Option pricing in multivariate stochastic volatility models of OU type

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We present a multivariate stochastic volatility model with leverage, which is flexible enough to recapture the individual dynamics as well as the interdependencies between several assets while still being highly analytically tractable.

First we derive the characteristic function and give conditions that ensure its analyticity and absolute integrability in some open complex strip around zero. Therefore we can use Fourier methods to compute the prices of multi-asset options efficiently. To show the applicability of our results, we propose a concrete specification, the OU-Wishart model, where the dynamics of each individual asset coincide with the popular Γ-OU BNS model. This model can be well calibrated to market prices, which we illustrate with an example using options on the exchange rates of some major currencies. Finally, we show that covariance swaps can also be priced in closed form.

AMS Subject Classification 2000: Primary: 91B28, Secondary: 60G51

Keywords: multivariate stochastic volatility models, Ornstein-Uhlenbeck type processes, option pricing

1. Introduction

This paper deals with the pricing of options depending on several underlying assets. While there is a vast amount of literature on the pricing of single-asset options, see e.g. Cont and Tankov (2004) or Schoutens (2003) for an overview, the amount of literature considering the multi-asset case is rather limited. This is most likely due to the fact that the trade-off between *flexibility* and *tractability* is particularly delicate in a multivariate setting. On the one hand, the model under consideration should be flexible enough to recapture stylized facts observed in real option prices. When dealing with multiple underlyings, this becomes challenging, since not only the individual assets but also their joint behaviour has to be taken into account. On the other hand, one needs enough mathematical structure to calculate option prices in the first place and to be able to calibrate the model to market prices. Due to

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an increasing number of state variables and parameters, this is also not an easy task in a multidimensional framework. In this article we want to propose a model which extends the one put forward in Pigorsch and Stelzer (2009) and seems to present a reasonable compromise between these competing requirements. The log-price processes $Y = (Y^1, \ldots, Y^d)$ of d financial assets are modelled as

$$dY_t = (\mu + \beta(\Sigma_t))\,dt + \Sigma_t^{\frac{1}{2}}\,dW_t + \rho(dL_t), \tag{1.1}$$

$$d\Sigma_t = (A\Sigma_t + \Sigma_t A^\mathsf{T})\,dt + dL_t, \tag{1.2}$$

where $\mu \in \mathbb{R}^d$, A is a real $d \times d$ matrix, and β, ρ are linear operators from the real $d \times d$ matrices to $\mathbb{R}^d$. Moreover, W is an $\mathbb{R}^d$-valued Wiener process and L is an independent matrix subordinator, i.e. a Lévy process which only has positive semidefinite increments. Hence the covariance process Σ is an Ornstein-Uhlenbeck (henceforth OU) type process with values in the positive semidefinite matrices, cf. Barndorff-Nielsen and Stelzer (2007). Thus we call (1.1), (1.2) the *multivariate stochastic volatility model of OU type*. The positive semidefinite OU type process Σ introduces a stochastic volatility and, what is difficult to achieve using several univariate models, a stochastic correlation between the assets. Moreover, Σ is mean reverting and increases only by jumps. The jumps represent the arrival of new information that results in positive shocks in the volatility and positive or negative shocks in the correlation of some assets. Due to the leverage term $\rho(dL_t)$ they are correlated with price jumps. The present model is a multivariate generalisation of the non-Gaussian OU type stochastic volatility model introduced by Barndorff-Nielsen and Shepard (2001) (henceforth BNS model). For one underlying, these models are found to be both flexible and tractable in Nicolato and Venardos (2003). The key reason is that the characteristic function of the return process can often be computed in closed form, which allows European options to be be priced efficiently using the Fourier methods introduced by Carr and Madan (1999b) and Raible (2000). In the present study, we show that a similar approach is also applicable in the multivariate case. Recently, Benth and Vos (2009) discussed a somewhat similar model in the context of energy markets. However, they do not establish rigorous conditions for the applicability of Fourier pricing and do not calibrate their model to market prices.

Alternatively, the covariance process Σ can also be modelled by other processes taking values in the positive semidefinite matrices. In particular, several authors have advocated to use a diffusion model based on the Wishart process, cf. e.g. Da Fonseca, Grasselli and Tebaldi (2007), Gourieroux (2007), and the references therein. This leads to a multivariate generalisation of the model of Heston (1993). However, the treatment of square-root processes on the cone of positive semidefinite matrices is mathematically quite involved (cf. e.g. Cuchiero, Filipović, Mayerhofer and Teichmann (2009)). Moreover, there exists empirical evidence suggesting that volatility jumps (together with the stock price), cf. Jacod and Todorov (2010), which cannot be recaptured by a diffusion model. Finally, none of the multivariate Heston models seems to have been successfully calibrated to market prices until now.

Another possible approach is to consider multivariate models based on a concatenation of univariate building blocks. This approach is taken e.g. in Luciano and Schoutens (2006) using Lévy processes, by Dimitroff, Lorenz and Szimayer (2009), who consider a multivariate Heston model, and by Hubalek and Nicolato (2005), who put forward a multifactor BNS model. However, all these models have either a somewhat limited capability to catch complex dependence structures or lead to tricky (factor) identification issues. Apart from models where all parameters are determined by single-asset options, we are not aware of successful calibrations of such models.

The remainder of this paper is organised as follows. Sections 2.1 and 2.2 serve as an introduction to the multivariate stochastic volatility model of OU type. Afterwards, we derive the joint characteristic function of (Y_t, Σ_t). We then show in Section 2.4 that a simple moment condition on L implies analyticity and absolute integrability of the moment generating function of Y_t in some open complex strip around zero. Equivalent martingale measures are discussed in Section 2.5, where we also present a subclass that preserves the structure of our model. In Section 3, we recall how to use Fourier methods to compute prices of multi-asset options efficiently. Subsequently, we propose the OU-Wishart model, where L is a compound Poisson process with Wishart distributed jumps. It turns out that the OU-Wishart model has margins which are in distribution equivalent to a Γ-OU BNS model, one of the tractable specifications commonly used in the univariate case. Moreover, the characteristic function can be computed in closed form, which makes option pricing and calibration particularly feasible. In an illustrative example we calibrate a bivariate OU-Wishart model to market prices. As a final application, we show that covariance swaps can also be priced in closed form in Section 5. The Appendix contains a simple result on multidimensional analytic functions which is proved to establish the regularity of the moment generating function in Section 2.4.

Notation

$M_{d,n}(\mathbb{R})$ (resp. $M_{d,n}(\mathbb{C})$) represent the $d \times n$ matrices with real (resp. complex) entries. We abbreviate $M_d(\cdot) = M_{d,d}(\cdot)$. $\mathbb{S}_d$ denotes the subspace of $M_d(\mathbb{R})$ of all symmetric matrices. We write $\mathbb{S}_d^+$ for the cone of all positive semidefinite matrices, and $\mathbb{S}_d^{++}$ for the open cone of all positive definite matrices. The identity matrix in $M_d(\mathbb{R})$ is denoted by I_d. $\sigma(A)$ denotes the set of all eigenvalues of $A \in M_d(\mathbb{C})$. We write $\mathrm{Re}(z)$ and $\mathrm{Im}(z)$ for the real or imaginary part of $z \in \mathbb{C}^d$ or $z \in M_d(\mathbb{C})$, which has to be understood componentwise. The components of a vector or matrix are denoted by subscripts, however for stochastic processes we use superscripts to avoid double indices.

On $\mathbb{R}^d$, we typically use the Euclidean scalar product, $\langle x, y\rangle_{\mathbb{R}^d} := x^\top y$, and on $M_d(\mathbb{R})$ or $\mathbb{S}_d$ the scalar products given by $\langle A, B\rangle_{M_d(\mathbb{R})} := \mathrm{tr}(A^\top B)$ or $\langle A, B\rangle_{\mathbb{S}_d} := \mathrm{tr}(AB)$ respectively. However, due to the equivalence of all norms on finite dimensional vector spaces, most results here hold true independently of the norm. We also write $\langle x, y\rangle = x^\top y$ for $x, y \in \mathbb{C}^d$, although this is only a bilinear form but not a scalar product on $\mathbb{C}^d$.

We denote by $\mathrm{vec} : M_d(\mathbb{R}) \to \mathbb{R}^{d^2}$ the bijective linear operator that stacks the columns of a matrix below one other. With the above norms, vec is a Hilbert space isometry. Likewise, for a symmetric matrix $S \in \mathbb{S}_d$ we denote by $\mathrm{vech}(S)$ the vector consisting of the columns of the upper-diagonal part including the diagonal.

Furthermore, we employ an intuitive notation concerning integration with respect to matrix-valued processes. For an $M_{m,n}(\mathbb{R})$-valued Lévy process L, and $M_{d,m}(\mathbb{R})$ resp. $M_{n,p}(\mathbb{R})$- valued processes X, Y integrable with respect to L, the term $\int_0^t X_s \, dL_s Y_s$ is to be understood as the $d \times p$ (random) matrix with (i, j)-th entry $\sum_{k=1}^m \sum_{l=1}^n \int_0^t X_s^{ik} \, dL_s^{kl} Y_s^{lj}$.

2. The multivariate stochastic volatility model of OU type

For the remainder of the paper, fix a filtered probability space $(\Omega, \mathscr{F}, (\mathscr{F}_t)_{t\in[0,T]}, P)$ in the sense of Jacod and Shiryaev (2003, I.1.3), where $\mathscr{F}_0 = \{\Omega, \emptyset\}$ and $T > 0$ is a a fixed terminal time.

2.1. Positive semidefinite processes of OU type

To formulate our model, we need to introduce the concept of matrix subordinators as studied in Barndorff-Nielsen and Pérez-Abreu (2008).

Definition 2.1. *An $\mathbb{S}_d$-valued Lévy Process $L = (L_t)_{t \in \mathbb{R}_+}$ is called* matrix subordinator, *if $L_t - L_s \in \mathbb{S}_d^+$ for all $t > s$.*

The characteristic function of a matrix subordinator L is given by $E(e^{i\mathrm{tr}(ZL_1)}) = \exp(\psi_L(Z))$ for the *characteristic exponent*

$$\psi_L(Z) = i\mathrm{tr}(\gamma_L Z) + \int_{\mathbb{S}_d^+} (e^{i\mathrm{tr}(XZ)} - 1)\, \kappa_L(dX), \quad Z \in M_d(\mathbb{R}),$$

where $\gamma_L \in \mathbb{S}_d^+$ and κ_L is a Lévy measure on $\mathbb{S}_d$ with $\kappa_L(\mathbb{S}_d \backslash \mathbb{S}_d^+) = 0$ and $\int_{\{||X|| \le 1\}} ||X||\, \kappa_L(dX) < \infty$.

