Aharonov-Anandan phases in Lipkin-Meskov-Glick model

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The Floquet theorem and decomposition of operator are generalized to calculate the non-abelian cyclic geometric phase. The general formula is achieved. Furthermore, the method is applied to calculate a concrete system named the Lipkin-Meskov-Glick model.

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I. INTRODUCTION

Geometric phase relating to quantum mechanics is one of the most interesting developments in the recent 25 years, which has been discovered by Berry [1] in the context of adiabatic, unitary, cyclic evolution of time-dependent quantum system. He demonstrated that besides the usual dynamical phase, an additional phase relating to geometry of the state space was generated. Soon Simon [2] give an geometrical interpretation of Berry's phase. Berry phase can be regarded as the holonomy of a line bundle L over the space of parameters M of the system, if L is endowed with a natural connection. Subsequently the degenerate case of Berry phase was generalized by Wilczek and Zee [3].

Discarding the assumption of adiabaticity, Aharonov and Anandan [4] generalized Berry's result. The dynamical phase was identified as the integral of the expectation value of the Hamiltonian. The Aharonov and Anandan phase (A-A phase) could be obtained by the difference between the total phase and the dynamical one and also be determined by the natural connection on a $U(1)$ principle fiber bundle over the space of projective Hilbert space. Soon after, Anandan [5] generalized the above one to the degenerate case.

Depending on the Pancharatnam's earlier work [6], Samuel and Bhandari [7] found a more general phase in the context of non-cyclic and non-unitary evolution of quantum mechanics. Furthermore, there are some reviewed papers [8, 9] and books [6, 10] about the theoretical developments, experiments and applications of geometric phase.

Recently, the study of geometric phase of a composite system of several spins has attracted a lot of attention. Sjövist [11] analyzed the non-cyclic and non-adiabatic two-particle geometric phase for a pair of entangled spins in a time-independent uniform magnetic field. Tong, Kwek and Oh [12] generalized the above case and calculated the geometric phase of the similar system in a rotating magnetic field. Yi, Wang and Zheng [13] investigated the adiabatic and cyclic geometric phase of two coupled spin-1/2 system, one of which is driven by a varying magnetic field. Xing [14, 15] studied further the adiabatic and cyclic geometric phase of two and three coupled spin-1/2 system with anisotropic interactions. Lately Sjövist, et. al. [16], analyzed the adiabatic geometric phase of ground state of finite-size Lipkin-Meskov-Glick type model (LMG) which consists of three spin-$\frac{1}{2}$ particles. In this paper, we calculate the non-adiabatic and cyclic geometric phase, namely Aharonov and Anandan phase, of this system.

The outline of the present paper is as follows. Section II. reviews the abelian and non-abelian Aharonov and Anandan phase. And a method of calculation for A-A phase is introduced in order to calculate the LMG model. Moreover, we generalize the above methods to the non-abelian case. In Sec. III., the cyclic state of the LMG model is solved. Furthermore, the Aharonov and Anandan phase for abelian and non-abelian case are calculated respectively. In Sec. IV., the abelian and non-abelian adiabatic phase, namely Berry phase, are analyzed respectively. And the connection between A-A phase and Berry phase is discussed according to quantum adiabatic theorem. At last, a conclusion was drawn.

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II. REVIEWS OF A-A PHASE AND ITS GENERALIZATION TO DEGENERACY CASE

Consider a general quantum system with time-dependent Hamiltonian $H(t)$, which is T- period, i.e., $H(T) = H(0)$. Given an initial state of the system $\psi(0)$, The evolution is determined by Schröinger equation, that is,

$$i\frac{d}{dt}|\psi(t)\rangle = H(t)|\psi(t)\rangle. \tag{1}$$

Choosing an initial state makes the evolution cyclic, i.e., $\psi(T) = e^{i\chi}\psi(0)$, where χ is the overall phase. Furthermore, the overall phase can be split into two parts, namely the dynamical phase and the geometric phase [4, 9, 10]. The dynamical phase has a natural definition, which is

$$\delta = -\int_0^T \langle\psi(t)|H(t)|\psi(t)\rangle dt. \tag{2}$$

Hence

$$\eta = \chi - \delta \tag{3}$$

is the geometric phase, which is determined by purely geometric property of evolution. In order to uncover the mystery of the geometric phase, we introduce a single-valued vector

$$|\tilde{\psi}(t)\rangle = e^{-i\theta(t)}|\psi(t)\rangle \tag{4}$$

such that $\theta(0) = 0$ and $\theta(T) = \chi$. Substituting Eq. (1) and Eq. (4) into Eq. (2), the dynamical can be rephrased into

$$\delta = \chi - i\int_0^T \langle\tilde{\psi}|\frac{d}{dt}|\tilde{\psi}\rangle dt.$$

