential. In 1941, Pauli extended this equation to include an additional nonminimal coupling term which takes into account the interaction caused by the anomalous magnetic moment of the charged particle [1]. This extended equation is usually called the Dirac–Pauli equation. Many works have been devoted to the investigation of exact solutions of this equation in various electromagnetic fields, say, a constant uniform magnetic field, an electromagnetic plane wave, and more complicated ones [2]. A constant central (spherically symmetric) electric field was also considered by some authors. In this case the Dirac–Pauli equation is separable in spherical coordinates, however, exact solutions of the radial equation have not been found in closed forms [2]. In these studies, the nonminimal coupling is conceptually taken as some correction to the minimal one (though the correction is considerable for protons), and the simultaneous presence of both couplings causes some mathematical difficulty.

In this paper we consider neutral fermions of spin $\frac{1}{2}$ with magnetic moment. Without electric charges, such particles can still interact with electromagnetic fields through nonminimal coupling, and can be well described by the Dirac–Pauli equation. On the one hand, without the minimal coupling, the Dirac–Pauli equation is simpler. On the other hand, the interaction solely from nonminimal coupling has not attracted enough attention, especially before the discovery of the Aharonov–Casher (AC) effect [3-5]. Since the AC effect is a consequence of the nonminimal coupling and has been observed in experiment [5], one may become interested in other consequences of the interaction. For instance, it may be of interest to study bound states of neutral fermions in external electromagnetic fields, especially when exact solutions are available. It appears that this problem was not considered in the literature as far as we know. The purpose of this paper is to deal with this problem. It is organized as follows.

In the next section we consider the Dirac–Pauli equation of a neutral fermion of spin $\frac{1}{2}$, with mass m_n and magnetic moment μ_n, interacting with an external electromagnetic field through nonminimal coupling. For spherically symmetric or central electromagnetic fields, it can be shown that the total angular momentum is a constant of motion. By separation of variables in spherical coordinates, the stationary Dirac–Pauli equation in a central electric field, which involves four partial differential equations, can be reduced to a system of two coupled ordinary differential equations (ODE) for two radial wave functions. Given a specific electric field, one can in principle solve the system of ODE to obtain the radial wave functions and determine the energy levels for bound states. For a wide variety of electric fields, one can find bound-state solutions of critical energy value m_n or $-m_\mathrm{n}$ in analytic forms. It turns out that these critical energy levels are infinitely degenerate. This is interesting because it reveals the possibility of condensing infinitely many fermions, say, neutrons, into a single energy level. Electric fields that support a finite number of critical bound states are also discussed. In Sec. III we study a radially constant field, in this case the system of ODE can be completely solved, and we have scattering as well as bound-state solutions. All bound-state solutions are given in closed forms. Only the critical energy level has infinite degeneracy. In Sec. IV we deal with radially linear electric fields. The system of ODE is also completely solvable. In this case we have only bound-state solutions, and many of the energy levels are infinitely degenerate. In Sec.

V we discuss the simultaneous presence of central magnetic and electric fields. In this case separation of variables is still possible in spherical coordinates. But the reduced system of ODE involves four coupled equations for four radial wave functions, and is thus much more difficult to solve. Some other remarks and discussins, say, the nonrelativistic limit, are also included in this section.

II. NEUTRAL FERMIONS IN CENTRAL ELECTRIC FIELDS

We work in (3+1)-dimensional space-time and use the natural units where $\hbar = c = 1$. Consider a neutral fermion of spin $\frac{1}{2}$ with mass $m_{\rm n}$ and magnetic monment $\mu_{\rm n}$, moving in an external electromagnetic field described by the field strength $F_{\mu\nu}$. The fermion is described by a four-component spinorial wave function Ψ obeying the Dirac–Pauli equation [2, 6]

$$(i\gamma^\mu\partial_\mu - \tfrac{1}{2}\mu_{\rm n}\sigma^{\mu\nu}F_{\mu\nu} - m_{\rm n})\Psi = 0, \tag{1}$$

where $\gamma^\mu = (\gamma^0, \boldsymbol{\gamma})$ are Dirac matrices satisfying

$$\{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu} \tag{2}$$

with $g^{\mu\nu} = \mathrm{diag}(1, -1, -1, -1)$, and

$$\sigma^{\mu\nu} = \frac{i}{2}[\gamma^\mu, \gamma^\nu]. \tag{3}$$

It can be shown that

$$\tfrac{1}{2}\sigma^{\mu\nu}F_{\mu\nu} = i\boldsymbol{\alpha}\cdot\mathbf{E} - \boldsymbol{\Sigma}\cdot\mathbf{B} \tag{4}$$

where $\mathbf{E}$ is the external electric field and $\mathbf{B}$ the magnetic one, $\boldsymbol{\alpha} = \gamma^0\boldsymbol{\gamma}$, and $\Sigma^k = \frac{1}{2}\epsilon^{kij}\sigma^{ij}$ where ϵ^{kij} is totally antisymmetric in its indices and $\epsilon^{123} = 1$. If both $\mathbf{E}$ and $\mathbf{B}$ are independent of the time t, one may set

$$\Psi(t, \mathbf{r}) = e^{-i\mathcal{E}t}\psi(\mathbf{r}), \tag{5}$$

and obtain a stationary Dirac–Pauli equation for ψ:

$$H\psi = \mathcal{E}\psi, \tag{6a}$$

where the Halmiltonian H is given by

$$H = \boldsymbol{\alpha}\cdot\mathbf{p} + i\mu_{\rm n}\boldsymbol{\gamma}\cdot\mathbf{E} - \mu_{\rm n}\gamma^0\boldsymbol{\Sigma}\cdot\mathbf{B} + \gamma^0 m_{\rm n}, \tag{6b}$$

where $\mathbf{p} = -i\nabla$ or $p^k = -i\partial_k$.

