Quantum Model of Bertrand Duopoly

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We present the quantum model of Bertrand duopoly and study the entanglement behaviour on the profit functions of the firms. Using the concept of optimal response of each firm to the price of the opponent, we found four Nash equilibria for maximally entangled initial state. We have shown that only one point among the four Nash equilibria has valid physical meaning. The very presence of quantum entanglement in the initial state gives payoffs higher to the firms than the classical payoffs at the physically valid point for higher values of substitution parameter.

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I. INTRODUCTION

In economics, oligopoly refers to a market condition in which sellers are so few that action of each seller has a measurable impact on the price and other market factors [1]. If the number of firms competing on a commodity in the market is just two, the oligopoly is termed as duopoly. The competitive behavior of firms in oligopoly makes it suitable to be analyzed by using the techniques of game theory. Cournot and Bertrand models are the two oldest and famous oligopoly models [2, 3]. In Cournot model of oligopoly firms put certain amount of homogeneous product simultaneously in the market and each firm tries to maximize its payoff by assuming that the opponent firms will keep their outputs constant. Later on Stackelberg introduced a modified form of Cournot oligopoly in which the oligopolistic firms supply their products in the market one after the other instead of their simultaneous moves. In Stackelberg duopoly the firm that moves first is called leader and the other firm is the follower [4]. In Bertrand model the oligopolistic firms compete on price of the commodity, that is, each firm tries to maximize its payoff by assuming that the opponent firms will not change the prices of their products. The output and price are related by the demand curve so the firms choose one of them to compete on leaving the other free. For a homogeneous product, if firms choose to compete on price rather than output, the firms reach a state of Nash equilibrium at which they charge a price equal to marginal cost. This result is usually termed as Bertrand paradox, because practically it takes the firms to ensure prices equal to marginal cost. One way to avoid this situation is to allow the firms to sell differentiated products [1].

For the last one decade quantum game theorists are attempting to study classical games in the domain of quantum mechanics [5-14]. Various quantum protocols have been introduced in this regard and interesting results have been obtained [15–22]. The first quantization scheme was presented by Meyer [15] in which he quantized a simple penny flip game and showed that a quantum player can always win against a classical player by utilizing quantum superposition. Eisert et al. have developed a general scheme for quantizing two players two strategy games and applied it to quantize the prisoner dilemma game by considering an unentangled initial state. They used an entangling gate and then the operators that define the players strategies. The payoffs were determined from the final state of the game obtained after applying an unentangling gate. They showed that in quantum case of the game the dilemma that exists in the classical version disappears. Later on Benjamin and Hayden [23] extended the protocol of Eisert et al. to quantize multiplayer games. Li et al. [18] developed a quantization protocol for games of continues strategic space and apply it to Cournot duopoly. They showed that for maximally entangled initial state the players can escape the frustrating dilemma-like situation, that is, the unique Nash equilibrium is inferior to Pareto optimal. Lo and Kiang [19, 22, 24] used the same protocol and quantized the Bertrand duopoly with differentiated products, Stackelberg duopoly and Stackelberg duopoly with incomplete information games. Marinatto and Weber [17] quantized the Battle of the Sexes game by giving the formal structure of Hilbert space to the strategic space. Iqbal and Toor [21] quantized the Stackelberg duopoly game by using the Marinatto and Weber protocol proposed for quantizing the Battle of the Sexes game. Lo and Kiang [19] used Li et al.'s [18] minimal quantization rules to quantize the game of Bertrand duopoly with differentiated products. In this paper, we extend the classical Bertrand duopoly with differentiated products to quantum domain by using the quantization scheme proposed by Marinatto and Weber [17]. Our results show that the classical game becomes a subgame of the quantum version. We found the four Nash equilibria for a maximally entangled initial state with one point having valid physical meaning. The detailed results of our calculations are presented in the next sections.