Positive semidefinite processes of OU type are a generalisation of nonnegative OU type processes (cf. Barndorff-Nielsen and Stelzer (2007)). Let L be a matrix subordinator and $A \in M_d(\mathbb{R})$. The positive semidefinite OU type process $\Sigma = (\Sigma_t)_{t \in \mathbb{R}_+}$ is defined as the unique strong solution to the stochastic differential equation

$$d\Sigma_t = (A\Sigma_t + \Sigma_t A^\mathsf{T})\, dt + dL_t, \quad \Sigma_0 \in \mathbb{S}_d^+. \tag{2.1}$$

It is given by

$$\Sigma_t = e^{At}\Sigma_0 e^{A^\mathsf{T} t} + \int_0^t e^{A(t-s)}\, dL_s\, e^{A^\mathsf{T}(t-s)}. \tag{2.2}$$

Since $\Sigma_t \in \mathbb{S}_d^+$ for all $t \in \mathbb{R}_+$, this process can be used to model the stochastic evolution of a covariance matrix. As in the univariate case there exists a closed form expression for the integrated volatility. Suppose

$$0 \notin \sigma(A) + \sigma(A). \tag{2.3}$$

Then the integrated OU type process Σ^+ is given by

$$\Sigma_t^+ := \int_0^t \Sigma_s\, ds = \mathbf{A}^{-1}(\Sigma_t - \Sigma_0 - L_t), \tag{2.4}$$

where $\mathbf{A} : X \mapsto AX + XA^\mathsf{T}$. Note that condition (2.3) implies that the operator $\mathbf{A}$ is invertible, cf. Horn and Johnson (1990, Theorem 4.4.5). In the case where Σ is *mean reverting*, i.e A only has eigenvalues with negative real part, condition (2.3) is trivially satisfied.

2.2. Definition and marginal dynamics of the model

The following model was introduced and studied in Pigorsch and Stelzer (2009) from a statistical point of view for $\beta, \rho = 0$. Here we extend it to allow for a more general drift and a leverage term in order to discuss its applicability to option pricing later on. The more general drift is necessary in order to incorporate exponential martingale dynamics in the present framework (cf. Theorem 2.10), whereas the leverage term significantly improves the calibration to market prices in Section 4.2.

Let L be a matrix subordinator with characteristic exponent ψ_L and W an independent $\mathbb{R}^d$-valued Wiener process. The *multivariate stochastic volatility model of OU type* is then given by

$$dY_t = (\mu + \beta(\Sigma_t))\, dt + \Sigma_t^{\frac{1}{2}}\, dW_t + \rho(dL_t), \quad Y_0 \in \mathbb{R}^d \tag{2.5}$$

$$d\Sigma_t = (A\Sigma_t + \Sigma_t A^\mathsf{T})\, dt + dL_t, \quad \Sigma_0 \in \mathbb{S}_d^+ \tag{2.6}$$

with linear operators $\beta, \rho : M_d(\mathbb{R}) \to \mathbb{R}^d$, $\mu \in \mathbb{R}^d$ and $A \in M_d(\mathbb{R})$ such that $0 \notin \sigma(A) + \sigma(A)$.

We have specified the *risk premium* β and the *leverage operator* ρ in a quite general form. The following specification turns out to be particularly tractable.

Definition 2.2. *We call β and ρ* diagonal *if, for $\beta_1,\dots,\beta_d \in \mathbb{R}$ and $\rho_1,\dots,\rho_d \in \mathbb{R}$,*

$$\beta(X) = \begin{pmatrix} \beta_1 X_{11} \\ \vdots \\ \beta_d X_{dd} \end{pmatrix}, \quad \rho(X) = \begin{pmatrix} \rho_1 X_{11} \\ \vdots \\ \rho_d X_{dd} \end{pmatrix}, \quad \forall X \in M_d(\mathbb{R}).$$

In the following, we will denote for each $i \in \{1,\dots,d\}$ by $\beta^i(X)$ and $\rho^i(X)$ the i-th component of the vector $\beta(X)$ or $\rho(X)$ respectively. The marginal dynamics of the individual assets have been derived in Barndorff-Nielsen and Stelzer (2009, Proposition 4.3).

Theorem 2.3. *Let $i \in \{1,\dots,d\}$. Then we have*

$$\left(Y_t^i\right)_{t\in\mathbb{R}_+} \overset{fidi}{=} \left(\mu_i t + \beta^i(\Sigma_t^+) + \int_0^t (\Sigma_s^{ii})^{\frac{1}{2}}\, dW_s^i + \rho^i(L_t)\right)_{t\in\mathbb{R}_+},$$

where $\overset{fidi}{=}$ denotes equality of all finite dimensional distributions.

Let us now consider the case where A is a diagonal matrix, $A = \begin{pmatrix} a_1 & & 0 \\ & \ddots & \\ 0 & & a_d \end{pmatrix}$, and β, ρ are diagonal as well. Then, for every $i \in \{1,\dots,d\}$, we have

$$dY_t^i \overset{fidi}{=} (\mu_i + \beta_i \Sigma_t^{ii})\, dt + \Sigma_t^{ii}\, dW_t^i + \rho_i\, dL_t^{ii}, \tag{2.7}$$

$$d\Sigma_t^{ii} = 2a_i \Sigma_t^{ii}\, dt + dL_t^{ii}. \tag{2.8}$$

Apparently, every diagonal element L^{ii}, $i = 1,\dots,d$, of a matrix subordinator L is a univariate subordinator, and thus Σ^{ii} is a nonnegative OU type process. Consequently, the model for the i-th asset is equivalent in distribution to a univariate BNS model.

2.3. Characteristic function

Let $\langle\cdot,\cdot\rangle_V$, $\langle\cdot,\cdot\rangle_W$ be bilinear forms as introduced in the notation, where V,W may be either $\mathbb{R}^d$, $\mathbb{C}^d$ or $M_d(\cdot)$. Given a linear operator $T : V \to W$, the *adjoint* $T^* : W \to V$ is the unique linear operator such that $\langle Tx, y\rangle_W = \langle x, T^* y\rangle_V$ for all $x \in V$ and $y \in W$. Directly by definition we obtain the following

Lemma 2.4. *Let $y \in \mathbb{R}^d$, $z \in M_d(\mathbb{R})$ and $t \in \mathbb{R}_+$. Then the adjoints of the linear operators*

$$\mathbf{A} : X \mapsto AX + XA^\top, \quad \mathscr{B}(t) : X \mapsto e^{At} X e^{A^\top t} - X,$$

$$\mathscr{C}(t) : X \mapsto e^{At} X e^{A^\top t} z + \beta(\mathbf{A}^{-1}(\mathscr{B}(t)X)) y^\top + \rho(X) y^\top + \frac{i}{2} y y^\top \mathbf{A}^{-1}(\mathscr{B}(t)X)$$

on $M_d(\mathbb{C})$ are given by

$$\mathbf{A}^* : X \mapsto A^\top X + XA, \quad \mathscr{B}(t)^* : X \mapsto e^{A^\top t} X e^{At} - X,$$

$$\mathscr{C}(t)^* : X \mapsto e^{A^\top t} X z^\top e^{At} + \rho^*(Xy) + \mathscr{B}(t)^* \mathbf{A}^{-*}\left(\beta^*(Xy) + \frac{i}{2} X y y^\top\right).$$

Our main objective in this section is to compute the joint characteristic function of (Y_t, Σ_t). This will pave the way for Fourier pricing of multi-asset options later on. Note that we use the scalar product

$$\langle (x_1, y_1), (x_2, y_2)\rangle := x_1^\top x_2 + \mathrm{tr}(y_1^\top y_2)$$

on $\mathbb{R}^d \times M_d(\mathbb{R})$.

Theorem 2.5 (Joint characteristic function). *For every $(y,z) \in \mathbb{R}^d \times M_d(\mathbb{R})$ and $t \in \mathbb{R}_+$, the joint characteristic function of (Y_t, Σ_t) is given by*

$$\begin{aligned} E\left[\exp\left(i\langle (y,z),(Y_t,\Sigma_t)\rangle\right)\right] = \exp\Big\{ & iy^\top (Y_0+\mu t) + i\mathrm{tr}(\Sigma_0 e^{A^\top t} z e^{At}) \\ & + i\mathrm{tr}\left(\Sigma_0\left(e^{A^\top t}\mathbf{A}^{-*}\left(\beta^*(y)+\frac{i}{2}yy^\top\right)e^{At} - \mathbf{A}^{-*}\left(\beta^*(y)+\frac{i}{2}yy^\top\right)\right)\right) \\ & + \int_0^t \psi_L\left(e^{A^\top s} z e^{As} + \rho^*(y) + e^{A^\top s}\mathbf{A}^{-*}\left(\beta^*(y)+\frac{i}{2}yy^\top\right)e^{As} - \mathbf{A}^{-*}\left(\beta^*(y)+\frac{i}{2}yy^\top\right)\right) ds \Big\}, \end{aligned}$$

where $\mathbf{A}^{-*} := (\mathbf{A}^{-1})^* = (\mathbf{A}^*)^{-1}$ denotes the inverse of the adjoint of $\mathbf{A} : X \mapsto AX + XA^\top$, that is the inverse of $\mathbf{A}^* : X \mapsto A^\top X + XA$.

Note that for $z = 0$ we obtain the characteristic function of Y_t.

