Quantization of Soliton Cellular Automata *

Demosthenes Ellinas† Elena P. Papadopoulou‡ and Yiannis G. Saridakis §

Department of Sciences
Section of Mathematics
Technical University of Crete
GR - 73 100 Chania Crete Greece

February 1, 2008

Abstract

A method of quantization of classical soliton cellular automata (QSCA) is put forward that provides a description of their time evolution operator by means of quantum circuits that involve quantum gates from which the associated Hamiltonian describing a quantum chain model is constructed. The intrinsic parallelism of QSCA, a phenomenon first known from quantum computers, is also emphasized.

Introduction. Soliton cellular automata (SCA), is a class of cellular automata[1] operating on binary sequences with an updating rule function f for each cell, that depends on past and present time cells, the number of which determines the *radius* r of the automaton[2]. Contrary to usual CA (that evolve their cells by means of past time cells only), SCA exhibit a variety of evolution patterns[4, 5, 6] that is mainly known to characterize the temporal behavior of solutions of non linear PDE's[7], namely: periodic evolution of *particles* (i.e localized groups of binary cells), or *solitonic* type of scattering of digital particles, or even *breathing* modes of oscillations between particles. All these properties have motivated a number of suggestive applications for a new kind of computational architecture that will utilize these evolution patterns of SCA in order to provide a "gateless" implementation of logical operations[2]. Towards a physical microscopic realization of these suggestions, envisaged in the context of the new paradigm of *Quantum Computing*, it is plausible to formulate SCA in terms of Quantum Mechanics and to investigate the possible quantum effects in their time evolution. To this end we put forward here the quantization of classical SCA, and point out their quantum parallelism.

Quantization. Let the product Hilbert space $\mathcal{H} = \otimes_{i\in\mathbb{Z}}\mathcal{H}_i$ where $\mathcal{H}_i = span\{|0\rangle, |1\rangle\} \approx \mathbb{C}^2$. Take $n \in \mathbf{N}$, and consider the subspace $H \subset \mathcal{H}$ where $H = \otimes_{k=-r}^{r}\mathcal{H}_{n-k}$, then define the vectors

$$\left|a^t\right\rangle \equiv \otimes_{i=r}^{1}\left|a_{n-i}^{t+1}\right\rangle \otimes \left|a_n^t\right\rangle \otimes_{j=1}^{r}\left|a_{n+j}^t\right\rangle \in H \ , \quad (1)$$

*Submitted to Journal of Nonlinear Mathematical Physics. Special Issue of Proccedings of NEEDS'99.

†ellinas@science.tuc.gr

‡elena@science.tuc.gr

§yiannis@science.tuc.gr

and the dual vectors $\langle a^t | \in \tilde{H}$, with orthogonormality relation

$$\begin{aligned} \left\langle a^t | b^t \right\rangle &= \Pi_{i=r}^{1} \left\langle a_{n-i}^{t+1} | b_{n-i}^{t+1} \right\rangle \cdot \left\langle a_n^t | b_n^t \right\rangle \cdot \Pi_{j=1}^{r} \left\langle a_{n+j}^t | b_{n+j}^t \right\rangle \\ &= \Pi_{i=r}^{1} \delta_{a_{n-i}^{t+1}, b_{n-i}^{t+1}} \cdot \delta_{a_n^t, b_n^t} \cdot \Pi_{j=1}^{r} \delta_{a_{n+j}^t, b_{n+j}^t} \,. \end{aligned} \tag{2}$$

Recall that for $(x, y) \in \mathbb{Z}_2^2$ the Kronecker delta function is defined as $\delta_{x,y} = 1 \oplus x \oplus y$, where $\oplus$ denotes `XOR`, i.e modulo-2 addition. Now take $\mathbb{Z}_2^{*2r+1} \equiv \mathbb{Z}_2^{2r+1} \setminus O^{2r+1}$, where O^{2r+1} is the $2r+1$-fold null string, $\mathbb{Z}_2^{\dagger 2r+1} \equiv \mathbb{Z}_2^{2r+1} \setminus \{a_0\}$ where $f(\{a_0\}) = O^{2r+1}$, the preimage of the null string, and define the one-to-one function[5] $f : \mathbb{Z}_2^{*2r+1} \to \mathbb{Z}_2^{\dagger 2r+1}$, as $f(a^t) = a^{t+1}$, where $a^t \equiv \{a_{n-r}^{t+1}, \ldots, a_{n-1}^{t+1}, a_n^t, a_{n+1}^t, \ldots, a_{n+r}^t\}$, and $a^{t+1} \equiv \{a_{n-r}^{t+1}, \ldots, a_{n-1}^{t+1}, a_n^{t+1}, a_{n+1}^t, \ldots, a_{n+r}^t\}$, where the updated bit takes the value $a_n^{t+1} = 1 \oplus_{i=r}^{1} a_{n-i}^{t+1} \oplus_{j=0}^{r} a_{n+j}^t$.