By substitution the above Equation into Eq. (3), one gets

$$\eta = i\int_0^T \mathcal{A}dt, \tag{5}$$

where $\mathcal{A} = \langle\tilde{\psi}|\frac{d}{dt}|\tilde{\psi}\rangle$ The above formula is gauge invariant and also has a mathematical counterpart called holonomy in $U(1)$ principle fiber bundle whose base manifold is the projective Hilbert space.

In the previous paragraph, the abelian Aharonov and Anandan phase is elucidated, moreover, let's talk about the its generalization, which is call the non-abelian A-A phase [5, 10]. As is known, the state vectors live in Hilbert space H. Now, we focus on the subspace $V_n(t)$ of H, where $n(n > 1)$ is the notation of dimension of the subspace. Furthermore, $V_n(t)$ undergoes cyclic evolution, such that $V_n(T) = V_n(0)$. Let $\{|\tilde{\psi}_{n\alpha}(t)\rangle, \alpha = 1, \cdots, f_n\}$ be an orthonormal basis of $V_n(t)$ with $|\tilde{\psi}_\alpha(T)\rangle = |\tilde{\psi}_\alpha(0)\rangle$ for every α, where we write $|\tilde{\psi}_\alpha(t)\rangle$ instead of $|\tilde{\psi}_{n\alpha}(t)\rangle$ for short. By comparison with the abelian case, it is easy to deduced that $|\tilde{\psi}_\alpha(t)\rangle$ is non-abelian generalization of of the single valued vector $|\tilde{\psi}(t)\rangle$. With the initial state $|\psi_\alpha(0)\rangle = |\tilde{\psi}_\alpha(0)\rangle$, substituting

$$|\psi_\alpha(t)\rangle = \sum_{\beta=1}^{n} U_{\beta\alpha}(t)|\tilde{\psi}_\beta(t)\rangle$$

into the Schröinger Eq. (1), we get

$$U(t) = \mathcal{T}exp\{\int_0^T i[\mathcal{A}(t) - \mathcal{E}(t)]dt\}, \tag{6}$$

where $\mathcal{A}_{\alpha\beta} = \langle\tilde{\psi}_\alpha(t)|\frac{d}{dt}|\tilde{\psi}_\beta(t)\rangle$ and $\mathcal{E}_{\alpha\beta} = \langle\tilde{\psi}_\alpha(t)|H(t)|\tilde{\psi}_\beta(t)\rangle$. As the matrix $\mathcal{A}$ and $\mathcal{E}$ do not generally commute, the total $U(T)$ can't be written as the product of the dynamical phase,

$$U^{dynamical} = \mathcal{T}exp(-i\int_0^T \mathcal{E}(t)dt), \tag{7}$$

and the geometrical phase,

$$U^{geometric} = \mathcal{T}exp(i\int_0^T \mathcal{A}(t)dt). \tag{8}$$

However, the latter quantity can be transformed into the path-order integral,

$$U^{geometric} = \mathcal{P}exp(i\oint_{\mathcal{C}} \mathcal{A}(t)dt). \tag{9}$$

Moreover, it can be regarded as the holonomy of $U(N)$ principle fiber bundle with a natural connection, whose base space is the grassmann manifold.