II. BERTRAND DUOPOLY WITH DIFFERENTIATED PRODUCTS

A. Classical version

Consider two firms A and B producing their products at a constant marginal cost c such that $c < a$, where a is a constant. Let p_1 and p_2 be the prices chosen by each firm for their products, respectively. The quantities q_A and q_B that each firm sells is given by the following key assumption of the Bertrand duopoly model

$$q_A = a - p_1 + bp_2$$

$$q_B = a - p_2 + bp_1 \tag{1}$$

where the parameter $0 < b < 1$ shows the amount of one firm's product substituted for the other firm's product. It can be seen from Eq. (1) that more quantity of the product is sold by the firm which has low price compare to the price chosen by his opponent. The profit function of the two firms are given by

$$u_A(p_1, p_2, b) = q_A(p_1 - c) = (a - p_1 + bp_2)(p_1 - c)$$

$$u_B(p_1, p_2, b) = q_B(p_2 - c) = (a - p_2 + bp_1)(p_2 - c) \tag{2}$$

In Bertrand duopoly the firms are allowed to change the quantity of their product to be put in the market and compete only in price. A firm changes the price of its product by assuming that the opponent will keep its price constant. Suppose that firm B has chosen p_2 as the price of his product, the optimal response of firm A to p_2 is obtained by maximizing its profit function with respect to its own product's price, that is, $\partial u_A/\partial p_1 = 0$, this leads to

$$p_1 = \frac{1}{2}(bp_2 + a + c) \tag{3}$$

Firm B response to a fixed price p_1 of firm A is obtained in a similar way and is given by

$$p_2 = \frac{1}{2}(bp_1 + a + c) \tag{4}$$

Solution of Eqs.(3 and 4) lead to the following optimal price level that defines the Nash equilibrium of the game

$$p_1^* = p_2^* = \frac{a + c}{2 - b} \tag{5}$$

The profit functions of the firms at the Nash equilibrium become

$$u_A^* = u_B^* = \left[\frac{a+c}{2-b} - c\right]^2 \tag{6}$$

From Eq. (6), we see that both firms can be made better off if they choose higher prices, that is, the Nash equilibrium is Pareto inefficient.

B. Quantization scheme

We consider that the game space of each firm is a two dimensional Hilbert space of basis vector $|0\rangle$ and $|1\rangle$, that is, the game consists of two qubits, one for each firm. The composite Hilbert space $\mathcal{H}$ of the game is a four dimensional space which is formed as a tensor product of the individual Hilbert spaces of the firms, that is, $\mathcal{H} = \mathcal{H}_A \otimes \mathcal{H}_B$, where $\mathcal{H}_A$ and $\mathcal{H}_B$ are the Hilbert spaces of firms A and B, respectively. To manipulate their respective qubits each firm can have only two strategies I and C. Where I is the identity operator and and C is the inversion operator also called Pauli spin flip operator. If x and $1-x$ stand for the probabilities of I and C that firm A applies and y, $1-y$, are the probabilities that firm B applies then the final state ρ_f of the game is given by [17]

$$\begin{aligned} \rho_f \;=\;& xyI_A \otimes I_B\ \rho_i\ I_A^\dagger \otimes I_B^\dagger + x(1-y)\, I_A \otimes C_B\ \rho_i\ I_A^\dagger \otimes C_B^\dagger \\ & + y(1-x)\, C_A \otimes I_B\ \rho_i\ C_A^\dagger \otimes I_B^\dagger \\ & + (1-x)(1-y)\, C_A \otimes C_B\ \rho_i\ C_A^\dagger \otimes C_B^\dagger \end{aligned} \tag{7}$$