Introduce now the quantization of classical cellular automaton by means of the following quantization diagram.

$$\begin{array}{ccccc} & \mathbb{Z}_2^{*2r+1} & \xrightarrow{f} & \mathbb{Z}_2^{\dagger 2r+1} & \\ \rho & \downarrow & & \downarrow & \rho \\ & H & \xrightarrow[U_f]{} & H & \end{array}$$

This diagram implies that:

$$U_f \circ \rho(a^t) = U_f \left| a^t \right\rangle = \left| f(a^t) \right\rangle = \left| a^{t+1} \right\rangle = \rho \circ f(a^t) = \rho(a^{t+1}) = \left| a^{t+1} \right\rangle \,. \tag{3}$$

The transition operator U_f implements in the space of qubits H the update rule f of the classical bits of the automaton, and acts trivially as the unit operator in the rest space. Since in order to establish the one-to-one property for f we have excluded the null string from its domain of values the associated operator $U_f \in End H$, is a partial isometry in $H_0^{\perp}$, the orthogonal complement space of $H_0 \equiv span\{|0\rangle\}$, assuming the decomposition $H = H_0 + H_0^{\perp}$. To be more specific, let $U_f = \sum_{a^t \in \mathbb{Z}_2^{*2r+1}} |f(a^t)\rangle\langle a^t|$, then its isometric property is due to the following relations:

$$\begin{aligned} U_f {U_f}^{\dagger} &= \sum_{(a^t,b^t) \in \mathbb{Z}_2^{*2r+1}} \left| f(a^t) \right\rangle \left\langle a^t | b^t \right\rangle \left\langle f(b^t) \right| = \sum_{a^t \in \mathbb{Z}_2^{2r+1} \setminus \{a_0^t\}} \left| a^t \right\rangle\!\left\langle a^t \right| \,, \\ {U_f}^{\dagger} U_f &= \sum_{(a^t,b^t) \in \mathbb{Z}_2^{*2r+1}} \left| b^t \right\rangle \left\langle f(b^t) | f(a^t) \right\rangle \left\langle a^t \right| = \sum_{a^t \in \mathbb{Z}_2^{2r+1} \setminus \{0\}} \left| a^t \right\rangle\!\left\langle a^t \right| \,, \end{aligned} \tag{4}$$

where the preimage of the null string for e.g the simplest case of $r = 2$ is $a_0^t = 00100$.

Choosing a basis of vectors in H, e.g the computational basis $\mathcal{B} \equiv \{|x_1\rangle \otimes \cdots \otimes |x_{2r+1}\rangle : \{x_i\}_{i=1}^{2r+1} \in \mathbb{Z}_2^{2r+1}\}$, we can obtain a $2^{2r+1} \times 2^{2r+1}$ matrix representation of U_f viz.

$$\pi_r(U_f) =$$

$$\sum_{a^t \in \mathbb{Z}^{2r+1}} \otimes_{i=r}^{1} \begin{pmatrix} \delta_{0,a_{n-i}^{t+1}} & 0 \\ 0 & \delta_{1,a_{n-i}^{t+1}} \end{pmatrix} \otimes \begin{pmatrix} \delta_{0,a_n^{t+1}} \delta_{0,a_n^t} & \delta_{0,a_n^{t+1}} \delta_{1,a_n^t} \\ \delta_{1,a_n^{t+1}} \delta_{0,a_n^t} & \delta_{1,a_n^{t+1}} \delta_{1,a_n^t} \end{pmatrix} \otimes_{j=1}^{r} \begin{pmatrix} \delta_{0,a_{n+j}^t} & 0 \\ 0 & \delta_{1,a_{n+j}^t} \end{pmatrix} \tag{5}$$