Now let us consider spherically symmetric or central fields

$$\mathbf{E} = E(r)\mathbf{e}_r, \quad \mathbf{B} = B(r)\mathbf{e}_r, \tag{7}$$

where r is one of the spherical coordinates (r, θ, ϕ) and $\mathbf{e}_r$ is the unit vector in the radial direction. As usual we define the the orbital angular momentum $\mathbf{L} = \mathbf{r} \times \mathbf{p}$. It is not difficult to calculate the commutator $[\mathbf{L}, H]$ and it turns out that $[\mathbf{L}, H] \neq 0$ even for free particles. For central fields, however, it can be shown that the total angular momentum $\mathbf{J} = \mathbf{L} + \mathbf{S}$ where $\mathbf{S} = \frac{1}{2}\mathbf{\Sigma}$ is a conserved quantity, i.e., $[\mathbf{J}, H] = 0$. Thus one can have a common set of eigenstates for $(H, \mathbf{J}^2, J_z)$. Because $\mathbf{S}^2 = \frac{3}{4}$ is a constant operator, it is also a conserved quantity. Unfortunately, $\mathbf{L}^2$ is not conserved and cannot have a common set of eigenstates with $(H, \mathbf{J}^2, J_z)$. If $B(r) = 0$, we have a further conserved quantity $K = \gamma^0(\mathbf{\Sigma} \cdot \mathbf{L} + 1)$ which commutes with both H and $\mathbf{J}$. In this case one can have a common set of eigenstates for $(H, \mathbf{J}^2, J_z, K, \mathbf{S}^2)$.

To solve the Dirac–Pauli equation, one should choose a specific representation for the γ matrices. Here we use the Dirac representation [6]. In this representation we have $\mathbf{\Sigma} = \mathrm{diag}(\boldsymbol{\sigma}, \boldsymbol{\sigma})$ where $\boldsymbol{\sigma}$ are Pauli matrices, and $\mathbf{J} = \mathrm{diag}(\mathbf{j}, \mathbf{j})$ where $\mathbf{j} = \mathbf{L} + \frac{1}{2}\boldsymbol{\sigma}$ is a 2×2 matrix. In this section we only consider a central electric field. The simultaneous presence of a central magnetic field will be discussed in Sec. V. We define $\psi = (\varphi, \chi)^\tau$, where τ denotes transpose, and both φ and χ are two-component spinors. The stationary Dirac–Pauli equation (6) then takes the form

$$\boldsymbol{\sigma} \cdot (\mathbf{p} - i\mu_{\mathrm{n}} E \mathbf{e}_r)\varphi = (\mathcal{E} + m_{\mathrm{n}})\chi, \tag{8a}$$

$$\boldsymbol{\sigma} \cdot (\mathbf{p} + i\mu_{\mathrm{n}} E \mathbf{e}_r)\chi = (\mathcal{E} - m_{\mathrm{n}})\varphi. \tag{8b}$$

Here four partial differential equations are involved. We are going to simplify these equations by separation of variables in spherical coordinates. Let us define the two-component spinors

$$f^+_{lm}(\theta, \phi) = \begin{pmatrix} \sqrt{\dfrac{l+m+1}{2l+1}} Y_{lm}(\theta, \phi) \\ \sqrt{\dfrac{l-m}{2l+1}} Y_{l,m+1}(\theta, \phi) \end{pmatrix}, \quad l = 0, 1, 2, \ldots; m = -(l+1), -l, \ldots, l; \tag{9a}$$

$$f^-_{lm}(\theta, \phi) = \begin{pmatrix} \sqrt{\dfrac{l-m+1}{2l+3}} Y_{l+1,m}(\theta, \phi) \\ -\sqrt{\dfrac{l+m+2}{2l+3}} Y_{l+1,m+1}(\theta, \phi) \end{pmatrix}, \quad l = 0, 1, 2, \ldots; m = -(l+1), -l, \ldots, l. \tag{9b}$$

Here $Y_{lm}(\theta, \phi)$ are spherical harmonics as defined in Ref. [7]. Both of them are common eigenstates of $(\mathbf{j}^2, j_z, \mathbf{L}^2, \mathbf{S}^2)$ with eigenvalues

$$((l + \tfrac{1}{2})(l + \tfrac{3}{2}), m + \tfrac{1}{2}, l(l+1), \tfrac{3}{4})$$

and

$$((l + \tfrac{1}{2})(l + \tfrac{3}{2}), m + \tfrac{1}{2}, (l+1)(l+2), \tfrac{3}{4}),$$

respectively. It can be shown that

$$\boldsymbol{\sigma} \cdot \mathbf{e}_r f^\pm_{lm}(\theta, \phi) = f^\mp_{lm}(\theta, \phi), \tag{10}$$

and

$$\boldsymbol{\sigma} \cdot \mathbf{L} f^+_{lm}(\theta, \phi) = l f^+_{lm}(\theta, \phi), \tag{11a}$$

$$\boldsymbol{\sigma}\cdot\mathbf{L}f_{lm}^{-}(\theta,\phi) = -(l+2)f_{lm}^{-}(\theta,\phi). \tag{11b}$$

The relation

$$\boldsymbol{\sigma}\cdot\mathbf{p} = -i(\boldsymbol{\sigma}\cdot\mathbf{e}_r)\partial_r + \frac{i}{r}(\boldsymbol{\sigma}\cdot\mathbf{e}_r)(\boldsymbol{\sigma}\cdot\mathbf{L}) \tag{12}$$

is also useful in the following. With these preparations we can simplify Eq. (8) for two different kinds of solutions.