From the above elucidation, to calculate the Aharonov and Anandan phase is to find the cyclic initial state and the single-valued vector. Now we concentrate on a periodic Hamiltonian $H(t)$ with period T. From the well-known Floquet theory, the time evolution operation $U(t)$ can be written as

$$U(t) = Z(t)e^{iMt}, \tag{10}$$

where $Z(t)$ is a unitary T-period operation with $Z(0) = 1$ and M is a Hermitian operator. Moore and Stedman [9, 17] had discussed application of the above result into the non-degenerate case. Specifically speaking, the connection one-form can be expressed as

$$\mathcal{A} = i\langle n|Z^{\dagger}(t)\frac{d}{dt}Z(t)|n\rangle, \tag{11}$$

where $|n\rangle$ is the nth eigenvector of M, which is also the cyclic initial state, hence $Z(t)|n\rangle$ is the single valued vector. Moreover, we point out that this theorem can also be used to calculate the degenerate case. For simplicity, suppose that M has mth eigenvalue, which is degenerate and spanned up a f_m subspace with orthonormal basis, and the the eigenvectors are expressed as $|m\alpha\rangle$. It is very easy to verify that the $\{|m\alpha\rangle, \alpha = 1, 2, \cdots, f_m\}$ are cyclic initial states and single-valued vector are $Z(t)|m\alpha\rangle, \alpha = 1, 2, \cdots, f_m\}$. Hence, we can calculate the connection matrix of degenerate case, which is

$$\mathcal{A}_{\alpha\beta} = i\langle m\alpha|Z^{\dagger}(t)\frac{d}{dt}Z(t)|m\beta\rangle. \tag{12}$$

Thus, substituting Eq. (11) and Eq. (12) into Eq. (5) and Eq. (8) respectively, both the abelian and non-abelian A-A phase can be obtained.

However, the time evolution operator is very hard to get according to Eq. (6).So generally we can't follow the procedure which is displayed in the previous paragraph to calculate the A-A phase.Nevertheless, don't be so discouraged and let's consider an important time-dependent Hamiltonian which has this form

$$H(t) = e^{-iAt}\tilde{H}e^{iAt}, \tag{13}$$

where A and $\tilde{H}$ are time independent. Substituting Eq. (13) into Eq. (1),then multiplying e^{iAt} at both side of the equation, one can get

$$ie^{iAt}\frac{d|\psi\rangle}{dt} = \tilde{H}e^{iAt}|\psi\rangle.$$

By use of the derivative formula $e^{iAt}d|\psi\rangle/dt = d(e^{iAt}|\psi\rangle)/dt - |\psi\rangle de^{iAt}/dt$, it is not very difficult to see that the time evolution operation can be written as

$$U(t) = e^{-iAt}e^{-iBt}, \tag{14}$$

where $B = \tilde{H} - A$ [9, 18, 19]. Again, suppose that the Hamiltonian is H is T-period. And it is a sufficient and necessary condition that e^{-iAT} commutes with B. Hence, they have a complete set of simultaneous eigenvectors, i.e.,

$$B\phi_n = B_n\phi_n, \tag{15}$$

$$e^{-iAT}\phi_n = e^{-i\theta_n}\phi_n, \tag{16}$$

where we suppose that B have non-degenerate eigenvalues. This case had already discussed by Moore [9, 19]. Its A-A phase [9, 19] is

$$\eta_n = \langle\phi_n|A|\phi_n\rangle T - \theta_n. \tag{17}$$

From above, we can see that the calculational methods for the abelian A-A phase was displayed. Moreover, there exists many non-abelian A-A phase in actual physical system. Nevertheless, few physicists consider about this problem, except Mostafazadeh [20]. He uses the methods of dynamical invariants to obtain the non-abelian geometric phase. But now let us fallow another line which is more direct and convenient and generalize it to the degenerate case. we still confer to the abelian case which is depicted above, so Eq. (15) and Eq. (16) can be expressed as

$$B\phi_{n\alpha} = b_n\phi_{n\alpha},$$

$$e^{-iAT}\phi_{n\alpha} = e^{-i\theta_n}\phi_{n\alpha},$$

where $\alpha = 1, 2, \cdots f_n$ and the degenerate space is a f_n-fold subspace. Subsequently, I want to transform Eq. (14) into the formula of Eq. (10). However, e^{-iAt} isn't generally T-periodic, so we introduce an operator Ω which is commute with B. Hence, they have the simultaneous eigenvectors, so we can get

$$\Omega\phi_{n\alpha} = \omega_n\phi_{n\alpha}. \tag{18}$$

Furthermore, let Ω still satisfy the following properties:

$$Z(t) = e^{-iAt}e^{i\Omega t/T},$$

$$M = -B - \Omega/T.$$

So the single-valued vector becomes $e^{-iAt}e^{i\Omega t/T}|\phi_{n\alpha}\rangle$. Thus we can get

$$\mathcal{A}_{\alpha\beta} = \langle\phi_{n\alpha}|A|\phi_{n\beta}\rangle - \frac{\omega_n}{T}\delta_{\alpha\beta}. \tag{19}$$