More explicitly if we partition the basis $\mathcal{B}$ into two orthogonal compliments corresponding to invariant and non invariant subspaces of U_f we will obtain for e.g the case of radius $r = 2$, the transition matrix :

$$\pi_2(U_f) = \begin{pmatrix} \mathbf{1} & \mathbf{0} \\ \mathbf{0} & \sigma_1^{'} \end{pmatrix} , \tag{6}$$

where $\mathbf{1}$,$\mathbf{0}$, stand for the square 16-dimensional unit and null matrices respectively, while $\sigma_1^{'}$, is a $N = 16$-dimensional matrix with elements along the main antidiagonal $(a_{1,N}, a_{2,N-1}, \dots, a_{N,1}) = (1, \dots, 1, 0)$, and all others been zero.

Fig. 1. The factorization of U_f in terms of quantum gates for radius $r = 2$.
Fig. 2. Quantum circuit implementation of FRT for three quantum BS's with $r = 2$.

Hamiltonian model. In order to construct a Hamiltonian model associated with the QSCA that generates the total evolution of the automaton, and in view of the fact that the time step evolution operator is determined by unitary NOT viz. $U_N^i |i\rangle = |1 \oplus i\rangle$ and CONTROL − NOT(CN) viz. $U_{CN}^{ij} |ij\rangle = |ii \oplus j\rangle$ gate operators, cf. its explicit factorization in Fig. 1, we recall here first the Hermitian operators corresponding to those gates [$|i\rangle \otimes |j\rangle$ is abbreviated to $|ij\rangle$]. The quantum negation of the i-th qubit is given by the matrix $U_N^i = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}_i$ acting non trivially only in the i-th subspace, with associated Hermitian generator given by $H_N^i = \frac{1}{2}(\sigma_z^i + \sigma_x^i)$, in terms of Pauli sigma matrices[8]. For the conditional negation of CN gate with i-control and j-target qubits and $i < j$ the unitary matrix $U_{CN}^{ij} = |0\rangle\langle 0| \otimes \mathbf{1} + |1\rangle\langle 1| \otimes \sigma_x = \exp(i\pi H_{CN}^{ij})$ is generated by the Hamiltonian $H_{CN}^{ij} = \frac{1}{2}(\mathbf{1} - \sigma_z^i)(\sigma_x^j - \mathbf{1})$. Correspondingly for $i > j$, $U_{CN}^{ij} = \mathbf{1} \otimes |0\rangle\langle 0| + \sigma_x \otimes |1\rangle\langle 1| = \exp(i\pi H_{CN}^{ij})$, and $H_{CN}^{ij} = \frac{1}{2}(\mathbf{1} - \sigma_z^j)(\sigma_x^i - \mathbf{1})$. These gate operators can be utilized to factorize the U_f that is used to update each qubit in a given string of qubits of a QSCA. To this end in Fig.1 we construct the quantum circuit that provides such a factorization for the special case $r = 2$; the generalization to any r is straightforward. Specifically the input state vector in Fig. 1 reads $|a^t\rangle \equiv \left|a_{n-2}^{t+1}, a_{n-1}^{t+1}, a_n^t, a_{n+1}^t, a_{n+2}^t\right\rangle$, and is tranformed by U_f to $|a^{t+1}\rangle \equiv U_f |a^t\rangle = \left|a_{n-2}^{t+1}, a_{n-1}^{t+1}, a_n^{t+1}, a_{n+1}^t, a_{n+2}^t\right\rangle = U_N^n U_{CN}^{n+2,n} U_{CN}^{n+1,n} U_{CN}^{n-1,n} U_{CN}^{n-2,n} |a_n^t\rangle$, where $|a_n^{t+1}\rangle = \left|1 \oplus a_{n-2}^{t+1} \oplus a_{n-1}^{t+1} \oplus a_n^t \oplus a_{n+1}^t \oplus a_{n+2}^t\right\rangle$. In Fig. 1 graphically the CN gate is indicated as a vertical line segment with a bullet (control qubit) and a crossed circle (target qubit) at its ends [10]. Then the total unitary evolution operator $U_f = \prod_i^\infty U_i^f \equiv \dots U_2^f U_1^f U_0^f$, is the product of update operators for each qubit in the automaton starting with the first one U_0^f, from which all later ones are determined by shifting i.e $U_{i+1}^f = \mathbf{1} \otimes U_i^f$. The total Hamiltonian $H_f = \lim_{t \to 0} \frac{1}{i} \frac{dU_f}{dt} = \sum_{i \geq 0} H_i^f$, is the sum of all Hamiltonians generationg each evolution operator $\{U_i^f, i \geq 0\}$. Since $U_i^f = U_N^i \prod_{k=1}^r U_{CN}^{i+k,i} \prod_{k=1}^r U_{CN}^{i-k,i}$, the corresponding Hamiltonian at each site i turns out to be $H_i^f = H_N^i + \sum_{k=1}^r (H_{CN}^{i+k,i} + H_{CN}^{i-k,i})$. In view of the generators of the quantum gates given above, the Hamiltonian model of the QSCA reads,