The first kind of solution to Eq. (8) is $\psi^+ = (\varphi^+, \chi^+)^\tau$, where

$$\varphi^+(r,\theta,\phi) = u^+(r)f_{lm}^{+}(\theta,\phi), \quad \chi^+(r,\theta,\phi) = iv^+(r)f_{lm}^{-}(\theta,\phi). \tag{13}$$

Note that ψ^+ is a common eigenstate of $(\mathbf{J}^2, J_z, K, \mathbf{S}^2)$ with eigenvalues $((l+\frac{1}{2})(l+\frac{3}{2}), m+\frac{1}{2}, l+1, \frac{3}{4})$, but it is not an eigenstate of $\mathbf{L}^2$. Using the relations (10-12), it is not difficult to show that Eq. (8) now reduces to a system of first-order ODE for the radial wave functions $u^+(r)$ and $v^+(r)$:

$$\frac{du^+}{dr} + \mu_{\rm n}Eu^+ - \frac{l}{r}u^+ = -(\mathcal{E}+m_{\rm n})v^+, \tag{14a}$$

$$\frac{dv^+}{dr} - \mu_{\rm n}Ev^+ + \frac{l+2}{r}v^+ = (\mathcal{E}-m_{\rm n})u^+. \tag{14b}$$

Because θ and ϕ are not defined at the origin, the appropriate boundary conditions for u^+ and v^+ are

$$|u^+(0)| < \infty \quad (l=0), \quad u^+(0) = 0 \quad (l \neq 0), \tag{15a}$$

$$v^+(0) = 0 \quad \forall l. \tag{15b}$$

Of course they should also satisfy appropriate boundary conditions at infinity. For bound-state solutions to be considered below, they should fall off rapidly enough when $r \to \infty$ such that ψ^+ is square integrable. For scattering problem, they should be finite at infinity. Given a specific form for $E(r)$, one can solve Eq. (14) at least numerically. This is much simpler than dealing with Eq. (8). For $\mathcal{E} \neq -m_{\rm n}$, one can express v^+ in terms of u^+ by using Eq. (14a), and substitute it into Eq. (14b) to obtain a second-order ODE solely for u^+:

$$\frac{d^2u^+}{dr^2} + \frac{2}{r}\frac{du^+}{dr} + \left[\mathcal{E}^2 - m_{\rm n}^2 + \mu_{\rm n}\frac{dE}{dr} - \mu_{\rm n}^2E^2 + 2(l+1)\mu_{\rm n}\frac{E}{r} - \frac{l(l+1)}{r^2}\right]u^+ = 0. \tag{16}$$

This is similar to the radial Schrödinger equation in a central potential. It can be exactly solved for some specific form of $E(r)$. This will be studied in the subsequent sections. When Eq. (16) is solved, it is easy to obtain v^+. If $\mathcal{E} = -m_{\rm n}$, one can directly solve Eq. (14) without difficulty.

The second kind of solution to Eq. (8) is $\psi^- = (\varphi^-, \chi^-)^\tau$, where

$$\varphi^-(r,\theta,\phi) = u^-(r)f_{lm}^{-}(\theta,\phi), \quad \chi^-(r,\theta,\phi) = iv^-(r)f_{lm}^{+}(\theta,\phi). \tag{17}$$

Note that ψ^- is also a common eigenstate of $(\mathbf{J}^2, J_z, K, \mathbf{S}^2)$ with eigenvalues $((l+\frac{1}{2})(l+\frac{3}{2}), m+\frac{1}{2}, -(l+1), \frac{3}{4})$, As before, Eq. (8) reduces to a system of first-order ODE for the radial wave functions $u^-(r)$ and $v^-(r)$:

$$\frac{du^-}{dr} + \mu_{\rm n}Eu^- + \frac{l+2}{r}u^- = -(\mathcal{E}+m_{\rm n})v^-, \tag{18a}$$

$$\frac{dv^-}{dr} - \mu_{\rm n} E v^- - \frac{l}{r} v^- = (\mathcal{E} - m_{\rm n}) u^-. \tag{18b}$$

This is similar to Eq. (14). If $\mathcal{E} \neq m_{\rm n}$, one can solve Eq. (18b) for u^-, and substitute it into Eq. (18a) to obtain a second-order ODE solely for v^-:

$$\frac{d^2 v^-}{dr^2} + \frac{2}{r}\frac{dv^-}{dr} + \left[\mathcal{E}^2 - m_{\rm n}^2 - \mu_{\rm n}\frac{dE}{dr} - \mu_{\rm n}^2 E^2 - 2(l+1)\mu_{\rm n}\frac{E}{r} - \frac{l(l+1)}{r^2}\right] v^- = 0. \tag{19}$$

This is similar to Eq. (16). Note that the appropriate boundary conditions for u^- and v^- at the origin are

$$u^-(0) = 0 \quad \forall l, \tag{20a}$$

$$|v^-(0)| < \infty \quad (l = 0), \quad v^-(0) = 0 \quad (l \neq 0). \tag{20b}$$

Thus Eqs. (16) and (19) have the same boundary conditions at the origin. Also note that they interchange if $E(r) \to -E(r)$. If $\mathcal{E} = m_{\rm n}$, Eq. (19) is invalid, and one can solve Eq. (18) directly.