Proof. Since Σ is adapted to the filtration generated by L, and by the independence of L and W,

$$\begin{aligned} E\left[\exp\left(\langle (y,z),(Y_t,\Sigma_t)\rangle\right)\right] &= e^{iy^\top (Y_0+\mu t)} E\left[e^{i\mathrm{tr}(z^\top \Sigma_t) + iy^\top(\beta(\Sigma_t^+)+\rho(L_t))} E\left(e^{iy^\top \int_0^t \Sigma_s^{\frac{1}{2}} dW_s} \Big| (L_s)_{s\in\mathbb{R}_+}\right)\right] \\ &= e^{iy^\top (Y_0+\mu t)} E\left[e^{i\mathrm{tr}(z^\top \Sigma_t) + iy^\top(\beta(\Sigma_t^+)+\rho(L_t))} \exp\left(-\frac{1}{2} y^\top \Sigma_t^+ y\right)\right]. \end{aligned}$$

By (2.4) and using the fact that the trace is invariant under cyclic permutations the last term equals

$$e^{iy^\top (Y_0+\mu t)} E\left[e^{i\mathrm{tr}(z^\top \Sigma_t + \beta(\mathbf{A}^{-1}(\Sigma_t - \Sigma_0 - L_t))y^\top + \rho(L_t)y^\top + \frac{i}{2}yy^\top \mathbf{A}^{-1}(\Sigma_t - \Sigma_0 - L_t))}\right].$$

In view of (2.2), we have

$$\Sigma_t - \Sigma_0 - L_t = \int_0^t \mathscr{B}(t-s)\, dL_s + \mathscr{B}(t)\Sigma_0,$$

for the linear operator $\mathscr{B}(t)$ from Lemma 2.4. Therefore

$$\begin{aligned} & E\left[\exp(i\langle (y,z),(Y_t,\Sigma_t)\rangle\right] \\ &= \exp\left(iy^\top (Y_0+\mu t) + i\mathrm{tr}\left(z^\top e^{At}\Sigma_0 e^{A^\top t} + \beta(\mathbf{A}^{-1}(\mathscr{B}(t)\Sigma_0))y^\top + \frac{i}{2}yy^\top \mathbf{A}^{-1}(\mathscr{B}(t)\Sigma_0)\right)\right) \\ &\quad \times E\left[\exp\left(i\mathrm{tr}\left(z^\top \int_0^t e^{A(t-s)}\, dL_s\, e^{A^\top (t-s)} + \beta\left(\mathbf{A}^{-1}\left(\int_0^t e^{A(t-s)}\, dL_s\, e^{A^\top (t-s)} - L_t\right)\right)y^\top \right.\right.\right. \\ &\qquad \left.\left.\left. + \rho(L_t)y^\top + \frac{i}{2}yy^\top \mathbf{A}^{-1}\left(\int_0^t e^{A(t-s)}\, dL_s\, e^{A^\top (t-s)} - L_t\right)\right)\right)\right] \\ &= \exp\left(iy^\top (Y_0+\mu t) + i\mathrm{tr}\left(z^\top e^{At}\Sigma_0 e^{A^\top t} + \beta(\mathbf{A}^{-1}(\mathscr{B}(t)\Sigma_0))y^\top + \frac{i}{2}yy^\top \mathbf{A}^{-1}(\mathscr{B}(t)\Sigma_0)\right)\right) \\ &\quad \times E\left[\exp\left(i\mathrm{tr}\left(\left(\int_0^t \mathscr{C}(t-s)\, dL_s\right)^\top I_d\right)\right)\right] \end{aligned}$$

with the linear operator $\mathscr{C}(t)$ from Lemma 2.4, since $\mathbf{A}^{-1}\left(\int_0^t e^{A(t-s)}\, dL_s\, e^{A^\top (t-s)} - L_t\right) \in \mathbb{S}_d$. An immediate multivariate generalisation of results obtained in Rajput and Rosinski (1989, Proposition 2.4) (see also Eberlein and Raible (1999, Lemma 3.1)) yields an explicit formula for the expectation above:

$$E\left[\exp\left(i\mathrm{tr}\left(\left(\int_0^t \mathscr{C}(t-s)\, dL_s\right)^\top I_d\right)\right)\right] = \exp\left(\int_0^t \psi_L\left(\mathscr{C}(s)^* I_d\right) ds\right).$$

By Lemma 2.4 we have

$$e^{\int_0^t \psi_L(\mathscr{C}(s)^* I_d)\,ds} = e^{\int_0^t \psi_L\left(e^{A^\top s} z^\top e^{As} + \rho^*(y) + e^{A^\top s}\mathbf{A}^{-*}\left(\beta^*(y)+\frac{i}{2}yy^\top\right)e^{As} - \mathbf{A}^{-*}\left(\beta^*(y)+\frac{i}{2}yy^\top\right)\right)ds}.$$

This expression is well-defined, because

$$e^{A^\top s} z^\top e^{As} + \rho^*(y) + e^{A^\top s}\mathbf{A}^{-*}\left(\beta^*(y)+\frac{i}{2}yy^\top\right)e^{As} - \mathbf{A}^{-*}\left(\beta^*(y)+\frac{i}{2}yy^\top\right) \in M_d(\mathbb{R}) + i\mathbb{S}_d^+,$$

for all $s \in [0,t]$. Indeed, this follows from

$$e^{A^\top s}\mathbf{A}^{-*}\left(yy^\top\right)e^{As} - \mathbf{A}^{-*}\left(yy^\top\right) = \int_0^s e^{A^\top u} yy^\top e^{Au}\,du \in \mathbb{S}_d^+. \tag{2.9}$$

Finally, we infer from Lemma 2.4 that

$$\operatorname{tr}\left(\beta(\mathbf{A}^{-1}(\mathscr{B}(t)\Sigma_0))y^\top + \frac{i}{2}yy^\top\mathbf{A}^{-1}(\mathscr{B}(t)\Sigma_0)\right) = \operatorname{tr}\left(\Sigma_0\left(\mathscr{B}(t)^*\mathbf{A}^{-*}\left(\beta^*(y)+\frac{i}{2}yy^\top\right)\right)\right),$$

which gives the desired result by noting that $\operatorname{tr}(z\Sigma_t) = \operatorname{tr}(z^\top\Sigma_t)$. □

2.4. Regularity of the moment generating function

In this section we provide conditions ensuring that the characteristic function of Y_t admits an analytic extension Φ to some open convex neighbourhood of 0 in $\mathbb{C}^d$. Afterwards, we show absolute integrability. The regularity results obtained in this section will allow us to apply Fourier methods in Section 3 to compute option prices efficiently.

Definition 2.6. *For any $t \in [0,T]$, the* moment generating function *of Y_t is defined as*

$$\Phi_{Y_t}(y) := E[\exp(y^\top Y_t)],$$

for all $y \in \mathbb{C}^d$ such that the expectation exists.

Note that Φ_{Y_t} may not exist anywhere but on $i\mathbb{R}^d$, where it coincides with the characteristic function of Y_t. The next lemma is a first step towards conditions for the existence and analyticity of the moment generating function Φ_{Y_t} in a complex neighbourhood of zero.

Lemma 2.7. *Let L be a matrix subordinator with* cumulant transform Θ_L, *that is*

$$\Theta_L(Z) = \psi_L(-iZ) = \operatorname{tr}(\gamma_L Z) + \int_{\mathbb{S}_d^+}(e^{\operatorname{tr}(XZ)}-1)\,\kappa_L(dX), \quad Z \in M_d(\mathbb{C}),$$

and let $\varepsilon > 0$. Then Θ_L is analytic on the strip

$$S_\varepsilon := \{Z \in M_d(\mathbb{C}) : ||\mathrm{Re}(Z)|| < \varepsilon\}, \tag{2.10}$$

if and only if

$$\int_{\{||X||\geq 1\}} e^{\operatorname{tr}(RX)}\,\kappa_L(dX) < \infty \quad \textit{for all } R \in M_d(\mathbb{R}) \textit{ with } ||R|| < \varepsilon. \tag{2.11}$$

Proof. If (2.11) holds, Duffie, Filipovic and Schachermayer (2003, Lemma A.2) implies that $Z \mapsto E\left(e^{\mathrm{tr}(ZL_1)}\right) = e^{\Theta_L(Z)}$ is analytic on S_ε. Due to Assumption (2.11), dominated convergence yields that Θ_L is continuous on S_ε. The claim now follows from Lemma A.1. Conversely, if Θ_L is analytic on S_ε, then Duffie et al. (2003, Lemma A.4) implies that $E\left(^{\mathrm{tr}(ZL_1)}\right) = e^{\Theta_L(Z)}$ for all $Z \in S_\varepsilon$. Thus, by Sato (1999, Theorem 25.17), Condition (2.11) holds. □

The next theorem generalises Nicolato and Venardos (2003, Theorem 2.2) to the multivariate case. It holds for all sub-multiplicative matrix norms on $M_d(\mathbb{R})$ that satisfy $\left|\left|yy^\mathsf{T}\right|\right| = ||y||^2$ for all $y \in \mathbb{R}^d$, where we use the Euclidean norm on $\mathbb{R}^d$. For example, this holds true for the Frobenius and the spectral norm (the operator norm associated to the Euclidean norm).

Theorem 2.8 (Strip of analyticity)**.** *Suppose the matrix subordinator L satisfies*

$$\int_{\{||X||\geq 1\}} e^{\mathrm{tr}(RX)}\, \kappa_L(dX) < \infty \quad \text{for all } R \in M_d(\mathbb{R}) \text{ with } ||R|| < \varepsilon, \tag{2.12}$$

for some $\varepsilon > 0$. Then the moment generating function Φ_{Y_t} of Y_t is analytic on the open strip

$$S_\theta := \{y \in \mathbb{C}^d : ||\mathrm{Re}(y)|| < \theta\},$$

where

$$\theta := -\frac{||\rho||}{(e^{2||A||t}+1)\,||\mathbf{A}^{-1}||} - ||\beta|| + \sqrt{\Delta} > 0 \tag{2.13}$$

with

$$\Delta := \left(\frac{||\rho||}{(e^{2||A||t}+1)\,||\mathbf{A}^{-1}||} + ||\beta||\right)^2 + \frac{2\varepsilon}{(e^{2||A||t}+1)\,||\mathbf{A}^{-1}||}.$$

Moreover,

$$\Phi_{Y_t}(y) = \exp\left(y^\mathsf{T}(Y_0 + \mu t) + \mathrm{tr}(\Sigma_0 H_y(t)) + \int_0^t \Theta_L\left(H_y(s) + \rho^*(y)\right) ds\right) \tag{2.14}$$

for all $y \in S$, where

$$H_y(s) := e^{A^\mathsf{T} s}\mathbf{A}^{-*}\left(\beta^*(y) + \frac{1}{2}yy^\mathsf{T}\right)e^{As} - \mathbf{A}^{-*}\left(\beta^*(y) + \frac{1}{2}yy^\mathsf{T}\right). \tag{2.15}$$

Proof. The main part of the proof is to show that the function

$$G(y) := \exp\left(y^\mathsf{T}(Y_0 + \mu t) + \mathrm{tr}(\Sigma_0 H_y(t)) + \int_0^t \Theta_L\left(H_y(s) + \rho^*(y)\right) ds\right)$$

is analytic on S_θ. First we want to find a θ such that for all $u \in \mathbb{R}^d$ with $||u|| < \theta$, it holds that $||H_u(s) + \rho^*(u)|| < \varepsilon$ for all $s \in [0,t]$. Since

$$\begin{aligned}||H_u(s) + \rho^*(u)|| &= \left|\left|e^{A^\mathsf{T} s}\mathbf{A}^{-*}\left(\beta^*(u) + \frac{1}{2}uu^\mathsf{T}\right)e^{As} - \mathbf{A}^{-*}\left(\beta^*(u) + \frac{1}{2}uu^\mathsf{T}\right) + \rho^*(u)\right|\right| \\ &\leq \frac{1}{2}(e^{2||A||t}+1)\left|\left|\mathbf{A}^{-1}\right|\right| ||u||^2 + \left(||\rho|| + (e^{2||A||t}+1)\left|\left|\mathbf{A}^{-1}\right|\right| ||\beta||\right)||u||,\end{aligned}$$

we have to find the roots of the polynomial

$$p(x) := \frac{1}{2}(e^{2||A||t}+1)\left|\left|\mathbf{A}^{-1}\right|\right| x^2 + \left(||\rho|| + (e^{2||A||t}+1)\left|\left|\mathbf{A}^{-1}\right|\right| ||\beta||\right)x - \varepsilon.$$