In Eq. (7) $\rho_i = |\Psi_i\rangle\langle\Psi_i|$ is the initial density matrix with initial state $|\Psi_i\rangle$, which is given by

$$|\Psi_i\rangle = \cos\gamma|00\rangle + \sin\gamma|11\rangle \tag{8}$$

where $\gamma \in [0, \pi]$ and represents the degree of entanglement of the initial state. In Eq. (8) the first qubit corresponds to firm A and the second qubit corresponds to firm B. The moves (prices) of the firms and the probabilities x, y of using the operators can be related as follows,

$$x = \frac{1}{1+p_1}, \qquad\qquad y = \frac{1}{1+p_2} \tag{9}$$

where the prices p_1 and $p_2 \in [0, \infty)$ and the probabilities x, $y \in [0, 1]$. By using Eqs. (7 - 9), the nonzero elements of the final density matrix are obtained as

$$\begin{aligned}
\rho_{11} &= \frac{(\cos^2\gamma + p_1 p_2 \sin^2\gamma)}{(1+p_1)(1+p_2)} \\
\rho_{14} &= \rho_{41} = \frac{(1+p_1 p_2)\cos\gamma\sin\gamma}{(1+p_1)(1+p_2)} \\
\rho_{22} &= \frac{p_2\cos^2\gamma + p_1\sin^2\gamma}{(1+p_1)(1+p_2)} \\
\rho_{23} &= \rho_{32} = \frac{(p_1+p_2)\cos\gamma\sin\gamma}{(1+p_1)(1+p_2)} \\
\rho_{33} &= \frac{p_1\cos^2\gamma + p_2\sin^2\gamma}{(1+p_1)(1+p_2)} \\
\rho_{44} &= \frac{p_1 p_2\cos^2\gamma + \sin^2\gamma}{(1+p_1)(1+p_2)}
\end{aligned} \tag{10}$$

The payoffs of the firms can be found by the following trace operations

$$\begin{aligned}
u_A(p_1, p_2, b) &= \text{Trace}\left(U_{A,\text{ oper}}\rho_f\right) \\
u_B(p_1, p_2, b) &= \text{Trace}\left(U_{B,\text{ oper}}\rho_f\right)
\end{aligned} \tag{11}$$

where $U_{A,\text{ oper}}$ and $U_{B,\text{ oper}}$ are payoffs operators of the firms, which we define as

$$\begin{aligned}
U_{A,\text{ oper}} &= \frac{q_A}{p_{12}}(k_B\rho_{11} - \rho_{22} + \rho_{33}) \\
U_{B,\text{ oper}} &= \frac{q_A}{p_{12}}(k_A\rho_{11} + \rho_{22} - \rho_{33})
\end{aligned} \tag{12}$$

where $k_A = p_1 - c$, $k_B = p_2 - c$ and $p_{12} = \frac{1}{(1+p_1)(1+p_2)}$. By using Eqs. (10 - 12), the payoffs of the firms are obtained as

$$\begin{aligned}
u_A(p_1, p_2, b) &= (a - p_1 + bp_2)[k_A\cos^2\gamma + \{p_2 + p_1(-1 - cp_2 + p_2^2)\}\sin^2\gamma] \\
u_B(p_1, p_2, b) &= (a - p_2 + bp_1)[k_B\cos^2\gamma + \{p_1 - p_2(1 + cp_1 - p_1^2)\}\sin^2\gamma]
\end{aligned} \tag{13}$$

One can easily see from Eq. (13) that the classical payoffs can be reproduced by setting $\gamma = 0$ in Eq. (13).

We proceed similar to the classical Bertrand duopoly to find the response of each firm to the price chosen by the opponent firm. For firm B, the price p_2, the optimal response of firm A is obtained by maximizing its own payoff (Eq. (13)) with respect to p_1. Similarly, the reaction function of firm B to a known p_1 is obtained. These reaction functions can be written as

$$\begin{aligned} p_1 &= \frac{k_B[-1+p_2(a+bp_2)]+[c+p_2+2bp_2-bp_2^2k_B+a\{2-p_2k_B\}]\cos 2\gamma}{(2-p_2k_B)\cos 2\gamma-2p_2k_B} \\ p_2 &= \frac{k_A[-1+p_1(a+bp_1)]+[c+p_1+2bp_1+bp_1^2k_A+a\{2+p_1k_A\}]\cos 2\gamma}{(2-p_1k_A)\cos 2\gamma+2p_1k_A} \end{aligned} \tag{14}$$

The results of Eq. (14) reduce to the classical results given in Eqs. (3 and 4) for the initially unentangled state, that leads to the classical Nash equilibrium. This shows that the classical game is a subgame of the quantum game.