Using the completeness relation of the spherical harmonics, it can be shown that the two-component spinors $f_{lm}^+(\theta,\phi)$ and $f_{lm}^-(\theta,\phi)$ constitute a complete set on the sphere. More specifically, we have

$$\sum_{l=0}^{\infty} \sum_{m=-(l+1)}^{l} [f_{lm}^+(\theta,\phi) f_{lm}^{+\dagger}(\theta',\phi') + f_{lm}^-(\theta,\phi) f_{lm}^{-\dagger}(\theta',\phi')] = \delta(\cos\theta - \cos\theta')\delta(\phi - \phi'). \tag{21}$$

Therefore all possible forms of solutions to Eq. (8) are included in Eqs. (13) and (17).

In the subsequent sections we are going to solve Eqs. (16) and (19) for radially constant and radially linear electric fields. Before dealing with these specific cases, we would like to give some special bound-state solutions for more general forms of the electric field. We assume that $E(r)$ behaves like $r^{-1+\delta_1}$ when $r \to 0$ and like $r^{-1+\delta_2}$ when $r \to \infty$ where δ_1 and δ_2 are positive numbers, and is regular everywhere except possibly at $r = 0$. If $\mu_{\rm n} E(r) > 0$ for $r > r_+$ where r_+ is some finite radius, we have the following solution to Eq. (14) with energy level $\mathcal{E} = m_{\rm n}$.

$$u_l^+(r) = A_l^+ r^l \exp\left[-\int_0^r \mu_{\rm n} E(r') dr'\right], \quad v_l^+(r) = 0, \tag{22}$$

where A_l^+ is a normalization constant. This obviously satisfies the boundary conditions (15) at the origin. It is a bound-state solution because it falls off rapidly enough to be square integrable. It is remarkable that the energy eigenvalue does not depend on the quantum numbers l and m (or $j = l + \frac{1}{2}$ and $m_j = m + \frac{1}{2}$). Thus the degeneracy of this energy level is numerably infinite. This is somewhat similar to the situation of a charged particle in a magnetic field with infinite flux in two dimensions [8]. To our knowledge other similar situations were not encountered previously in the realistic three-dimensional space. This is interesting because it reveals the possibility of condensing infinitely many fermions, say, neutrons, into a single energy level. If $\mu_{\rm n} E(r) < 0$ for $r > r_-$ where r_- is some finite radius, then we have the following solution to Eq. (18) with energy level $\mathcal{E} = -m_{\rm n}$.

$$u_l^-(r) = 0, \quad v_l^-(r) = A_l^- r^l \exp\left[\int_0^r \mu_{\rm n} E(r') dr'\right], \tag{23}$$

where A_l^- is a normalization constant. This is also a bound-state solution, and the energy level is infinitely degenerate. Here we have a negative-energy solution, and will have more in the following sections. The presence of negative-energy eigenvalues and eigenstates is a quite general feature of relativistic quantum mechanics. Though these solutions are nonphysical in the one-particle theory, it is well known that they correspond to antiparticles after second quantization. Therefore we will not exclude these solutions in this paper.

If $E(r) \sim \kappa/r$ for large r where κ is a constant, the situation is of special interest. In this case the nonvanishing component of the critical solutions [$u_l^+(r)$ for $\mu_\mathrm{n}\kappa > 0$ or $v_l^-(r)$ for $\mu_\mathrm{n}\kappa < 0$] does not fall off exponentially at large r, but behaves like $r^{l-|\mu_\mathrm{n}\kappa|}$. To be normalizable, one should have $l < |\mu_\mathrm{n}\kappa| - \frac{3}{2}$. Therefore to have at least one critical bound state, one should have $|\mu_\mathrm{n}\kappa| > \frac{3}{2}$. When $|\mu_\mathrm{n}\kappa| - \frac{3}{2}$ is a natural number, we have $|\mu_\mathrm{n}\kappa| - \frac{3}{2}$ critical boumd states (degeneracy over m has not been taken into account). When $|\mu_\mathrm{n}\kappa| - \frac{3}{2}$ is not a natural number, the number of critical bound states is $[|\mu_\mathrm{n}\kappa| - \frac{1}{2}]$, where the square bracket denotes the integral part of a number. This is similar to the situation of a charged particle in a magnetic field with finite flux in two dimensions [8].

For a specific electric field, critical bound states with $\mathcal{E} = m_\mathrm{n}$ and those with $\mathcal{E} = -m_\mathrm{n}$ do not appear simultaneously. This spectral asymmetry is also similar to that of a charged particle in a magnetic field in two dimensions. Therefore vacuum polarization similar to those in two dimensions for charged particles [9-11] or neutral ones [12] may be expected for the present system after second quantization.

To conclude this section we write down the normalization condition for bound-state (or so called square integrable) solutions:

$$\int d\mathbf{r}\, \psi^{\pm\dagger}(\mathbf{r})\psi^{\pm}(\mathbf{r}) = \int_0^\infty [u^\pm(r)]^2 r^2\, dr + \int_0^\infty [v^\pm(r)]^2 r^2\, dr = 1. \tag{24}$$

The normalization constants in Eqs. (22) and (23), and those in the following sections are to be determined by this condition.

III. RADIALLY CONSTANT ELECTRIC FIELDS

In this section we consider radially constant electric fields $E(r) = E_0$ where E_0 is a constant. As pointed out before, Eqs. (16) and (19) interchange when $E(r) \to -E(r)$. So we need only consider a positive E_0 or a negative E_0. For convenience we assume that $\mu_\mathrm{n} E_0 > 0$. Now Eq. (16) takes the form

$$\frac{d^2u^+}{dr^2} + \frac{2}{r}\frac{du^+}{dr} + \left[\mathcal{E}^2 - m_\mathrm{n}^2 - \mu_\mathrm{n}^2 E_0^2 + \frac{2(l+1)\mu_\mathrm{n}E_0}{r} - \frac{l(l+1)}{r^2}\right] u^+ = 0. \tag{25}$$

This has the same form as the radial Schrödinger equation in an attractive Coulomb field. The difference is that the "Coulomb field" (the next to the last term in the square bracket) here depends on the quantum number l. Thus the energy levels will depend on l as well as a principal quantum number or a radial quantum number. As Eq. (25) is familiar in quantum mechanics, we will give the solutions only. Remember that Eq. (25) is invalid for $\mathcal{E} = -m_\mathrm{n}$.