The positive one is given by θ as stated in (2.13). Note that $\theta > 0$, because p is a cup-shaped parabola with $p(0) = -\varepsilon < 0$.
Now let $y \in S_\theta$, i.e. $y = u + iv$ with $||u|| < \theta$. Using $\operatorname{Re}(yy^\top) = uu^\top - vv^\top$ and (2.9) we get

$$\begin{aligned}\operatorname{Re}(H_y(s) + \rho^*(y)) &= H_u(s) + \rho^*(u) - \frac{1}{2}\left(e^{A^\top s}\mathbf{A}^{-*}(vv^\top)e^{As} - \mathbf{A}^{-*}(vv^\top)\right)\\ &= H_u(s) + \rho^*(u) - \frac{1}{2}\int_0^s e^{A^\top r} vv^\top e^{Ar}\,dr.\end{aligned}$$

Because of $\int_0^s e^{A^\top r} vv^\top e^{Ar}\,dr \in \mathbb{S}_d^+$, we have

$$\begin{aligned}\int_{\{||X||\geq 1\}} e^{\operatorname{tr}(\operatorname{Re}(H_y(s)+\rho^*(y))X)}\,\kappa_L(dX) &= \int_{\{||X||\geq 1\}} e^{\operatorname{tr}((H_u(s)+\rho^*(u))X)} e^{-\frac{1}{2}\operatorname{tr}\left(\left(\int_0^s e^{A^\top r} vv^\top e^{Ar}\,dr\right)X\right)}\,\kappa_L(dX)\\ &\leq \int_{\{||X||\geq 1\}} e^{\operatorname{tr}((H_u(s)+\rho^*(u))X)}\,\kappa_L(dX) < \infty\end{aligned}$$

by Assumption (2.12), since $||H_u(s) + \rho^*(u)|| < \varepsilon$. Thus, by Lemma 2.7 the function

$$S_\theta \in y \mapsto \Theta_L(H_y(s) + \rho^*(y))$$

is analytic on S for every $s \in [0,t]$. An application of Fubini's and Morera's theorem shows that integration over $[0,t]$ preserves analyticity, cf. Königsberger (2004, p. 228), hence G is analytic on S_θ. Obviously, we have $\Phi_{Y_t}(iy) = G(iy)$ for all $y \in \mathbb{R}^d$ by Theorem 2.5 and the definition of G. Thus, Duffie et al. (2003, Lemma A.4), finally implies $\Phi_{Y_t} \equiv G$ on S_θ. ☐

With Theorem 2.8 at hand, we can show

Theorem 2.9 (Absolute integrability). *If (2.12) holds for some $\varepsilon > 0$, the mapping $w \mapsto \Phi_{Y_t}(y+iw)$ is absolutely integrable, for all $y \in \mathbb{R}^d$ with $||y|| < \theta$, where θ is given as in Theorem 2.8.*

Proof. As in the proof of Theorem 2.8, we obtain from

$$\operatorname{Re}(H_{y+iw}(s)) = H_y(s) - \frac{1}{2}\int_0^s e^{A^\top s} ww^\top e^{As}\,ds$$

and $\operatorname{Re}(e^{\operatorname{tr}(Z)}) \leq |e^{\operatorname{tr}(Z)}| = e^{\operatorname{Re}(\operatorname{tr}(Z))} = e^{\operatorname{tr}(\operatorname{Re}(Z))}$ for $Z \in M_d(\mathbb{C})$, that

$$\operatorname{Re}\left(\int_0^t \int_{\mathbb{S}_d^+}\left(e^{\operatorname{tr}((H_{y+iw}(s)+\rho^*(y+iw))X)} - 1\right)\kappa_L(dX)ds\right) \leq \int_0^t \int_{\mathbb{S}_d^+}\left(e^{\operatorname{tr}((H_y(s)+\rho^*(y))X)} - 1\right)\kappa_L(dX)ds.$$

Using this inequality yields

$$\begin{aligned}|\Phi_{Y_t}(y+iw)| &\leq \Phi_{Y_t}(y) e^{-\frac{1}{2}\operatorname{tr}(\Sigma_0(e^{A^\top t}\mathbf{A}^{-*}(ww^\top)e^{At} - \mathbf{A}^{-*}(ww^\top))) - \frac{1}{2}\int_0^t \operatorname{tr}(\gamma_L(e^{A^\top s}\mathbf{A}^{-*}(ww^\top)e^{As} - \mathbf{A}^{-*}(ww^\top)))\,ds}\\ &= \Phi_{Y_t}(y) e^{-\frac{1}{2}\left\langle\left(\mathbf{A}^{-1}\mathscr{B}(t)(\Sigma_0) + \int_0^t \mathbf{A}^{-1}\mathscr{B}(s)(\gamma_L)\,ds\right)w, w\right\rangle}\end{aligned}$$

with $\mathscr{B}(t)$ as in Lemma 2.4. Note that $\mathbf{A}^{-1}\mathscr{B}(t)(\Sigma_0) + \int_0^t \mathbf{A}^{-1}\mathscr{B}(s)(\gamma_L)\,ds \in \mathbb{S}_d^+$, hence

$$\int_{\mathbb{R}^d} |\Phi_{Y_t}(y+iw)|\,dw \leq \Phi_{Y_t}(y)\int_{\mathbb{R}^d} e^{-\frac{1}{2}\left\langle\left(\mathbf{A}^{-1}\mathscr{B}(t)(\Sigma_0) + \int_0^t \mathbf{A}^{-1}\mathscr{B}(s)(\gamma_L)\,ds\right)w, w\right\rangle}\,dw < \infty,$$

by Theorem 2.8 and because the integrand is proportional to the density of a multivariate Normal distribution. ☐

2.5. Martingale Conditions and Equivalent Martingale Measures

For notational convenience, we work in this section with the model

$$
\begin{aligned}
dY_t &= (\mu + \beta(\Sigma_t))\,dt + \Sigma_t^{\frac{1}{2}}\,dW_t + \rho(dL_t), \quad Y_0 \in \mathbb{R}^d, && (2.16)\\
d\Sigma_t &= (\gamma_L + A\Sigma_t + \Sigma_t A^{\mathsf{T}})\,dt + dL_t, \quad \Sigma_0 \in \mathbb{S}_d^{++}, && (2.17)
\end{aligned}
$$

where L is a drift-less matrix subordinator with Lévy measure κ_L. Clearly, this is our multivariate stochastic volatility model of OU type (2.5), (2.6), except that μ in (2.5) is replaced by $\mu - \rho(\gamma_L)$, such that there is no deterministic drift from the leverage term $\rho(dL_t)$.

In Mathematical Finance, Y is used to model the joint dynamics of the log-returns of d assets with price processes $S_t^i = S_0^i e^{Y_t^i}$, where we set $Y_0^i = 0$ from now on and, hence, S_0 denotes the vector of initial prices.

The martingale property of the *discounted stock prices* $(e^{-rt}S_t)_{t\in[0,T]}$ for a constant interest rate $r > 0$ can be characterised as follows.

Theorem 2.10. *The discounted price process $(e^{-rt}S_t)_{t\in[0,T]}$ is a martingale if and only if, for $i = 1,\dots,d$,*

$$\int_{\{||X||>1\}} e^{\rho^i(X)}\,\kappa_L(dX) < \infty, \tag{2.18}$$

and

$$\beta^i(X) = -\frac{1}{2}X_{ii}, \quad X \in \mathbb{S}_d^+, \tag{2.19}$$

$$\mu_i = r - \int_{\mathbb{S}_d^+} (e^{\rho^i(X)} - 1)\,\kappa_L(dX). \tag{2.20}$$

Proof. Define $\widehat{S_t} := e^{-rt}S_t$ for all $t \in [0,T]$ and let $i \in \{1,\dots,d\}$. By Itô's formula and Jacod and Shiryaev (2003, III.6.35), $\widehat{S}^i$ is a local martingale if and only if (2.18), (2.19) and (2.20) hold. Thus it remains to show that it is actually a true martingale under the stated assumptions. Since $\widehat{S}$ is a positive local martingale, it is a supermartingale and hence a martingale if and only if $E(\widehat{S}_T^i) = \widehat{S}_0^i$ for all $i \in \{1,\dots,d\}$. This can be seen as follows. By Theorem 2.3, (2.19) and (2.20) we have

$$
\begin{aligned}
E(\widehat{S}_T^i) &= \widehat{S}_0^i E\left(\exp\left((\mu^i - r)T + \beta^i(\Sigma_T^+) + \int_0^T (\Sigma_s^{ii})^{\frac{1}{2}}\,dW_s^i + \rho^i(L_T)\right)\right)\\
&= \widehat{S}_0^i e^{-T\int_{\mathbb{S}_d^+}(e^{\rho^i(X)}-1)\,\kappa_L(dX)} E\left(e^{-\frac{1}{2}(\Sigma_T^+)^{ii} + \rho^i(L_T)} E\left(e^{\int_0^T (\Sigma_s^{ii})^{\frac{1}{2}}\,dW_s^i}\,\middle|\,(L_s)_{s\in[0,T]}\right)\right)\\
&= \widehat{S}_0^i e^{-T\int_{\mathbb{S}_d^+}(e^{\rho^i(X)}-1)\,\kappa_L(dX)} E\left(e^{\rho^i(L_T)}\right)\\
&= \widehat{S}_0^i.
\end{aligned}
$$

This proves the assertion. □

As in Nicolato and Venardos (2003, Theorem 3.1), it is possible to characterise the set of all equivalent martingale measures (henceforth EMMs), if the underlying filtration is generated by W and L. More specifically, it follows from the Martingale Representation Theorem (cf. Jacod and Shiryaev (2003, III.4.34)), that the density process $Z_t = E(\frac{dQ}{dP}|\mathscr{F}_t)$ of any equivalent martingale measure Q can be written as

$$Z = \mathscr{E}\left(\int_0^{\cdot} \psi_s dW_s + (Y-1) * (\mu^L - \nu^L)\right) \tag{2.21}$$

for suitable processes ψ and Y in this case. Here μ^L resp. ν^L denote the random measure of jumps resp. its compensator (cf. Jacod and Shiryaev (2003, II.1) for more details). Under an arbitrary EMM, L may not be a Lévy process, and W and L may not be independent. However, there is a subclass of *structure preserving* EMMs under which L remains a Lévy process independent of W. This translates into the following specifications of ψ and Y (cf. Nicolato and Venardos (2003, Theorem 3.2) for the univariate case):