$$H_f = \sum_{i \geq 0} \frac{1}{2} [(\sigma_x^i + \sigma_z^i) + \sum_{k=1}^r (\mathbf{1}^{i+k} - \sigma_z^{i+k})(\sigma_x^i - \mathbf{1}^i) + (\mathbf{1}^{i-k} - \sigma_z^{i-k})(\sigma_x^i - \mathbf{1}^i)]. \tag{7}$$

This can be interpreted as the Hamiltonian operator of a infinite quantum spin chain model with free boundaries and interactions ranged over r neighbors. Obvious modifications such that

QCSA with periodic boundary conditions, or higher dimensions are in the present formalism appealing and worth of further study.

Quantum Fast Rule Theorem. We turn now to study QSCA in terms of the so called Fast Rule Theorem(FRT) [3] that provides analytic tools for the classification and the study of types of dynamical evolution of classical SCA. In the limited space available here we shall confine ourselves to the special case of periodic particles and provide a quantum circuit of operators that governs the time development of periodic qubit-particles, i.e the quantum analogue of periodic particles of classical CA theory[4, 5, 6].

We recall that a particle is an integer multiple of $r+1$ consecutive sites and it can be a collection of the so called *basic strings* (BS), which are $r+1$ consecutive sites starting with a boxed site. If a particle consists of a single BS, we call it a *simple particle*.
Then a basic theorem states the following[4]: Let a single particle $OA^1A^2\cdots A^LO$ with $A^1 \neq O, A^L \neq O$, consisting of L basic strings $A^1, A^2, \cdots, A^L$ (here O denotes the null BS consisting of $r+1$ zeros), and let $\oplus$ denotes sitewise `XOR` among bits of the BS. Suppose that the BS's $A^1, A^1 \oplus A^2, A^2 \oplus A^3, \cdots, A^{L-1} \oplus A^L, A^l$, contain $l_0, l_1, \cdots, l_L$ 1's respectively. Then if such a particle does not split or loose any BS's from its right end at all times during its evolution, then at times $t = l_0 + l_1 + \ldots l_m, \quad m \leq k \leq L$, it becomes $A^{m+1} \oplus (A^{m+2}A^{m+3}\cdots A^LA^0A^1\cdots A^m)$. Especially for $k = L$, i.e at time $t = l_0 + l_1 + \ldots l_L$ it returns to its initial state $A^1A^2\cdots A^L$.

To illustrate the workings of the quantum FRT i.e the quantum adaptation of the preceding theorem, let us consider an initial particle of $L = 3$ classical BS's $A^1A^2A^3$. Such a classical particle by means of the quantization map ρ becomes the qubit word $|A^1A^2A^3OOOO\rangle$. Let us assume that the classical word labelling this state vector evolves under the previously stated conditions so that the classical FRT applies and assures that it has a periodic behaviour. Then the quantization map induces into the quantum state vector the periodic evolution of its classical label word. Namely the quantum state vector evolves periodically, and we seek to determine the evolution operator that implements this periodic motion. Assuming we have a sufficient number of null quantum BS's from each side of the initial particle state i.e $|A^1A^2A^3OOOO\rangle$, let us consider the following two operators. First, the operator $\mathcal{P}_0^I = \sum_{x\in\mathbb{Z}^{*r+1}} |0^{r+1}\rangle\langle x|$, which projects each $r+1$-fold tensor product BS state vector onto the $r+1$-fold null state vector. With the upper index I denotes the fact that the $(r+1)$-fold input vector is placed in the I-th position of a given chain of tensor product of state vector, and that $\mathcal{P}_0^I$ acts non trivial only in that I-th subspace. Second, let us define the collective `CN` gate operator $\mathcal{U}_{CN}^{IJ} \equiv U_{CN}^{I_{r+1}J_{r+1}}\cdots U_{CN}^{I_2J_2}U_{CN}^{I_1J_1}$, with control BS placed in the I-th position and target BS placed in the J-th position. This collective `CN` gate is actually defined as a succesion of $(r+1)$ `CN` gates for the corresponding qubits of the $(r+1)$-fold tensor product control and the target BS states vectors.