We have the bound-state energy levels

$$\mathcal{E}_{0l} = m_{\rm n}, \quad n_r = 0, \tag{26a}$$

$$\mathcal{E}_{n_r l\pm} = \pm\left[m_{\rm n}^2 + \mu_{\rm n}^2 E_0^2 \frac{(n_r+l+1)^2-(l+1)^2}{(n_r+l+1)^2}\right]^{\frac{1}{2}}, \quad n_r = 1, 2, \ldots. \tag{26b}$$

Here n_r is a radial quantum number. When $n_r = 0$ we have a positive critical energy level given in Eq. (26a). Though it is independent of l, we keep the subscript l to make a clear correspondence to the corresponding wave functions below. For $n_r \neq 0$ we have positive- and negative-energy levels, indicated by the subscript $\pm$ in Eq. (26b). The corresponding radial wave functions are

$$u^+_{n_r l\pm}(r) = A_{n_r l\pm}\rho^l e^{-\rho/2} L^{2l+1}_{n_r}(\rho), \tag{27a}$$

$$v^+_{n_r l\pm}(r) = A_{n_r l\pm}\frac{\mu_{\rm n} E_0}{(n_r+l+1)(\mathcal{E}_{n_r l\pm}+m_{\rm n})}\rho^{l+1} e^{-\rho/2} L^{2l+3}_{n_r-1}(\rho) \tag{27b}$$

for $n_r \neq 0$, and

$$u^+_{0l}(r) = A_{0l}\rho^l e^{-\rho/2}, \tag{27c}$$

$$v^+_{0l}(r) = 0 \tag{27d}$$

for $n_r = 0$, where

$$\rho = \alpha_{n_r l} r, \quad \alpha_{n_r l} = \frac{2(l+1)\mu_{\rm n} E_0}{(n_r+l+1)}, \tag{28}$$

and $L^{2l+1}_{n_r}(\rho)$, etc., are Laguerre polynomials defined in Ref. [13], which are different from those used in Ref. [7]. Note that the superscript + indicates the first kind of solutions (13), while the subscript $\pm$ indicates the sign of the energy levels. It is seen from Eq. (27) that negative- and positive-energy solutions have the same functional form, but the coefficients are different. The normalization constants are

$$A_{n_r l\pm} = \frac{(\mu_{\rm n} E_0)^{\frac{3}{2}}}{(n_r+l+1)^2}\left[\frac{2(l+1)^3 n_r!}{(n_r+2l+1)!}\right]^{\frac{1}{2}}\left(\frac{\mathcal{E}_{n_r l\pm}+m_{\rm n}}{\mathcal{E}_{n_r l\pm}}\right)^{\frac{1}{2}} \tag{29a}$$

for $n_r \neq 0$ and

$$A_{0l} = \frac{(2\mu_{\rm n} E_0)^{\frac{3}{2}}}{\sqrt{(2l+2)!}} \tag{29b}$$

for $n_r = 0$. The degeneracy of the energy level $\mathcal{E}_{n_r l+}$ or $\mathcal{E}_{n_r l-}$ is $2l+2$. As the energy level $\mathcal{E}_{0l} = m_{\rm n}$ is actually independent of l, its degeneracy is numerably infinite. Indeed, the solution (28) is a specific realization of the solution (22) discussed before.

When $\mathcal{E} = -m_{\rm n}$, Eq. (25) is invalid. In this case one should deal with Eq. (14) directly. It is easy to show that this energy value corresponds to a trivial solution. Thus all first-kind solutions are included in Eq. (27), and the corresponding energy levels are given by Eq. (26). Note that all energy levels have absolute values less than $\sqrt{m_{\rm n}^2 + \mu_{\rm n}^2 E_0^2}$. When $|\mathcal{E}|$ exceeds this value, we have scattering solutions to Eq. (25). This will not be discussed here.

Now we turn to Eq. (19), which in the present case becomes

$$\frac{d^2v^-}{dr^2}+\frac{2}{r}\frac{dv^-}{dr}+\left[\mathcal{E}^2-m_{\rm n}^2-\mu_{\rm n}^2E_0^2-\frac{2(l+1)\mu_{\rm n}E_0}{r}-\frac{l(l+1)}{r^2}\right]v^-=0. \tag{30}$$

Since $\mu_{\rm n}E_0>0$, this is equivalent to the radial Schrödinger equation in a repulsive Coulomb field. In this case only scattering solutions are available. These scattering solutions have energy $\mathcal{E}>\sqrt{m_{\rm n}^2+\mu_{\rm n}^2E_0^2}$ or $\mathcal{E}<-\sqrt{m_{\rm n}^2+\mu_{\rm n}^2E_0^2}$. If $\mathcal{E}=m_{\rm n}$, Eq. (30) is invalid. Then we may deal with Eq. (18) directly. It turns out that this energy value corresponds to a trivial solution. We thus conclude that there is no bound state of the second kind in the present case.