Theorem 2.11 (Structure preserving EMMs)**.** *Let $y : \mathbb{S}_d^+ \to (0,\infty)$ such that*

(i) $\int_{\mathbb{S}_d^+} (\sqrt{y(X)} - 1)^2 \,\kappa_L(dX) < \infty$,

(ii) $\int_{\{||X||>1\}} e^{\rho^i(X)} \,\kappa_L^y(dX) < \infty, \quad i = 1,\dots,d,$

where $\kappa_L^y(B) := \int_B y(X)\,\kappa_L(dX)$ for $B \in \mathscr{B}(\mathbb{S}_d^+)$. Define the $\mathbb{R}^d$-valued process $(\psi_t)_{t\in[0,T]}$ as

$$\psi_t = -\Sigma_t^{-\frac{1}{2}} \left(\mu + \beta(\Sigma_t) + \frac{1}{2} \begin{pmatrix} \Sigma_t^{11} \\ \vdots \\ \Sigma_t^{dd} \end{pmatrix} + \begin{pmatrix} \int_{\mathbb{S}_d^+} (e^{\rho^1(X)} - 1)\,\kappa_L^y(dX) \\ \vdots \\ \int_{\mathbb{S}_d^+} (e^{\rho^d(X)} - 1)\,\kappa_L^y(dX) \end{pmatrix} - \mathbf{1}r \right),$$

where $\mathbf{1} = (1,\dots,1)^\top \in \mathbb{R}^d$. Then $Z = \mathscr{E}(\int_0^\cdot \psi_s dW_s + (y-1) * (\mu^L - \nu^L))$ is a density process, and the probability measure Q defined by $\frac{dQ}{dP} = Z_T$ is an equivalent martingale measure. Moreover, $W^Q := W - \int_0^\cdot \psi_s ds$ is a Q-standard Brownian motion, and L is an independent driftless Q-matrix subordinator with Lévy measure κ_L^y. The Q-dynamics of (Y,Σ) are given by

$$\begin{aligned} dY_t^i &= \left(r - \int_{\mathbb{S}_d^+} (e^{\rho^i(X)} - 1)\,\kappa_L^y(dX) - \frac{1}{2}\Sigma_t^{ii} \right) dt + \left(\Sigma_t^{\frac{1}{2}}\, dW_t^Q \right)^i + \rho^i(dL_t), \quad i = 1,\dots,d, \\ d\Sigma_t &= (\gamma_L + A\Sigma_t + \Sigma_t A^\top)\, dt + dL_t. \end{aligned}$$

Proof. Since $y - 1 > -1$, Z is strictly positive by Jacod and Shiryaev (2003, I.4.61). The martingale property of Z follows along the lines of the proof of Nicolato and Venardos (2003, Theorem 3.2).

The remaining assertions follow from Kallsen (2006, Proposition 1) and the Lévy-Khintchine formula by applying the Girsanov-Jacod-Memin Theorem as in Kallsen (2006, Proposition 4) to the $\mathbb{R}^{\frac{1}{2}d(d+1)}$-valued process

$$\widetilde{L} = \begin{pmatrix} W^Q \\ 0 \end{pmatrix} + \operatorname{vech}(L),$$

where $W^Q := W - \int_0^\cdot \psi_s ds$. □

The previous theorem shows that it is possible to use a model of the same type under the real-world probability measure P and some EMM Q, e.g. to do option pricing and risk management within the same model class. The model parameters under Q can be determined by calibration, the model parameters under P by statistical methods.

3. Option pricing using integral transform methods

In this section we first recall results of Eberlein, Glau and Papapantoleon (2009) on Fourier pricing in general multivariate semimartingale models. To this end, let $S = (S_0^1 e^{Y^1}, \dots, S_0^d e^{Y^d})$ be a d-dimensional semimartingale such that the discounted price process $(e^{-rt}S_t)_{t\in[0,T]}$ is a martingale under some pricing measure Q, for some constant instantaneous interest rate $r > 0$.

We want to determine the price $E_Q(e^{-rT} f(Y_T - s))$ of a European option with payoff $f(Y_T - s)$ at maturity T, where $f : \mathbb{R}^d \to \mathbb{R}_+$ is a measurable function and $s := (-\log(S_0^1), \dots, -\log(S_0^d))$. Denote by $\widehat{f}$ the *Fourier transform* of f. The following theorem is from Eberlein et al. (2009, Theorem 3.2) and represents a multivariate generalisation of integral transform methods first introduced in the context of option pricing by Carr and Madan (1999b) and Raible (2000).

Theorem 3.1 (Fourier Pricing)**.** *Fix $R \in \mathbb{R}^d$, let $g(x) := e^{-\langle R,x\rangle} f(x)$ for $x \in \mathbb{R}^d$, and assume that*

(i) $g \in L^1 \cap L^\infty$, *(ii)* $\Phi_{Y_T}(R) < \infty$, *(iii)* $w \mapsto \Phi_{Y_T}(R + iw)$ *belongs to* L^1.

Then

$$E_Q(e^{-rT} f(Y_T - s)) = \frac{e^{-\langle R,s\rangle - rT}}{(2\pi)^d} \int_{\mathbb{R}^d} e^{-i\langle u,s\rangle} \Phi_{Y_T}(R + iu) \widehat{f}(iR - u)\, du. \tag{3.1}$$

Observe that Theorems 2.8 and 2.9 show that Conditions *(ii)* and *(iii)* are satisfied for our multivariate stochastic volatility model of OU type (2.5), (2.6) if condition (2.12) holds, i.e. if L has enough exponential moments. More specifically, the vector R has to lie in the intersection of the domains of Φ_{Y_T} and $\widehat{f}$.

We now present some examples. As is well-known, the Fourier transform of the payoff function of a *plain vanilla call option* with strike $K > 0$, $f(x) = (e^x - K)^+$ is given by

$$\widehat{f}(z) = \frac{K^{1+iz}}{iz(1+iz)} \tag{3.2}$$

for $z \in \mathbb{C}$ with $\mathrm{Im}(z) > 1$. The Fourier transforms of many other single-asset options like barrier, self-quanto and power options as well as multi-asset options like worst-of and best-of options can be found e.g. in the survey Eberlein et al. (2009). From Hubalek and Nicolato (2005) we have the following formulae for basket and spread options.

Example 3.1. (i) The Fourier transform of $f(x) = (K - \sum_{j=1}^d e^{x_j})^+$, $K > 0$, that is the payoff function of a *basket* put option, is given by

$$\widehat{f}(z) = K^{1+i\sum_{j=1}^d z_j} \frac{\prod_{j=1}^d \Gamma(iz_j)}{\Gamma(2 + i\sum_{j=1}^d z_j)}$$

for all $z \in \mathbb{C}$ with $\mathrm{Im}(z_j) < 0$, $j = 1, \dots, d$. The price of the corresponding call can easily be derived using the put-call-parity $(K - x)^+ = (x - K)^+ - x + K$. Since we have separated the initial values s in (3.1), we can use FFT methods to compute the prices of weighted baskets for several weights efficiently.

(ii) The Fourier transform of the payoff function of a *spread* call option, $f(x) = (e^{x_1} - e^{x_2} - K)^+$, $K > 0$, is given by

$$\widehat{f}(z) = \frac{K^{1+iz_1+iz_2}}{iz_1(1+iz_1)} \frac{\Gamma(iz_2)\Gamma(-iz_1 - iz_2 - 1)}{\Gamma(-iz_1 - 1)}$$

for all $z \in \mathbb{C}$ with $\mathrm{Im}(z_1) > 1$, $\mathrm{Im}(z_2) < 0$ and $\mathrm{Im}(z_1 + z_2) > 1$.

Since the Fourier transform of $(e^{x_1} - e^{x_2})^+$ does not exist anywhere, we cannot use Theorem 3.1 to price zero-strike spread options. Nevertheless, we can derive a similar formula directly.

Proposition 3.2 (Spread options with zero strike)**.** *Suppose that*

$$\Phi_{(Y_T^1,Y_T^2)}(R,1-R)<\infty \quad \textit{for some } R>1.$$

Then the price of a zero-strike spread option *with payoff* $(S_0^1 e^{Y_T^1}-S_0^2 e^{Y_T^2})^+$ *is given by*

$$E_Q(e^{-rT}(S_T^1-S_T^2)^+)=\frac{e^{R(s_2-s_1)-s_2-rT}}{2\pi}\int_{\mathbb{R}} e^{iu(s_2-s_1)}\frac{\Phi_{(Y_T^1,Y_T^2)}(R+iu,1-R-iu)}{(R+iu)(R+iu-1)}\,du,$$

where $s_1=-\ln(S_0^1)$ *and* $s_2=-\ln(S_0^2)$.

Observe that unlike for $K>0$, we only have to compute a one-dimensional integral to determine the price of a zero-strike spread option. This will be advantageous in the calibration procedure in Section 4.

Proof. Let $R>1$ and define $f_K(x)=(e^x-K)^+$ for $K>0$, and $g_K(x)=e^{-Rx}f_K(x)$. By Fourier inversion and (3.2) we have

$$f_{e^y}(x)=\frac{1}{2\pi}\int_{\mathbb{R}}\frac{e^{(R+iu)x}e^{(1-R-iu)y}}{(R+iu)(R+iu-1)}\,du,$$

for all $y\in\mathbb{R}$. Hence, for the function $h_{e^y}(x):=(S_0^1e^x-S_0^2e^y)^+=f_{e^{y-s_2}}(x-s_1)$ we get

$$h_{e^y}(x)=\frac{1}{2\pi}e^{R(s_2-s_1)-s_2}\int_{\mathbb{R}} e^{iu(s_2-s_1)}\frac{e^{(R+iu)x}e^{(1-R-iu)y}}{(R+iu)(R+iu-1)}\,du.$$

Finally, by Fubini's theorem

$$\begin{aligned}E_Q(h_{e^{Y_T^2}}(Y_T^1))&=\frac{e^{R(s_2-s_1)-s_2}}{2\pi}\int_{\mathbb{R}^2}\int_{\mathbb{R}} e^{iu(s_2-s_1)}\frac{e^{(R+iu)x}e^{(1-R-iu)y}}{(R+iu)(R+iu-1)}\,du\,P_{(Y_T^1,Y_T^2)}(dx,dy)\\&=\frac{e^{R(s_2-s_1)-s_2}}{2\pi}\int_{\mathbb{R}} e^{iu(s_2-s_1)}\frac{\Phi_{(Y_T^1,Y_T^2)}(R+iu,1-R-iu)}{(R+iu)(R+iu-1)}\,du,\end{aligned}$$

where the application of Fubini's theorem is justified by

$$\begin{aligned}\int_{\mathbb{R}^2}\int_{\mathbb{R}}\left|\frac{e^{(R+iu)x}e^{(1-R-iu)y}}{(R+iu)(R+iu-1)}\right|du\,P_{(Y_T^1,Y_T^2)}(dx,dy)&=\int_{\mathbb{R}^2}e^{Rx}e^{(1-R)y}\int_{\mathbb{R}}|\widehat{g}_1(u)|\,du\,P_{(Y_T^1,Y_T^2)}(dx,dy)\\&\le||\widehat{g}_1||_{\mathrm{L}^1}\,\Phi_{(Y_T^1,Y_T^2)}(R,1-R)<\infty,\end{aligned}$$

since $||\widehat{g}_1||_{\mathrm{L}^1}<\infty$ as shown in Eberlein et al. (2009, Example 5.1). □