In Fig. 2 we indicate graphically the sequence of operations that tranform the initial state vector $|A^1A^2A^3OOOO\rangle$ into the final state $|OOOOA^1A^2A^3\rangle$, indicating in this way its periodic *propagation*. Each parallel wire in the figure represents an $(r+1)$-fold tensor product of qubits and the operators $\mathcal{P}_0$ are indicated by boxed zeros, while operators $\mathcal{U}_{CN}^{IJ}$ are indicated a double circled `CN` gate symbol. Explicitly the sequence of operations of the quantum circuit in Fig. 2 reads:

$$\begin{aligned} \left|A^1A^2A^3OOOO\right\rangle &\rightarrow \\ \left|OA^1 \oplus A^2A^1 \oplus A^3A^1OOO\right\rangle &= \mathcal{P}_0^1\mathcal{U}_{CN}^{14}\mathcal{U}_{CN}^{13}\mathcal{U}_{CN}^{12}\left|A^1A^2A^3OOOO\right\rangle \rightarrow \end{aligned}$$

$$\left|OOA^2 \oplus A^3A^2A^1 \oplus A^2OO\right\rangle = \mathcal{P}_0^2\mathcal{U}_{CN}^{25}\mathcal{U}_{CN}^{24}\mathcal{U}_{CN}^{23}\left|OA^1 \oplus A^2A^1 \oplus A^3A^1OOO\right\rangle \rightarrow$$
$$\left|OOOA^3A^1 \oplus A^3A^2 \oplus A^3O\right\rangle = \mathcal{P}_0^3\mathcal{U}_{CN}^{36}\mathcal{U}_{CN}^{35}\mathcal{U}_{CN}^{34}\left|OOA^2 \oplus A^3A^2A^1 \oplus A^2OO\right\rangle \rightarrow$$
$$\left|OOOOA^1A^2A^3\right\rangle = \mathcal{P}_0^4\mathcal{U}_{CN}^{47}\mathcal{U}_{CN}^{46}\mathcal{U}_{CN}^{45}\left|OOOA^3A^1 \oplus A^3A^2 \oplus A^3O\right\rangle . \quad (8)$$

Quantum parallelism. The quantum description of the cells of a CA in terms of qubits allows for having a superposition $a|0\rangle + b|1\rangle$ at each cell. According to the standard interpretation of Quantum Mechanics[8] this combination means that we have the state $|0\rangle(|1\rangle)$ with probability $|a|^2(|b|^2)$. This is the inherent probabilistic character of QCA that entails two important advantages of QCA over classical probabilistic (noisy) CA (see e.g [1]). First, there is an exponential overhead in storing cell values in QCA over classical CA: an r-radious probabilistic SCA needs at each discrete time step to store $N = 2^{2r+1}$ input words in order to process them later on and to update the current cell value. On the contrary a QSCA may form and admit as input a superposition state vector $|\psi\rangle = \frac{1}{\sqrt{N}}\sum_{x\in\mathbb{Z}_2^{*2r+1}}|x\rangle \equiv \frac{1}{\sqrt{N}}\sum_{x\in\mathbb{Z}_2^{*2r+1}}|x_1\cdots x_{2r+1}\rangle$, which with resources linear in r i.e $2r+1$ qubits only can constructs an exponential in r register of 2^{2r+1} equiprobable input states. Second, there is an exponential overhead in the updating time of the cells. Once a superposition of all input states has been prepared as the single state vector $|\psi\rangle$, then by acting the linear evolution operator U_f only once on it i.e $U_f|\psi\rangle = \frac{1}{\sqrt{N}}\sum_{x\in\mathbb{Z}_2^{*2r+1}}U_f|x\rangle = \frac{1}{\sqrt{N}}\sum_{x\in\mathbb{Z}_2^{*2r+1}}|f(x)\rangle$, we can update all the N state vectors simultaneously. This same exponential acceleration in storing space and processing time of quantum information has been known in quantum computation, where it is utilized in a number of tasks such as fast quantum algorithms, information encoding, cryptography etc[9].