To finish this section we estimate the "Bohr radius" of the neutron in this radially constant field. It is roughly equal to $\alpha_{n_rl}^{-1}$. For the critical-energy state, $n_r=0$, and $\alpha_{0l}^{-1}=(2\mu_{\rm n}E_0)^{-1}$. In the MKS system it reads

$$\alpha_{0l}^{-1}=\frac{\hbar c^2}{2\mu_{\rm n}E_0}. \tag{31}$$

We take $|E_0|=5.15\times10^{11}$ V/m, the electric field strength at the Bohr radius of the hydrogen atom. For neutrons, we have $\alpha_{0l}^{-1}=9.5\times10^{-4}$ m. This is a macroscopic length scale but rather small. However, it might be not easy to realize a radially constant central electric field with the above magnitude in the laboratory. We do not know whether there exists some such field somewhere in the universe.

IV. RADIALLY LINEAR ELECTRIC FIELDS

In this section we turn to another exactly solvable field, the radially linear electric field $E(r)=\beta r$ where β is a constant. The electric charge density that produces this field is $\rho_{\rm c}=3\beta/4\pi$ in the Gaussian units, which is a constant. To realize the above central field, however, the electric charge density should become zero outside some large sphere where the particle under consideration cannot reach practically. Otherwise the electric field would be zero everywhere. In the region of interest (inside the large sphere) the field is then radially linear. For reasons given earlier, we need only consider one sign of β. For convenience we assume that $\beta\mu_{\rm n}>0$. Eq. (16) then takes the form

$$\frac{d^2u^+}{dr^2}+\frac{2}{r}\frac{du^+}{dr}+\left[\mathcal{E}^2-m_{\rm n}^2+(2l+3)\beta\mu_{\rm n}-\beta^2\mu_{\rm n}^2r^2-\frac{l(l+1)}{r^2}\right]u^+=0. \tag{32}$$

This is not valid for $\mathcal{E}=-m_{\rm n}$. In the latter case one can solve Eq. (14) directly and obtain a trivial solution. Thus all nontrivial solutions of the first kind are those arise from Eq. (32). The equation (32) has the same form as the radial Schrödinger equation for an isotropic harmonic oscillator. The difference is that the "energy" here depends on the quantum number l. Thus the dependence of the energy levels on the quantum numbers will be different from that of the isotropic harmonic oscillator. Since Eq. (32) is also familiar in quantum mechanics, we will give the solutions only. There are only bound-state solutions. The energy levels are

$$\mathcal{E}_0^+=m_{\rm n},\quad n_r=0 \tag{33a}$$

$$\mathcal{E}^{+}_{n_r\pm} = \pm\sqrt{m_{\rm n}^2 + 4n_r\beta\mu_{\rm n}}, \quad n_r = 1, 2, \ldots, \tag{33b}$$

where n_r is a radial quantum number. Note that the superscript $+$ for $\mathcal{E}$ indicates the first kind of solutions, while the subscript $\pm$ indicates the sign of the energy. As before, we have negative- as well as positive-energy levels. The corresponding radial wave functions are

$$u^{+}_{0l}(r) = A^{+}_{0l}\rho^l e^{-\rho^2/2}, \quad v^{+}_{0l}(r) = 0 \tag{34}$$

for $n_r = 0$, and

$$u^{+}_{n_r l\pm}(r) = A^{+}_{n_r l\pm}\rho^l e^{-\rho^2/2} L^{l+1/2}_{n_r}(\rho^2), \tag{35a}$$

$$v^{+}_{n_r l\pm}(r) = A^{+}_{n_r l\pm}\frac{2\sqrt{\beta\mu_{\rm n}}}{\mathcal{E}^{+}_{n_r\pm} + m_{\rm n}}\rho^{l+1} e^{-\rho^2/2} L^{l+3/2}_{n_r-1}(\rho^2) \tag{35b}$$

for $n_r \neq 0$, where

$$\rho = \sqrt{\beta\mu_{\rm n}}r \tag{36}$$

and $L^{l+1/2}_{n_r}(\rho^2)$, etc., are Laguerre polynomials as employed in Sec. III but the argument here is ρ^2. The normalization constants are determined by Eq. (24) and are given by

$$A^{+}_{0l} = \frac{\sqrt{2}(\beta\mu_{\rm n})^{\frac{3}{4}}}{\sqrt{\Gamma(l + 3/2)}}, \quad n_r = 0 \tag{37a}$$

$$A^{+}_{n_r l\pm} = (\beta\mu_{\rm n})^{\frac{3}{4}}\left[\frac{n_r!}{\Gamma(n_r + l + 3/2)}\right]^{\frac{1}{2}}\left(\frac{\mathcal{E}^{+}_{n_r\pm} + m_{\rm n}}{\mathcal{E}^{+}_{n_r\pm}}\right)^{\frac{1}{2}}, \quad n_r = 1, 2, \ldots. \tag{37b}$$

It is remarkable that all the above energy levels are independent of the quantum number l, and thus all of them are infinitely degenerate. The critical-energy solution (34) is another realization of the solution (22).