4. Calibration of the OU-Wishart model

We now put forward a specific parametric specification of the model discussed in Section 2. To this end, let $n\in\mathbb{N}$, $\Theta\in\mathbb{S}_d^+$ and let X be a $d\times n$ random matrix with i.i.d standard normal entries. Then the matrix $M:=\Theta^{\frac{1}{2}}XX^{\top}\Theta^{\frac{1}{2}}$ is said to be *Wishart distributed*, written $M\sim\mathscr{W}_d(n,\Theta)$. Note that this definition can be extended to noninteger $n>d-1$ using the characteristic function

$$Z\mapsto\det(I_d-2iZ\Theta)^{-\frac{1}{2}n}, \tag{4.1}$$

see Gupta and Nagar (2000, Theorem 3.3.7). Since $M \in \mathbb{S}_d^+$ almost surely, we can define a compound Poisson matrix subordinator L with intensity λ and $\mathscr{W}_d(n,\Theta)$ distributed jumps. We call the resulting multivariate stochastic volatility model of OU type *OU-Wishart model*.

Since we have $\int_{\mathbb{S}_d^+} e^{\mathrm{tr}(RX)}\,\kappa_L(dX) = \det(I_d - 2R\Theta)^{-\frac{1}{2}n}$ by (4.1), we see that L has exponential moments as long as $||R|| < \frac{1}{2||\Theta||}$, where $||\cdot||$ denotes the spectral norm. That means (2.12) holds for $\varepsilon := \frac{1}{2||\Theta||}$, and we can apply the integral transform methods from the previous section to compute prices of multi-asset options.

By (2.7) and (2.8), each asset follows a BNS model at the margins, if A, β and ρ are chosen to be diagonal. In particular, for $n = 2$ we see that L^{ii}, $i = 1,\dots,d$, is a compound Poisson subordinator with exponentially distributed jumps, thus we have in distribution the Γ-*OU BNS model* with stationary Gamma distribution at the margins, cf. e.g. Nicolato and Venardos (2003, Section 2.2). In particular, the characteristic functions of the single assets are known in closed form. Note that while the characteristic function of the stationary distribution of the marginal OU type process is still known for $n \neq 2$, it no longer corresponds to a Gamma distribution in this case.

Remark 4.1. There exists a subclass of structure preserving EMMs Q (cf. Theorem 2.11) such that we have an OU-Wishart model under both P and Q. This means that L is a compound Poisson process with $\mathscr{W}_d(n,\Theta)$ distributed jumps and intensity λ under P, and $\mathscr{W}_d(\widetilde{n},\widetilde{\Theta})$ distributed jumps with intensity $\widetilde{\lambda}$ under Q. We only need to assume that the Wishart distribution under both P and Q has a Lebesgue density, i.e. $n,\widetilde{n} > d-1$ and $\Theta,\widetilde{\Theta} \in \mathbb{S}_d^{++}$. Then one simply has to take y as the quotient of the according Lévy densities. Hence, by Gupta and Nagar (2000, 3.2.1), y has to be defined as

$$y(X) = \frac{\widetilde{\lambda}}{\lambda}\left(2^{\frac{1}{2}(\widetilde{n}-n)d}\,\frac{\Gamma_d\left(\frac{1}{2}\widetilde{n}\right)}{\Gamma_d\left(\frac{1}{2}n\right)}\,\frac{\det(\widetilde{\Theta})^{\frac{1}{2}\widetilde{n}}}{\det(\Theta)^{\frac{1}{2}n}}\right)^{-1}\det(X)^{\frac{1}{2}(\widetilde{n}-n)}e^{-\frac{1}{2}\mathrm{tr}\left((\widetilde{\Theta}^{-1}-\Theta^{-1})X\right)},\quad X \in \mathbb{S}_d^+.$$

4.1. The OU-Wishart model in dimension 2

We work directly under a pricing measure Q. In two dimensions, and for diagonal A, ρ, our model is given by

$$\begin{aligned} d\begin{pmatrix} Y_t^1 \\ Y_t^2 \end{pmatrix} &= \left(\begin{pmatrix} \mu_1 \\ \mu_2 \end{pmatrix} - \frac{1}{2}\begin{pmatrix} \Sigma_t^{11} \\ \Sigma_t^{22} \end{pmatrix}\right)dt + \begin{pmatrix} \Sigma_t^{11} & \Sigma_t^{12} \\ \Sigma_t^{12} & \Sigma_t^{22} \end{pmatrix}^{\frac{1}{2}} d\begin{pmatrix} W_t^1 \\ W_t^2 \end{pmatrix} + \begin{pmatrix} \rho_1\,dL_t^{11} \\ \rho_2\,dL_t^{22} \end{pmatrix} \\ d\begin{pmatrix} \Sigma_t^{11} & \Sigma_t^{12} \\ \Sigma_t^{12} & \Sigma_t^{22} \end{pmatrix} &= \left(\begin{pmatrix} \gamma_1 & 0 \\ 0 & \gamma_2 \end{pmatrix} + \begin{pmatrix} 2a_1\Sigma_t^{11} & (a_1+a_2)\Sigma_t^{12} \\ (a_1+a_2)\Sigma_t^{12} & 2a_2\Sigma_t^{22} \end{pmatrix}\right)dt + d\begin{pmatrix} L_t^{11} & L_t^{12} \\ L_t^{12} & L_t^{22} \end{pmatrix} \end{aligned}$$

with initial values

$$Y_0 = \begin{pmatrix} 0 \\ 0 \end{pmatrix},\quad \Sigma_0 = \begin{pmatrix} \Sigma_0^{11} & \Sigma_0^{12} \\ \Sigma_0^{12} & \Sigma_0^{22} \end{pmatrix} \in \mathbb{S}_2^{++},$$

and parameters $\gamma_1,\gamma_2 \geq 0$, $a_1,a_2 < 0$, $\rho_1,\rho_2 \in \mathbb{R}$. L is a compound Poisson process with intensity λ and $\mathscr{W}_2(n,\Theta)$-jumps, where $n = 2$ and

$$\Theta = \begin{pmatrix} \Theta_{11} & \Theta_{12} \\ \Theta_{12} & \Theta_{22} \end{pmatrix} \in \mathbb{S}_2^+.$$

Single-asset option pricing

Since the margins are (in distribution) Γ-OU BNS models, we have a closed form expression for the moment generating function of Y^1, which can be used to price single-asset options. It is given by

$$E[e^{y_1 Y_t^1}] = \exp\left\{ y_1\mu_1 t + \frac{e^{2a_1 t}-1}{4a_1}(y_1^2-y_1)\Sigma_0^{11} + \frac{1}{4a_1}\left(\frac{1}{2a_1}(e^{2a_1 t}-1)-t\right)(y_1^2-y_1)\gamma_1 \right.$$
$$\left. + \frac{\lambda}{2a_1(f_2-\xi)}\left(\xi \ln\left(\frac{\xi-f_1}{\xi-\rho_1 y_1}\right) - 2a_1 f_2 t\right)\right\}$$

with $\xi = \frac{1}{2\Theta_{11}}$ and

$$f_1 = \frac{1}{4a_1}(e^{2a_1 t}-1)(y_1^2-y_1)+\rho_1 y_1, \quad f_2 = -\frac{1}{4a_1}(y_1^2-y_1)+\rho_1 y_1,$$

which can be obtained by simple integration (also cf. Nicolato and Venardos (2003, Table 2.1) corrected for a typo in f_1 and f_2). Note that one can use the recursion formula stated in Gradshteyn and Ryzhik (2007, 2.155) to obtain a closed form expression for $\mathscr{W}_2(n,\Theta)$-jumps with $n \in 2\mathbb{N}$, too.

Multi-asset option pricing

By (2.14) and (4.1), the joint moment generating function of (Y^1, Y^2) is given by

$$E[e^{y^\top Y_t}] = \exp\left(y^\top \mu t + \mathrm{tr}(\Sigma_0 H_y(t)) + \int_0^t \mathrm{tr}(\gamma_L H_y(s))ds + \lambda\int_0^t \frac{1}{\det(I_2 - 2(H_y(s)+\rho^*(y))\Theta)}ds - \lambda t\right)$$

with H_y as in (2.15), $A = \begin{pmatrix} a_1 & 0 \\ 0 & a_2\end{pmatrix}$, $\gamma_L = \begin{pmatrix} \gamma_1 & 0 \\ 0 & \gamma_2\end{pmatrix}$ and $\rho^*(y) = \begin{pmatrix} \rho_1 y_1 & 0 \\ 0 & \rho_2 y_2\end{pmatrix}$. Whereas the first integral above is easy to calculate, the second one is not. More specifically, it is not possible to obtain a closed form expression in terms of ordinary functions, unless one sets $a_1 = a_2 =: a$. In this case

$$E[e^{y_1 Y_t^1 + y_2 Y_t^2}] = \exp\left\{ y_1\mu_1 t + y_2\mu_2 t + \frac{e^{2at}-1}{4a}\mathrm{tr}\left(\Sigma_0\begin{pmatrix} y_1^2-y_1 & y_1y_2 \\ y_1y_2 & y_2^2-y_2\end{pmatrix}\right)\right.$$
$$+\frac{1}{4a}\left(\gamma_1(y_1^2-y_1)+\gamma_2(y_2^2-y_2)\right)\left(\frac{1}{2a}(e^{2at}-1)-t\right)$$
$$+\frac{\lambda}{2ab_0}\left[\frac{b_1}{d}\left(\arctan\left(\frac{2b_2+b_1}{d}\right) - \arctan\left(\frac{2b_2e^{2at}+b_1}{d}\right)\right)\right.$$
$$\left.\left.+\frac{1}{2}\ln\left(\frac{b_0+b_1+b_2}{b_2e^{4at}+b_1e^{2at}+b_0}\right)\right] + \frac{\lambda}{b_0}t - \lambda t\right\}$$

with coefficients

$$b_0 := 1 + 4\det(B-C) + 2\mathrm{tr}(B-C),$$
$$b_1 := -8\det(B) + 4\mathrm{tr}(B)\mathrm{tr}(C) - 4\mathrm{tr}(BC) - 2\mathrm{tr}(B),$$
$$b_2 := 4\det(B),$$

and matrices

$$B := \frac{1}{4a}\begin{pmatrix} y_1^2-y_1 & y_1y_2 \\ y_1y_2 & y_2^2-y_2\end{pmatrix}\Theta, \quad C := \begin{pmatrix}\rho_1 y_1 & 0 \\ 0 & \rho_2 y_2\end{pmatrix}\Theta.$$

Using $\det(A+B) = \det(A) + \det(B) + \operatorname{tr}(A)\operatorname{tr}(B) - \operatorname{tr}(AB)$ for $A, B \in M_2(\mathbb{R})$, this follows from

$$\det(I_2 - 2(H_y(s) + \rho^*(y))\Theta) = \det(I_2 - 2(e^{2as} - 1)B - 2C) = b_0 + b_1 e^{2as} + b_2 e^{4as},$$

and straightforward integration. Likewise, one can also derive a closed form expression for $n = 4, 6, \ldots$ using Gradshteyn and Ryzhik (2007, 2.18(4)).