Harnessing the space-time resources made available by the quantum parallelism of QCSA in various applications is an open challenge. The research program initiated here will require further studies of QSCA both as a new computational machine and as an implementable physical system. Some of these questions however will be addressed elsewhere[11].

Discussion. We close by discussing some prospects of our work. Towards a realization of QSCA as a physical process that could be implemented experimentally and so would potentially provide a novel quantum computational machine, we may construct an optical circuit made of elementary passive optical elements i.e light beam splitters and phase shifters that are assembled so that they implement the operator U_f of a QSCA. For that purpose we may utilize the theorem of embedding the $SU(2)$ Lie group into $SU(N)$, which recently[12] has been used in the form of expressing every unitary matrix by a sequence of embedded 2×2 elementary unitary matrices of only two types : one matrix that realizes a beam splitter with transmission and reflection coefficients determined by the matrix elements and one that similarly realizes a phase shifter of a classical propagating light wave. By virtue of this embedding we may factorize the U_f matrix of a $r=2$, say QSCA into elementary matrices of beam splitters and phase shifters. Assuming a two-state encoding of the qubits of the QSCA cells (e.g taking the vertical and horizontal polarization states of a laser beam as $|0>$ and $|1>$), we may construct the quantum optical analog of the U_f of the QSCA.

Finally, we should mention a number of ramifications of our QSCA formalism that are currently also under investigation, namely: QSCA beyond the binary case (this would involve higher dimensional representations of the discrete Heisenberg group); differential versus matrix formulation of QSCA (i.e partial differential operators acting on a space of multivariable complex polynomials realizing the respective action of the U_f matrix); and also the case of *noisy QSCA* (i.e QSCA evolution that occurs in the presence of of quantum mechanical noise, exemplified by errors due to random bit-flipping and phase shifting in the qubits of the QSCA string); this would require the description of QSCA in terms of trace-preserving operators instead of the unitary operators as in the noiseless case, and the respective quantization of the classical SCA to be carried out not by Hilbert-space state vectors but by employing the ρ-density operator formalism of Quantum Mechanics.

References

[1] S. Wolfram, *Theory and Applications of Cellular Automata* (World Scientific, Singapore 1986).

[2] J. K. Park, K. Steiglitz and W. P. Thurston, Physica D **19**, 423 (1986);
K. Steiglitz, I. Kamal and A. Watson, IEEE Trans. on Comp. **37**, 138 (1988).
M. Jakubowski, K. Steiglitz and R. Squier, PhysComp96, (New England Complex Systems Institute, 1996), Eds. T. Toffoli et. al., p. 171.

[3] T. S. Papatheodorou, M. J. Ablowitz and Y. G. Saridakis, Stud. Appl. Math. **79**, 173 (1988).

[4] A. S. Fokas, E. P. Papadopoulou, Y. G. Saridakis, Physica D **41**, 297 (1990);
A. S. Fokas, E. P. Papadopoulou, Y. G. Saridakis, Phys. Lett. A **147**, 369 (1990);
A. S. Fokas, E. P. Papadopoulou, Y. G. Saridakis, Complex Syst. **3**, 615 (1989).

[5] M. J. Ablowitz, J. M. Keiser and L. A. Takhtajan, Phys. Rev. A. **44**, 6909 (1991).

[6] M. Brushi, P. M. Santini and O. Ragnisco, Phys. Lett. A. **169**, 151 (1992).

[7] F. Calogero and A. Degasperis, *Spectral Transform and Solitons* (North-Holland, Amsterdam 1982).

[8] A. Peres, *Quantum Theory: Concepts and Methods* (Kluwer, Dordrecht 1995).

[9] A. K. Ekert and R. Jozsa, Rev. Mod. Phys. **68**, 733 (1996);
Eds. H.-K Lo, S. Popescu and T. Spiller, *Introduction to Quantum Computation and Information*, (World Scientific, Singapore 1998), and references therein.

[10] A. Barenco, et. al Phys. Rev. A **52**, 3457 (1995).

[11] D. Ellinas, E. P. Papadopoulou and Y. G. Saridakis, forthcoming.

[12] M. Reck, A. Zeilinger et. al., Phys. Rev. Lett. **73**, 58 (1994).