Now we consider the second kind of solutions (17). It is easy to show that Eq. (18) gives a trivial solution when $\mathcal{E} = m_{\rm n}$. Thus all nontrivial solutions arise from Eq. (19) which is valid for $\mathcal{E} \neq m_{\rm n}$ and in the present case becomes

$$\frac{d^2v^-}{dr^2} + \frac{2}{r}\frac{dv^-}{dr} + \left[\mathcal{E}^2 - m_{\rm n}^2 - (2l+3)\beta\mu_{\rm n} - \beta^2\mu_{\rm n}^2 r^2 - \frac{l(l+1)}{r^2}\right]v^- = 0. \tag{38}$$

This is very similar to Eq. (32). The only difference lies in the sign of the third term in the square bracket. This difference, however, will render the energy levels quite different from those obtained above. As before, we only give the results here. The energy levels are

$$\mathcal{E}^{-}_{N\pm} = \pm\sqrt{m_{\rm n}^2 + (4N+6)\beta\mu_{\rm n}}, \quad N = n_r + l = 0, 1, 2, \ldots, \tag{39}$$

where $n_r = 0, 1, 2, \ldots$ is a radial quantum number and N is a principal quantum number. The superscript $-$ of $\mathcal{E}$ indicates the second kind of solutions, while the subscript $\pm$ indicates the sign of the energy. The spectrum obtained here has no overlap with that in Eq. (33). The corresponding radial wave functions are

$$u^{-}_{n_r l\pm}(r) = -A^{-}_{n_r l\pm}\frac{2\sqrt{\beta\mu_{\rm n}}}{\mathcal{E}^{-}_{N\pm} - m_{\rm n}}\rho^{l+1} e^{-\rho^2/2} L^{l+3/2}_{n_r}(\rho^2), \tag{40a}$$

$$v^{-}_{n_r l\pm}(r) = A^{-}_{n_r l\pm}\rho^l e^{-\rho^2/2} L^{l+1/2}_{n_r}(\rho^2), \tag{40b}$$

where ρ is given by Eq. (36), and $n_r = 0, 1, 2, \ldots$ is the radial quantum number. The normalization constants are given by

$$A^{-}_{n_r l\pm} = (\beta\mu_{\rm n})^{\frac{3}{4}} \left[\frac{n_r!}{\Gamma(n_r + l + 3/2)}\right]^{\frac{1}{2}} \left(\frac{\mathcal{E}^{-}_{N\pm} - m_{\rm n}}{\mathcal{E}^{-}_{N\pm}}\right)^{\frac{1}{2}}, \quad n_r = 0, 1, 2, \ldots. \tag{41}$$

The energy levels $\mathcal{E}^{-}_{N+}$ and $\mathcal{E}^{-}_{N-}$ depend only on the principal quantum number N. Given N, l may vary from 0 to N. For a given l, there are $2l + 2$ different solutions. Therefore the degeneracy of the level $\mathcal{E}^{-}_{N+}$ or $\mathcal{E}^{-}_{N-}$ is

$$d_N = \sum_{l=0}^{N}(2l + 2) = (N + 1)(N + 2). \tag{42}$$

In conclusion, in the radially linear electric field, we have two sets of bound-state energy levels. The first set is given in Eq. (33), corresponding to the first kind of solutions. The second set is given in Eq. (39), corresponding to the second kind of solutions. There is no scattering solution here. In contrast, the radially constant electric field studied in Sec. III admits both scattering and bound-state solutions, though there exists no bound state of the second kind. Finally we estimate the "Bohr radius" of the neutron in the present case. This is roughly equal to $(\beta\mu_{\rm n})^{-\frac{1}{2}}$, or $(3/4\pi\rho_{\rm c}\mu_{\rm n})^{\frac{1}{2}}$ where $\rho_{\rm c}$ is the electric charge density producing the field. In the MKS system this reads

$$\left(\frac{3\hbar}{4\pi\mu_0\rho_{\rm c}\mu_{\rm n}}\right)^{\frac{1}{2}},$$

where μ_0 is the permeability of the vacuum. We take $\rho_{\rm c} = e/a_0^3$ where e is the electron charge and a_0 is the Bohr radius of the hydrogen atom. For neutrons the above "Bohr radius" has the value 4.4×10^{-8} m. This is rather small. However, it may be difficult to achieve the above electric charge density.

V. SUMMARY AND DISCUSSIONS

In the preceding sections we have studied the Dirac–Pauli equation of a neutral fermion with nonminimal coupling to a central electric field. By separation of variables in spherical coordinates, the stationary Dirac–Pauli equation which involves four partial differential equations can be reduced to a system of ODE which involves two coupled first-order ODE for two radial wave functions. There are two different kinds of solutions, and thus two independent systems of ODE. Bound states of critical energy values can be obtained analytically for a quite general class of electric fields, where the degeneracy of the critical energy level turns out to be numerably infinite. This reveals the possibility of condensing infinitely many fermions into a single energy level. We also discussed a special form of the electric field that supports a finite number of critical bound states. Two specific electric fields, one radially constant and the other radially linear, are studied in detail and all the bound-state solutions are obtained in closed forms. In the first case bound states exist only for the first kind of solutions, while scattering states exist for both kinds. Scattering states are not discussed in detail. In the second case, we have two sets of discrete energy levels corresponding to the two kinds of solutions. There is no scattering state. It turns out that the energy levels in the first set are all infinitely degenerate. In both fields we have negative as well as positive energy levels. Critical energy levels are also admitted in both cases, which may be positive or negative depending on the signs of $\mu_{\rm n}$ and the electric fields. Note that the two critical energy levels are not admitted at the same time, however. This spectral asymmetry may likely lead to vacuum polarization after second quantization.