Now, we discuss the behavior of the entanglement on the game dynamics. It can be seen from Eq. (14) that the optimal responses of the firms to a fixed price of the opponent firm, for a maximally entangled state, are given by

$$\begin{aligned} p_1 &= \frac{bp_2^2+ap_2-1}{2p_2} \\ p_2 &= \frac{bp_1^2+ap_1-1}{2p_1} \end{aligned} \tag{15}$$

Solving these equations, we can obtain the optimal price levels and the corresponding payoffs of each firm. In this case the following four Nash equilibria are obtained

$$\begin{aligned} p_1^*(1) &= p_2^*(1) = \frac{a+\sqrt{a^2+4\beta}}{-2\beta} \\ p_1^*(2) &= p_2^*(2) = \frac{2}{a+\sqrt{a^2+4\beta}} \\ p_1^*(3,4) &= \frac{2b}{a\sqrt{2+b}\left(\sqrt{2+b}\pm\gamma\right)} \\ p_2^*(3,4) &= -\frac{1}{2b}\left[a\pm\frac{\gamma}{\sqrt{2+b}}\right] \end{aligned} \tag{16}$$

where the numbers in the parentheses correspond to the respective Nash equilibrium points (the symbols $\pm$ correspond to the equilibrium points 3 and 4 respectively). The payoffs of the firms corresponding to these points are given by

$$
\begin{aligned}
u_A(1) &= u_B(1) = \frac{1}{4\beta^4}[a^4 + 2\alpha^2 + 2a^2b\beta + a^3c\beta - a\{(\beta-2)\,\beta - 3\}c\beta^2 \\
&\quad + \sqrt{a^2+4\beta}(a^3 + 2a\alpha + c\alpha^2 + a^2c\beta)] \\
u_A(2) &= u_B(2) = -\frac{4}{(a+\sqrt{a^2+4\beta})^4}[a^5c + a(-1+b)\{(-9+5b)c - 2\sqrt{a^2+4\beta}\} \\
&\quad - a^3\{(8-5b)c + \sqrt{a^2+4\beta}\} + (-1+b)^2(-2+c\sqrt{a^2+4\beta}) \\
&\quad + a^4(-1+c\sqrt{a^2+4\beta}) + a^2\{6 - 4c\sqrt{a^2+4\beta} + b(-4+3c\sqrt{a^2+4\beta})\}] \\
u_A(3,4) &= \frac{(1+b)^2(a^2(2+b)^{3/2} + a(2+b)(b\sqrt{2+b}c \pm \Gamma) + b(2b\sqrt{2+b} \pm c\Gamma\,(2+b)))}{(2+b)^{3/2}(a(2+b) \pm \sqrt{2+b}\Gamma)^2} \\
u_B(3,4) &= -\frac{(1+b)^2\sqrt{2+b}\left[2ac + abc - 2b \pm \sqrt{2+b}\Gamma c\right]}{4b(2+b)^{5/2}} \qquad (17)
\end{aligned}
$$

The new parameters introduced in Eqs. (16 and 17) are defined as $\beta = b - 2$, $\alpha = 2 - 3b + b^2$, $\Gamma = \sqrt{4b^2 + a^2(2+b)}$.