Substituting the above Eq. (19) into Eq. (8) or Eq. (9), we can get the corresponding A-A phase. Another quantity can also be achieved by the similar calculation, i.e.,

$$\mathcal{E}_{\alpha\beta} = \langle\phi_{n\alpha}|\tilde{H}|\phi_{n\beta}\rangle. \tag{20}$$

Hence, the dynamical phase can be obtained by substitution Eq. (20) into Eq. (7).

III. THE AHARONOV AND ANANDAN PHASE OF LMG

The previous section has introduced methods to calculate the abelian and non-abelian A-A phase. The methods will be use to calculate a concrete model called three qubits LMG [16] in this section. Its Hamiltonian reads

$$\tilde{H} = -\frac{1}{3}(S_x^2 + \gamma S_y^2) - hS_z,$$

where γ is an anisotropy parameter, h is the strength of an external magnetic field along the z direction, and $S_\alpha = \frac{1}{2}\sum_{k=1}^{N}\sigma_\alpha^k$ is the αth component of the spin operator(for simplicity, we set $\hbar = 1$ from now on) with σ_x^k, σ_y^k and σ_z^k are Pauli operators of the $k$the qubit in the representation of σ_z [16]. Ignoring the trivial and constant term $-\frac{1}{4}(1+\gamma)$ of $\tilde{H}$, the Hamiltonian becomes

$$\tilde{H} = -\frac{1}{6}[\sigma_x^1\sigma_x^2 + \sigma_x^2\sigma_x^3 + \sigma_x^1\sigma_x^3 + \gamma(\sigma_y^1\sigma_y^2 + \sigma_y^2\sigma_y^3 + \sigma_y^1\sigma_y^3)] - \frac{h}{2}(\sigma_z^1 + \sigma_z^2 + \sigma_z^3).$$

Moreover, let's consider about the isospectral one-parameter Hamiltonian family

$$H = e^{-i\phi S_z}\tilde{H}e^{i\phi S_z}, \tag{21}$$

where $\phi = \omega t$ is a varying parameter and ω is the angular velocity. By a glance at Eq. (13), a conclusion can be drawn that the two Hamiltonian have a similar form. Hence, we can take the similar procedure to calculate the corresponding A-A phase. The time evolution operator becomes

$$U(t) = e^{-iAt} e^{-iBt}, \tag{22}$$

where $A = \omega S_z$ and $B = \tilde{H} - \omega S_z$. Next, we intend to calculate the cyclic initial state which is the eigenvectors of B in this case. Thus, at first, the operator B must be represented in a concrete basis, which are $\{|000\rangle, |011\rangle, |101\rangle, |110\rangle, |111\rangle, |100\rangle, |010\rangle, |001\rangle\}$, where $|0\rangle$ represents spin up and $|1\rangle$ represents spin down. So B takes the block-diagonal form

$$B(\gamma, h, \omega) = \begin{pmatrix} P(\gamma, h, \omega) & 0 \\ 0 & P(\gamma, -h, -\omega) \end{pmatrix},$$

where

$$P(\gamma, h, \omega) = \begin{pmatrix} -\frac{3}{2}(h+\omega) & -\frac{1}{6}(1-\gamma) & -\frac{1}{6}(1-\gamma) & -\frac{1}{6}(1-\gamma) \\ -\frac{1}{6}(1-\gamma) & \frac{1}{2}(h+\omega) & -\frac{1}{6}(1+\gamma) & -\frac{1}{6}(1+\gamma) \\ -\frac{1}{6}(1-\gamma) & -\frac{1}{6}(1+\gamma) & \frac{1}{2}(h+\omega) & -\frac{1}{6}(1+\gamma) \\ -\frac{1}{6}(1-\gamma) & -\frac{1}{6}(1+\gamma) & -\frac{1}{6}(1+\gamma) & \frac{1}{2}(h+\omega) \end{pmatrix},$$