In Sec. II we have shown that the total angular momentum $\mathbf{J}$ is a conserved quantity in the simultaneous presence of a central magnetic field and a central electric field. But we have not discussed the solutions of the Dirac–Pauli equation in this case. In the Dirac representation, the stationary Dirac–Pauli equation (6) takes the form

$$\boldsymbol{\sigma}\cdot(\mathbf{p}-i\mu_{\rm n}E\mathbf{e}_r)\varphi=(\mathcal{E}+m_{\rm n}-\mu_{\rm n}B\boldsymbol{\sigma}\cdot\mathbf{e}_r)\chi, \tag{43a}$$

$$\boldsymbol{\sigma}\cdot(\mathbf{p}+i\mu_{\rm n}E\mathbf{e}_r)\chi=(\mathcal{E}-m_{\rm n}+\mu_{\rm n}B\boldsymbol{\sigma}\cdot\mathbf{e}_r)\varphi. \tag{43b}$$

These equations are similar to Eq. (8) but more complicated. They are still separable in spherical coordinates. We set

$$\varphi(r,\theta,\phi)=u^+(r)f^+_{lm}(\theta,\phi)+u^-(r)f^-_{lm}(\theta,\phi), \tag{44a}$$

$$\chi(r,\theta,\phi)=iv^+(r)f^-_{lm}(\theta,\phi)+iv^-(r)f^+_{lm}(\theta,\phi). \tag{44b}$$

Substituting these ansatz into Eq. (43) and using the relations (10-12) we obtain the following system of ODE for the four radial wave functions.

$$\frac{du^+}{dr}+\mu_{\rm n}Eu^+-\frac{l}{r}u^+=-(\mathcal{E}+m_{\rm n})v^++\mu_{\rm n}Bv^-, \tag{45a}$$

$$\frac{dv^+}{dr}-\mu_{\rm n}Ev^++\frac{l+2}{r}v^+=(\mathcal{E}-m_{\rm n})u^++\mu_{\rm n}Bu^-, \tag{45b}$$

$$\frac{du^-}{dr}+\mu_{\rm n}Eu^-+\frac{l+2}{r}u^-=-(\mathcal{E}+m_{\rm n})v^-+\mu_{\rm n}Bv^+, \tag{45c}$$

$$\frac{dv^-}{dr}-\mu_{\rm n}Ev^--\frac{l}{r}v^-=(\mathcal{E}-m_{\rm n})u^-+\mu_{\rm n}Bu^+. \tag{45d}$$

If $B(r)=0$, one may set $u^-=v^-=0$ which reduces Eq. (45) to Eq. (14) for the first kind of solutions, or set $u^+=v^+=0$ which reduces Eq. (45) to Eq. (18) for the second kind of solutions. This is what we have done before for a pure electric field. When a magnetic field is present at the same time, this is not allowed, however. The essential reason is that K is no longer a conserved quantity in this case. All the four ODE in Eq. (45) are coupled to each other. It seems difficult to solve them even for a pure magnetic field. We will not go into further details of these equations here.

Let us briefly discuss the nonrelativistic limit of the Dirac–Pauli equation. Consider the stationary case with a pure electric field. We can solve Eq. (8a) for χ, and substitute it into Eq. (8b) to obtain an equation for φ:

$$[\boldsymbol{\sigma}\cdot(\mathbf{p}+i\mu_{\rm n}\mathbf{E})][\boldsymbol{\sigma}\cdot(\mathbf{p}-i\mu_{\rm n}\mathbf{E})]\varphi=(\mathcal{E}^2-m_{\rm n}^2)\varphi. \tag{46}$$

This holds for any value of $\mathcal{E}$ except $\mathcal{E}=-m_{\rm n}$, and is valid for noncentral electric field as well. To discuss the nonrelativistic limit we consider only positive $\mathcal{E}$ and set

$$\mathcal{E}=m_{\rm n}+\mathcal{E}'.$$

When $\mathcal{E}'\ll m_{\rm n}$ we get the nonrelativistic limit of Eq. (46):

$$[\boldsymbol{\sigma}\cdot(\mathbf{p}+i\mu_{\rm n}\mathbf{E})][\boldsymbol{\sigma}\cdot(\mathbf{p}-i\mu_{\rm n}\mathbf{E})]\varphi=2m_{\rm n}\mathcal{E}'\varphi. \tag{47}$$

This has essentially the same form as Eq. (46), and thus the same solutions. However, it should be remarked that even when $|\mu_{\rm n}\mathbf{E}|\ll m_{\rm n}$, Eq. (47) is not valid for those $\mathcal{E}'$ comparable with $m_{\rm n}$. For example, in the radially constant field with $|\mu_{\rm n}E_0|\ll m_{\rm n}$, Eq. (47) is good for all bound states, but not for scattering ones with large $\mathcal{E}$, say, $\mathcal{E}=2m_{\rm n}$. On the other hand, even if $|\mathbf{E}|$ is unbounded, Eq. (47) is still valid for small $\mathcal{E}'$. For example, in the radially linear field, Eq. (47) may be good for lower levels if $|\beta\mu_{\rm n}|\ll m_{\rm n}$. Since Eq. (47) is not simpler, it is more convenient to deal with Eq. (46) directly. The nonrelativistic limit with both magnetic and electric fields can be similarly discussed, though the situation is more complicated. We will not give further details here.

We have pointed out in Sec. III that the radially constant electric field admit scattering solutions of both kinds. Though Eqs. (25) and (30) can be solved to give partial wave solutions, the scattering problem is difficult to handle in this case since these equations involve long-range "Coulomb potentials". An easier situation for the scattering problem may be the field $E(r)\propto r^{-1}$. This will be studied subsequently.

In this paper we have dealt with (3+1)-dimensional problems. The Dirac–Pauli equation (1) has a much simpler form in a (2+1)-dimensional space-time. Indeed, the situation for the AC effect is equivalent to a (2+1)-dimensional problem because of the specific field configuration. Recently, we have calculated the probability of neutral particle-antiparticle pair creation in the vacuum by external electromagnetic fields in 2+1 dimensions, based on the nonminimal coupling [14]. Both scattering and bound-state problems in external fields are easier in 2+1 dimensions. These and other consequences of the nonminimal coupling will also be studied subsequently.

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