Consequently, one faces a tradeoff at this point. One possibility is to retain the flexibility of different mean reversion speeds a_i by evaluating the remaining integral using numerical integration. Alternatively, one can restrict attention to identical mean reversion speeds in order to have a closed-form expression of the moment generating function at hand. The impact of this decision on the calibration performance is discussed in Section 4.2 below.

Remark 4.2 (High Dimensionality)**.** The above model can also be defined for $d > 2$, but of course, the Fourier formula (3.1) is numerically infeasible in high dimensions. Nevertheless, the calibration of a high dimensional OU-Wishart model is possible by only evaluating options on *two* underlyings. Using zero strike spread options and provided the characteristic function is known explicitly, this means that one only has to evaluate single integrals numerically, as in the univariate case. Indeed, combining Barndorff-Nielsen and Stelzer (2009, Proposition 4.5) and the fact that every symmetric sub-matrix of a Wishart distributed matrix is again Wishart distributed, cf. Gupta and Nagar (2000, Theorem 3.3.10), it follows that the joint dynamics of each pair of assets follows a 2-dimensional OU-Wishart model as above. Hence, we can calibrate the model using only two-asset options (e.g. spread options). The price to pay is that the resulting model only incorporates pairwise dependencies, since the respective covariances completely determine the underlying Wishart distribution.

4.2. Empirical illustration

The aim of this subsection is to show that a calibration of the OU-Wishart model to market prices is feasible. To the best of our knowledge, this has not been done for any of the other multivariate stochastic volatility models with non-trivial dependence structure proposed in the literature. Since multi-asset options are mostly traded over-the-counter, it is difficult to obtain real price quotes. To circumvent this problem, we proceed as in Taylor and Wang (2009) and consider *foreign exchange rates* instead, where a call option on some exchange rate can be seen as a spread option between two others. Let us emphasise that our calibration routine should not be seen as a finished product, but much rather as a first test.

We consider a 2-dimensional OU-Wishart model as above where our first asset is the EUR/USD exchange rate $S^{\$/€} = S_0^{\$/€} e^{Y^1}$, that is the price of 1 € in \$, and our second asset is the GBP/USD exchange rate $S^{\$/£} = S_0^{\$/£} e^{Y^2}$, i.e. the price of 1 £ in \$. Since we model directly under a martingale measure, we set

$$\mu_1 = r_{\$} - r_{€} - \frac{\lambda \rho_1}{\frac{1}{2\Theta_{11}} - \rho_1}, \quad \mu_2 = r_{\$} - r_{£} - \frac{\lambda \rho_2}{\frac{1}{2\Theta_{22}} - \rho_2}.$$

By Hull (2003, 13.4), it follows that the price in \$ of a plain vanilla call option on $S^{\$/€}$ or $S^{\$/£}$ is given by $e^{-(r_{\$}-r_{€})T} E((S_T^{\$/€} - K)^+)$ or $e^{-(r_{\$}-r_{£})T} E((S_T^{\$/£} - K)^+)$, respectively. Now observe that the \$-payoff of a plain vanilla call option on the EUR/GBP exchange rate $S^{£/€}$ is given by $S_T^{\$/£}(S_T^{£/€} - K)^+ = (S_T^{\$/€} - K S_T^{\$/£})^+$, hence it can be regarded as a spread option on $S^{\$/€} - S^{\$/£}$ where the initial value of the second asset is replaced by $K S_0^{\$/£}$. Since it is a zero-strike spread option, we can use Proposition 3.2 to valuate it.

We obtained the option price data on September 11, 2009 from EUWAX. The EUR/USD exchange rate at that time was $S_0^{\$/€} = 1.4578\$$, the GBP/USD exchange rate was $S_0^{\$/£} = 1.6683\$$ and the EUR/GBP exchange rate was 0.8738£. As a proxy for the instantaneous riskless interest rate we took the 3-month LIBOR for each currency, viz. $r_€ = 0.732\%$, $r_£ = 0.299\%$ and $r_\$ = 0.627\%$. All call options here are plain vanilla call options of European style. We used 153 call options on the EUR/USD exchange rate, 37 call options on the GBP/USD exchange rate, and 88 call options on the EUR/GBP exchange rate, all of them for different strikes and 4-5 different maturities. We always used the mid-value between bid and ask price.

The calibration was performed by choosing the model parameters so as to minimise the mean squared error between market and model prices. The results can be found in Table 4.1. The overall root mean squared error (RMSE) is 0.0586. If one considers only the marginal models for EUR/USD and GBP/USD one has a RMSE of 0.0683 and 0.0425 respectively. For comparison, we calibrated two independent Γ-OU BNS models to the margins separately and obtained a slightly lower RMSE of 0.0610 and 0.0320 respectively. This stems from the fact that the additional dependence parameters do not enter the pricing formulas for single asset options, whereas the intensity of the compound Poisson process is the same for all assets in our multivariate framework, unlike when using two univariate models. This means that we are *not overfitting* the marginal distributions with an excessive amount of additional parameters, but much rather using a simplified version of a standard model. Nevertheless, the calibration does not appear to be worsened too much by using this simplification.

To depict the good fit visually we provide Figure 1, where market and model prices are compared for a sample of different strikes and maturities. In Figure 2 we compare the corresponding Black-Scholes implied volatilities.

If one sets the mean reversion parameter of both assets equal, $a := a_1 = a_2 < 0$, one has a closed form solution for the moment generating function of (Y^1, Y^2). This decreases the runtime considerably. The corresponding calibrated parameters can be found in the second row of Table 4.1. The overall RMSE is 0.0591 and therefore only differs from the one for different mean reversion speeds by about 0.85 %. At the margins, we have 0.0686 or 0.0439 respectively. Therefore this specification appears to be an appealing alternative if computation time is an issue.

Finally, we also examined the impact of the leverage term. For $\rho_1 = 0 = \rho_2$, the overall RMSE increases to the considerably higher value of 0.1191, that means by about 101.5 %. At the margins we have 0.1164 and 0.0639, respectively. The calibrated parameters can be found in the third row of Table 4.1. These empirical results suggest that it is highly advisable to include a leverage operator ρ in the model. This is in line with statistical studies under P.

Table 4.1: Calibrated parameters. First row: different mean reversion ($a_1 \neq a_2$). Second row: same mean reversion ($a_1 = a_2$). Third row: no leverage case ($\rho_1 = 0 = \rho_2$)

λ	a_1	ρ_1	Θ^{11}	Σ_0^{11}	γ_1	a_2	ρ_2	Θ^{22}	Σ_0^{22}	γ_2	Θ^{12}	Σ_0^{12}
0.415	-0.313	1.538	0.012	0.017	0	-0.606	-0.211	0.036	0.016	0.003	0.017	0.012
0.369	-0.405	1.388	0.014	0.018	0.003	-0.405	-0.274	0.033	0.015	0	0.017	0.012
0.891	-0.680	/	0.014	0.019	0	-0.680	/	0.020	0.015	0	0.010	0.012

We tested the sensitivity of the calibration with respect to the initial values of the optimisation routine. In particular, we found that a calibration to model prices could recover the true parameters very well from a broad range of initial values.

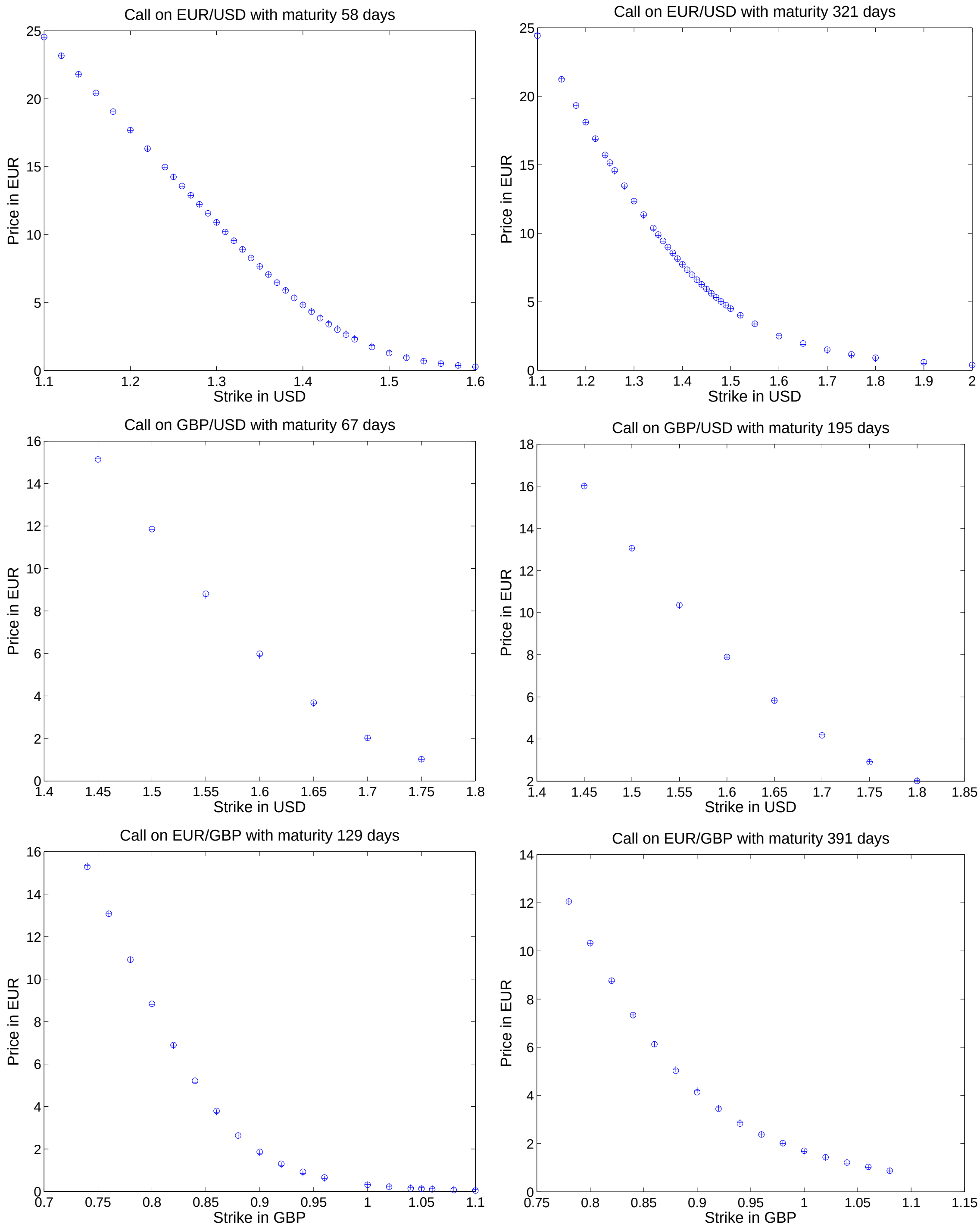

Figure 1: Calibration of the OU-Wishart model. Market prices (circle) against model prices (plus).