and 0 is the 4×4 null matrix. Because $B(\gamma, h, \omega) = P(\gamma, h, \omega) \oplus P(\gamma, -h, -\omega)$, we can investigate $P(\gamma, h, \omega)$ and $P(\gamma, -h, -\omega)$ respectively. However, the similar information from $P(\gamma, -h, -\omega)$ can be obtain from $P(\gamma, h, \omega)$. So we can focus on the subspace of solutions which the sub-matrix $P(\gamma, h, \omega)$ acts to simplify the problem. Thus the two eigenvalues of $P(\gamma, h, \omega)$ are

$$p_1 = -\frac{1}{2}(\omega + h) - \frac{1}{6}(1+\gamma) - \frac{1}{3}\sqrt{r},$$

$$p_2 = -\frac{1}{2}(\omega + h) - \frac{1}{6}(1+\gamma) + \frac{1}{3}\sqrt{r}$$

and

$$p_3 = p_4 = \frac{1}{2}(h+\omega) + \frac{1}{6}(1+\gamma),$$

where

$$r = 9h^2 + 9\omega^2 + \gamma^2 + 18h\omega - 3h\gamma - 3\gamma\omega - 3\omega - 3h - \gamma + 1.$$

From above calculation, we can draw a conclusion that p_1 and p_2 correspond to the non-degenerate case whereas $p_3 = p_4$ corresponds to the degenerate case. However, its A-A phase factor is proved to be trivial. So we only focus on the non-degenerate case. And the corresponding two orthogonal eigenvectors are

$$|\phi_1\rangle = \frac{1}{\sqrt{n_1}} \begin{pmatrix} \gamma + 1 - 6(\omega+h) - 2\sqrt{r} & \gamma - 1 & \gamma - 1 & \gamma - 1 \end{pmatrix}^T \tag{23}$$

and

$$|\phi_2\rangle = \frac{1}{\sqrt{n_2}} \begin{pmatrix} \gamma + 1 - 6(\omega+h) + 2\sqrt{r} & \gamma - 1 & \gamma - 1 & \gamma - 1 \end{pmatrix}^T, \tag{24}$$

where

$$n_1 = 3(\gamma - 1)^2 + [-6(\omega + h) + \gamma + 1 - 2\sqrt{r}]^2,$$

$$n_2 = 3(\gamma - 1)^2 + [-6(\omega + h) + \gamma + 1 + 2\sqrt{r}]^2,$$

and T denotes the transpose operation of matrix.

We have already calculated the cyclic initial state of the system. Moreover, from Eq. (21), it is very easy to verify that the system has $2\pi/\omega$-periodic Hamiltonian. In the following paragraphs, we will calculated the corresponding A-A phase of the non-degenerate case and degenerate case.

Substituting $T = 2\pi/\omega$ into $e^{-i\omega T S_z}$, by use of Eq. (16), we can choose that

$$\theta_n = \pi. \tag{25}$$

Subsequently, we represent A in the given basis, which is expressed as

$$A = \frac{1}{2}\omega \begin{pmatrix} 3 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}. \tag{26}$$

Substituting the above Eq. (25), Eq. (26) and Eq. (23) into Eq. (17), one can get the A-A phase corresponding to the first cyclic initial state, which reads

$$\eta_1 = \frac{3\pi}{n_1}\{[1+\gamma-6(\omega+h)-2\sqrt{r}]^2-(1-\gamma)^2\}-\pi. \tag{27}$$

By the similar procedure, the second geometric phase is

$$\eta_2 = \frac{3\pi}{n_2}\{[1+\gamma-6(\omega+h)+2\sqrt{r}]^2-(1-\gamma)^2\}-\pi. \tag{28}$$

The above calculation is all about A-A phase of the non-degenerate case. Next, A-A phase of the degenerate case will be discussed. Nevertheless, the non-abelian one is trivial one, so it is necessary to reconsider another system whose Hamiltonian is

$$H = e^{-i\phi S_x}\tilde{H}e^{i\phi S_x}.$$

According to the operator decomposition, the time evolution operator is

$$U(t) = e^{-iAt}e^{-iBt},$$

where $A = \omega S_x$ and $B = \tilde{H} - \omega S_x$. Moreover, B can be represented by the given basis as