III. DISCUSSION AND CONCLUSION

We present a quantization scheme for the Bertrand duopoly with differentiated products. To analyze the effect of quantum entanglement on the game dynamics, we plot the payoffs of the firms (Eq. 17) against the substitution parameter b in figure (1). The values of parameters a and c are arbitrarily chosen to be 3 and 1, respectively. For a physically valid solution, the payoffs of both firms must be positive. The payoffs of the firms corresponding to the second equilibrium point are shown in the dashed line ($u_A^Q(2)$). It is seen that the payoffs remain negative for an entire range of the substitution parameter b. Therefore, this has no physical meaning and non of the firms will prefer this equilibrium point. The payoff of firm A corresponding to the third equilibrium point is shown in the dotted line ($u_A^Q(3)$) and the payoff of firm B in the dashed-dotted line ($u_B^Q(3)$). The payoffs of the firms ($u_A^Q(3)$, $u_B^Q(3)$) are interchanged at the fourth equilibrium point. In each of these cases, the payoffs are significantly damped and is negative for the whole range of substitution parameter b for one firm, while for the other firm it increases slightly and linearly against b. The payoffs of the firms corresponding to the first equilibrium point are shown in the solid line ($u_A^Q(1)$). The classical payoffs of the firms corresponding to the classical Nash equilibrium points (u_A^C) are represented by the dashed-dashed-dotted line. It is clear from the figure that the classical and quantum payoffs (at the first Nash equilibrium point) of the firms increase with substitution parameter b. For lower values of the substitution parameter b, the effect of maximally entangled initial state is to diminish the quantum payoffs below the classical limit. However for the higher values of the substitution parameter b, the quantum payoffs are significantly enhanced. This could be the best equilibrium point for the two firms.

In conclusion, we have used the Marinatto and Weber quantization scheme to find the quantum version of Bertrand duopoly with differentiated products. We have studied the entanglement behaviour on the payoffs of the firms for a maximally entangled initial state. We found that for large values of substitution parameter b, both firms can achieve higher payoffs as compared to the classical payoffs. Furthermore, the dilemma-like situation in the classical Bertrand duopoly game is resolved.

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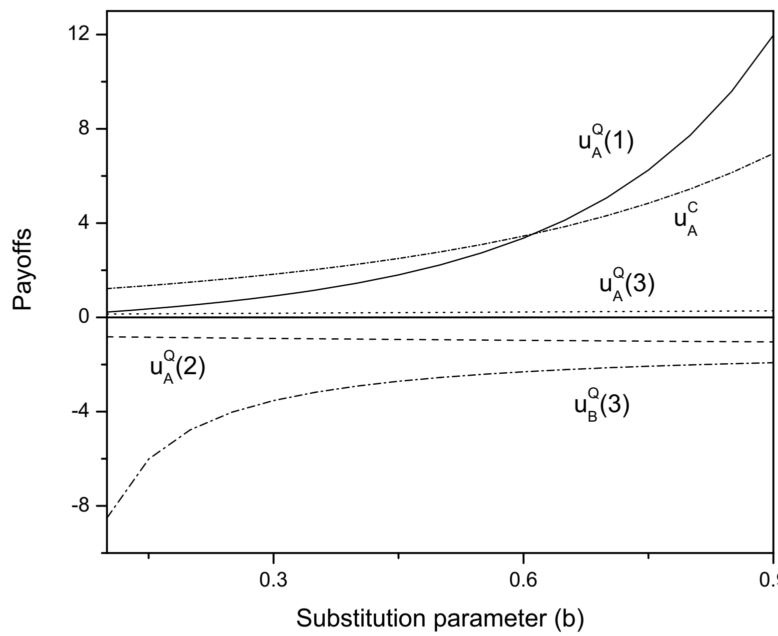
Figure caption

Figure 1. We have plotted the payoffs of the firms as a function of the substitution parameter b. The values of the parameter a and the marginal cost c are chosen as 3 and 1, respectively. The superscripts C and Q of u represent the classical and quantum cases, respectively. The subscripts A and B correspond to the firms A and B respectively. The numbers in the parentheses represent the corresponding Nash equilibrium points.



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