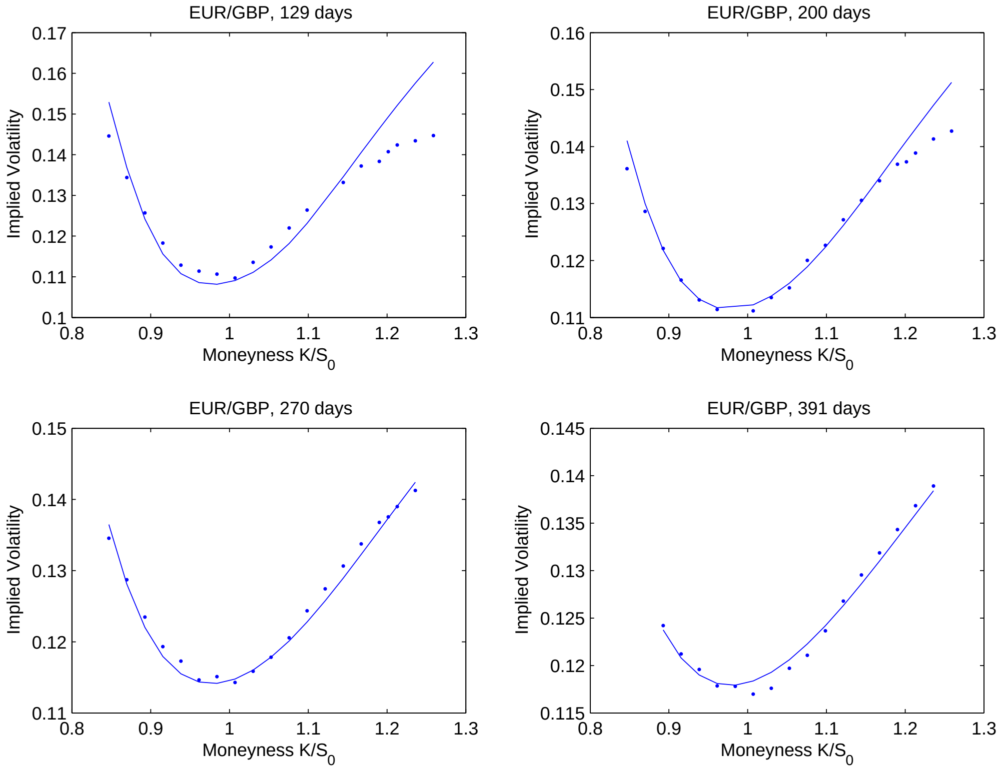

Figure 2: Comparison of the Black-Scholes implied volatility of market prices (dot) and model prices (solid line).

5. Covariance swaps

In this final section, we show that it is possible to price swaps on the covariance between different assets in closed form. This serves two purposes. On the one hand, options written on the realised covariance represent a family of payoffs that only make sense in models where covariances are modeled as stochastic processes rather than constants. On the other hand, the ensuing calculations exemplify once more the analytical tractability of the present framework.

We consider again our multivariate stochastic volatility model of OU type under an EMM Q. In addition, we suppose that the matrix subordinator L is square integrable, i.e. $\int_{\{||X||>1\}} ||X||^2 \kappa_L(dX) < \infty$. The pricing of options written on the realised variance resp. the quadratic variation as its continuous-time limit have been studied extensively in the literature, cf. e.g. Carr and Lee (2008) and the references therein. Since we have a nontrivial correlation structure in our model, one can also consider *covariance swaps* on two assets $i, j \in \{1, \ldots, d\}$, i.e. contracts with payoff $[Y^i, Y^j]_T - K$ with *covariance swap rate* $K = E([Y^i, Y^j]_T)$ (see e.g. Carr and Madan (1999a), Da Fonseca, Grasselli and Ielpo (2008) or Swischuk (2005) for more background on these products). Now we show how to compute the covariance swap rate. We have

$$[Y^i, Y^j]_T = [Y^i, Y^j]^c_T + \sum_{s \leq T} \Delta Y^i_s \Delta Y^j_s = (\Sigma^+_T)^{ij} + \rho^i(X)\rho^j(X) * \mu^L_T(dX),$$

and since $\kappa_L(dX)dt$ is the compensator of μ^L, this yields

$$E([Y^i, Y^j]_T) = (E(\Sigma_T^+))^{ij} + T \int_{\mathbb{S}_d^+} \rho^i(X)\rho^j(X)\kappa_L(dX). \tag{5.1}$$

Note that by Pigorsch and Stelzer (2009, Proposition 2.4) resp. since $|\rho^i(X)\rho^j(X)| \leq ||\rho||^2||X||^2$, our integrability assumption on L implies that the expectation is finite. The first summand can be calculated as follows. By setting $y = 0$ in Theorem 2.5 we obtain the characteristic function of Σ_t. Differentiation yields

$$E(\Sigma_T) = e^{AT}\Sigma_0 e^{A^\top T} + e^{AT}\mathbf{A}^{-1}(E(L_1))e^{A^\top T} - \mathbf{A}^{-1}(E(L_1)),$$

where $E(L_1) = \gamma_L + \int_{\mathbb{S}_d^+} X\, \kappa_L(dX)$. Using Equation (2.4) we obtain

$$E(\Sigma_T^+) = \mathbf{A}^{-1}(E(\Sigma_T) - TE(L_1) - \Sigma_0),$$

so we only need to know $E(L_1)$. The second summand in (5.1) can analogously be computed by differentiating the characteristic function of the matrix subordinator L.

In our OU-Wishart model, where L is a compound Poisson matrix subordinator plus drift with $\mathscr{W}_d(n,\Theta)$-distributed jumps, we have by Gupta and Nagar (2000, Theorem 3.3.15) that

$$E(L_1) = \gamma_L + \lambda n \Theta.$$

If ρ is diagonal, the second term in (5.1) simplifies to

$$T\rho_i\rho_j \int_{\mathbb{S}_d^+} X_{ii}X_{jj}\, \nu(dX) = T\rho_i\rho_j\lambda n\left(2\Theta_{ij}^2 + n\Theta_{ii}\Theta_{jj}\right),$$

again by Gupta and Nagar (2000, Theorem 3.3.15). Thus we have a closed form expression for the covariance swap rate:

$$K = \left(\mathbf{A}^{-1}\left[e^{AT}(\Sigma_0 + \mathbf{A}^{-1}(\gamma_L + \lambda n\Theta))e^{A^\top T} - \mathbf{A}^{-1}(\gamma_L + \lambda n\Theta) - T(\gamma_L + \lambda n\Theta) - \Sigma_0\right]\right)^{ij} + T\rho_i\rho_j\lambda n\left(2\Theta_{ij}^2 + n\Theta_{ii}\Theta_{jj}\right).$$

For example, in the 2-dimensional OU-Wishart model from Section 4.1 we have for $i = 1$ and $j = 2$

$$K = \frac{1}{a_1 + a_2}\left[\left(e^{(a_1+a_2)T} - 1\right)\left(\Sigma_0^{12} + \frac{\lambda n\Theta_{12}}{a_1 + a_2}\right) - T\lambda n\Theta_{12}\right] + T\rho_1\rho_2\lambda n\left(2\Theta_{12}^2 + n\Theta_{11}\Theta_{22}\right).$$

As in Carr and Lee (2008), pricing of options on the covariance can be dealt with using the Fourier methods from Section 3, since the joint characteristic function of $(\Sigma^+, \rho^i(X)\rho^j(X) * \mu^L(dX))$ can be calculated similarly as in the proof of Theorem 2.5.

Acknowledgements

The first author gratefully acknowledges support from the FWF (Austrian Science Fund) under grant P19456. The second and third author greatly appreciate the support of the Technische Universität München - Institute of Advanced Study funded by the German Excellence Initiative.

A. Appendix

The following simple result on multidimensional analytic functions is needed in the proof of Lemma 2.7.

Lemma A.1. *Let $D_\varepsilon = \{z \in \mathbb{C}^n : ||\mathrm{Re}(z)|| < \varepsilon\}$ for some $\varepsilon > 0$. Suppose $f : D_\varepsilon \to \mathbb{C}$ is an analytic function of the form $f = e^F$, where $F : D_\varepsilon \to \mathbb{C}$ is continuous. Then F is analytic in D_ε.*

Proof. Let $z = (z_1, z_2, \ldots, z_n) \in D_\varepsilon$ and define $z_{-1} = (z_2, \ldots, z_n)$. Then $f_{z_{-1}} : w \mapsto f(w, z_{-1})$ defines an analytic function without zeros on the open convex set $D_{\varepsilon, z_{-1}} := \{w \in \mathbb{C} : (w, z_{-1}) \in D_\varepsilon\}$. By e.g. Fischer and Lieb (1994, Satz V.1.4), there exists an analytic function $g^1_{z_{-1}} : D_{\varepsilon,z_{-1}} \to \mathbb{C}$ such that $\exp(g^1_{z_{-1}}) = f_{z_{-1}}$. Hence $F(w, z_{-1}) - g^1_{z_{-1}}(w) \in 2\pi i \mathbb{Z}$ on $D_{\varepsilon,z_{-1}}$. Since both F and g are continuous, their difference is constant and it follows that $w \mapsto F(w, z_{-1})$ is analytic on $D_{\varepsilon,z_{-1}}$. Analogously, one shows analyticity of F in all other components. The assertion then follows from Hartog's Theorem (cf. e.g. Hörmander (1967, Theorem 2.2.8)). □

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