$$B = \begin{pmatrix} -\frac{3}{2}h & \frac{1}{6}(\gamma-1) & \frac{1}{6}(\gamma-1) & \frac{1}{6}(\gamma-1) & 0 & -\frac{\omega}{2} & -\frac{\omega}{2} & -\frac{\omega}{2} \\ \frac{1}{6}(\gamma-1) & \frac{h}{2} & -\frac{1}{6}(\gamma+1) & -\frac{1}{6}(\gamma+1) & -\frac{\omega}{2} & 0 & -\frac{\omega}{2} & -\frac{\omega}{2} \\ \frac{1}{6}(\gamma-1) & -\frac{1}{6}(\gamma+1) & \frac{h}{2} & -\frac{1}{6}(\gamma+1) & -\frac{\omega}{2} & -\frac{\omega}{2} & 0 & -\frac{\omega}{2} \\ \frac{1}{6}(\gamma-1) & -\frac{1}{6}(\gamma+1) & -\frac{1}{6}(\gamma+1) & \frac{h}{2} & -\frac{\omega}{2} & -\frac{\omega}{2} & -\frac{\omega}{2} & 0 \\ 0 & -\frac{\omega}{2} & -\frac{\omega}{2} & -\frac{\omega}{2} & \frac{3}{2}h & \frac{1}{6}(\gamma-1) & \frac{1}{6}(\gamma-1) & \frac{1}{6}(\gamma-1) \\ -\frac{\omega}{2} & 0 & -\frac{\omega}{2} & -\frac{\omega}{2} & \frac{1}{6}(\gamma-1) & -\frac{h}{2} & -\frac{1}{6}(\gamma+1) & -\frac{1}{6}(\gamma+1) \\ -\frac{\omega}{2} & -\frac{\omega}{2} & 0 & -\frac{\omega}{2} & \frac{1}{6}(\gamma-1) & -\frac{1}{6}(\gamma+1) & -\frac{h}{2} & -\frac{1}{6}(\gamma+1) \\ -\frac{\omega}{2} & -\frac{\omega}{2} & -\frac{\omega}{2} & 0 & \frac{1}{6}(\gamma-1) & -\frac{1}{6}(\gamma+1) & -\frac{1}{6}(\gamma+1) & -\frac{h}{2} \end{pmatrix}.$$

Now we only focus on its two degenerate eigenvalues which are

$$B_1 = \frac{1}{6}(1+\gamma-3\sqrt{h^2+\omega^2}),$$

and

$$B_2 = \frac{1}{6}(1+\gamma+3\sqrt{h^2+\omega^2}).$$

And the corresponding eigenvectors are

$$|\phi_{11}\rangle = \frac{1}{\sqrt{N_{11}}}\begin{pmatrix} 0 & \frac{1}{\omega}(\sqrt{h^2+\omega^2}-h) & 0 & \frac{1}{\omega}(h-\sqrt{h^2+\omega^2}) & 0 & -1 & 0 & 1 \end{pmatrix}^T, \tag{29}$$

$$|\phi_{12}\rangle = \frac{1}{\sqrt{N_{12}}}\begin{pmatrix} 0 & \frac{1}{2\omega}(\sqrt{h^2+\omega^2}-h) & \frac{1}{\omega}(h-\sqrt{h^2+\omega^2}) & \frac{1}{2\omega}(\sqrt{h^2+\omega^2}-h) & 0 & -\frac{1}{2} & 1 & -\frac{1}{2} \end{pmatrix}^T, \tag{30}$$

and

$$|\phi_{21}\rangle = \frac{1}{\sqrt{N_{21}}} \left(0 \;\; -\tfrac{1}{\omega}(h+\sqrt{h^2+\omega^2}) \;\; 0 \;\; \tfrac{1}{\omega}(h+\sqrt{h^2+\omega^2}) \;\; 0 \;\; -1 \;\; 0 \;\; 1 \right)^T,$$

$$|\phi_{22}\rangle = \frac{1}{\sqrt{N_{22}}} \left(0 \;\; -\tfrac{1}{2\omega}(h+\sqrt{h^2+\omega^2}) \;\; \tfrac{1}{\omega}(h+\sqrt{h^2+\omega^2}) \;\; -\tfrac{1}{2\omega}(h+\sqrt{h^2+\omega^2}) \;\; 0 \;\; -\tfrac{1}{2} \;\; 1 \;\; -\tfrac{1}{2} \right)^T,$$

where $N_{11} = \frac{4}{1+h/\sqrt{h^2+\omega^2}}$, $N_{12} = \frac{3}{1+h/\sqrt{h^2+\omega^2}}$, $N_{21} = 4 + \frac{4h(h+\sqrt{h^2+\omega^2})}{\omega^2}$, and $N_{22} = 3 + \frac{3h(h+\sqrt{h^2+\omega^2})}{\omega^2}$.

It is very easy to verify that $e^{-i\frac{1}{2}\sigma_x \omega t}$ is not a $2\pi/\omega$ period unitary operator, but $e^{-i\frac{1}{2}\sigma_x \omega t} e^{i\Omega t/T}$ is so, where

$$\Omega = \pi \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}. \tag{31}$$

In order to calculate the non-abelian connection matrix according to Eq. (19), we must represent A in a matrix form, that is

$$A = \frac{1}{2}\omega \begin{pmatrix} 0&0&0&0&0&1&1&1 \\ 0&0&0&0&1&0&1&1 \\ 0&0&0&0&1&1&0&1 \\ 0&0&0&0&1&1&1&0 \\ 0&1&1&1&0&0&0&0 \\ 1&0&1&1&0&0&0&0 \\ 1&1&0&1&0&0&0&0 \\ 1&1&1&0&0&0&0&0 \end{pmatrix}. \tag{32}$$

Substituting Eq. (29), Eq. (30), Eq. (32) and Eq. (31) into Eq. (19), the connection matrix becomes

$$\mathcal{A} = \frac{\omega}{2}\left(\frac{\omega}{\sqrt{h^2+\omega^2}} - 1\right) \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}. \tag{33}$$

Further, by substitution Eq. (33) into Eq. (8), one can get the non-abelian A-A phase factor

$$U^{Geometric} = \exp[i\pi(\frac{\omega}{\sqrt{h^2+\omega^2}} - 1)] \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

By the similar procedure, the second non-abelian A-A phase factor reads

$$U^{Geometric} = \exp[-i\pi(\frac{\omega}{\sqrt{h^2+\omega^2}} + 1)] \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

The A-A phase factor have obtained already. Furthermore, we can still get the dynamical phase factor according to Eq. 20 and Eq. 7.

IV. DISCUSSION AND CONCLUSION

In previous section, we have calculated the abelian and non-abelian A-A phase respectively. For integrity, it is necessary for us to concentrate on the Adiabatic phase, namely Berry phase, though Sjövist et. al. [16] had already calculated the Berry phase of the first model. But their paper probably has a minor bug of the choice of the instantaneous eigenvectors, which I will explain in the following paragraphs. Moreover their article only focus on the energetic ground state of LMG. So it is necessary for us to calculated the Berry phase for every eigenstates. But the non-abelian Berry phase factor is still trivial, so we only concentrate on the other abelian Berry phase. The Hamiltonian (21) we focus on have a unified form expressed by Eq. (13). According Berry [1], Wilczek and Zee [3], in order to determine the Berry phase, we must calculate the instantaneous eigenstate first. The eigen equation reads

$$e^{-iAt}\tilde{H}e^{iAt}|n\rangle = \lambda_n|n\rangle. \tag{34}$$

It is difficult to solve the above Equation, the equation can be transformed

$$\tilde{H}|n\rangle' = \lambda_n|n\rangle', \tag{35}$$

where $|n\rangle' = e^{iAt}|n\rangle$. For the LMG model, the above expression can be transformed to be $|n\rangle = e^{-i\frac{1}{2}\sigma_z \omega t}|n\rangle'$. In order to calculate Berry phase, we must choose the instantaneous eigenvectors to satisfy $|n(T)\rangle = |n(0)\rangle$. However, the condition can't be satisfied when the period $T = 2\pi/\omega$. After a minor modification, the eigenvector can be chosen to be

$$|n\rangle = e^{-i\frac{1}{2}\sigma_z \omega} e^{i\pi t/T}|n\rangle', \tag{36}$$

which is still satisfy Eq. (34). Next, we can calculate the non-degenerate connection one-form according to Berry [1], which is expressed as

$$\mathcal{A} = i\langle n|\frac{d}{dt}|n\rangle. \tag{37}$$

Substituting Eq. (36) into Eq. (37), one can get

$$\mathcal{A} = \frac{1}{2}\omega\langle n|'\sigma_z|n\rangle' - \frac{\pi}{T}.$$

Thus the Berry phase becomes

$$\eta_n = \int_0^T \mathcal{A}dt. \tag{38}$$

According to Wilczek and Zee [3], by similar calculations, the non-abelian Berry phase factor can be expressed as

$$U^{Geometric} = \mathcal{T}exp(i\int_0^T \mathcal{A}(t)dt), \tag{39}$$

where $\mathcal{A}_{\alpha\beta} = i\langle n\alpha|\frac{d}{dt}|n\beta\rangle = \frac{1}{2}\omega\langle n\alpha|'\sigma_z|m\beta\rangle' - \frac{\pi}{T}\delta_{\alpha\beta}$. At first let us represent $\tilde{H}$ in the given basis, which is

$$\tilde{H} = \begin{pmatrix} -\frac{3}{2}h & \frac{1}{6}(\gamma-1) & \frac{1}{6}(\gamma-1) & \frac{1}{6}(\gamma-1) \\ \frac{1}{6}(\gamma-1) & \frac{1}{2}h & -\frac{1}{6}(\gamma+1) & -\frac{1}{6}(\gamma+1) \\ \frac{1}{6}(\gamma-1) & -\frac{1}{6}(\gamma+1) & \frac{1}{2}h & -\frac{1}{6}(\gamma+1) \\ \frac{1}{6}(\gamma-1) & -\frac{1}{6}(\gamma+1) & -\frac{1}{6}(\gamma+1) & \frac{1}{2}h \end{pmatrix}.$$

It is not very complicated to get the eigenvalues

$$\lambda_1 = -\frac{1}{6}(1+\gamma+3h+2\sqrt{q}),$$

$$\lambda_2 = -\frac{1}{6}(1+\gamma+3h-2\sqrt{q}),$$

and the corresponding eigenvectors are

$$|1\rangle = \frac{1}{\sqrt{N_1}}\begin{pmatrix} 1+\gamma-6h-2\sqrt{q} & \gamma-1 & \gamma-1 & \gamma-1 \end{pmatrix}^T, \tag{40}$$

$$|2\rangle = \frac{1}{\sqrt{N_2}}\begin{pmatrix} 1+\gamma-6h+2\sqrt{q} & \gamma-1 & \gamma-1 & \gamma-1 \end{pmatrix}^T, \tag{41}$$

where $q = 9h^2+\gamma^2-3h\gamma-3h-\gamma+1$, $N_1 = 3(\gamma-1)^2+(1+\gamma-6h-2\sqrt{q})$ and $N_2 = 3(\gamma-1)^2+(1+\gamma-6h+2\sqrt{q})$. Substituting Eq. (40) into Eq. (38), we can get

$$\eta_1 = \frac{3\pi}{N_1}[(1+\gamma-6h-2\sqrt{q})^2-(\gamma-1)^2]-\pi. \tag{42}$$

Similarly, we can also get

$$\eta_2 = \frac{3\pi}{N_2}[(1+\gamma-6h+2\sqrt{q})^2-(\gamma-1)^2]-\pi. \tag{43}$$

Now, let's analyze the connection between A-A phase and Berry phase further. Above all, the condition of the adiabatic theorem reads

$$\left| \frac{\langle m|\frac{d}{dt}H(t)|n\rangle}{E_n - E_m} \right| \ll 1. \tag{44}$$

Substituting Eq. (34) into above Eq. (44), one can simplify the above condition to

$$\omega \left|\langle m|\sigma_z|n\rangle\right| \ll 1.$$

By a simple calculation, one can know $|\langle m|\sigma_z|n\rangle| \sim 1$, so the condition becomes

$$\omega \ll 1. \tag{45}$$

Substituting the above Eq. (45) into Eq. (27) and Eq. (28), A-A phase can be reduced to Berry phase which are Eq. (42) and Eq. (43). Moreover, for the non-abelian A-A phase of the second model, we will get the similar conclusion.

To sum up, we have generalized the Floquet theorem and decomposition of operator to calculate the non-abelian cyclic geometric phase. The general formula is achieved. Furthermore, the methods is applied to calculate a concrete system named LMG.

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