

An Economic analogy to Electrodynamics

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Abstract. - In this note, we would like to find the laws of electrodynamics in simple economic systems. In this direction, we identify the chief economic variables and parameters, scalar and vector, which are amenable to be put directly into the crouch of the laws of electrodynamics, namely Maxwell's equations. Moreover, we obtain Phillip's curve, recession and Black-Scholes formula, as sample applications.

Introduction. – Physicists have tried to understand the complexity of economics from time immemorial, starting from Copernicus, through Isaac Newton to Eugene Stanley [1]. There have been continuous efforts in recent times to understand statistical mechanics [2] and thermodynamics [3] of economics.

The question keeps coming, can we understand economics as simply as mechanics [4]? Can we comprehend force laws behind economic developments as simply as four force laws in physics? Though there are few interesting attempts [5–7], direct attacks to answer the questions probably are missing.

In this letter, we will refer to the easily available books on electrodynamics [8] and economics [9] while trying to separate, step by step, one kind of force law in action in economics. We do this in the following way. First we describe the Maxwell's equations of electrodynamics as well as continuity equation and Lorentz force law. Then we introduce the chief economic variables and formulate the correspondence of the economic variables to the standard electrodynamic variables and parameters. After that we verify how equations of electrodynamics are holding good in economic systems. We also consider analogue of materials in economics. Potential formulation of electrodynamics is a powerful solution technique. We will see how that too descends down to us in economics.

Unemployment, inflation of prices are day to day headache. Recession was the first word of the song for the day to start with until a year back. What is less heard that there is an empirical graphical relation between

inflation rate and unemployment rate, in the short run. The name of the line is Phillip's curve. We derive sort of Phillip's curve using the rules, describe the recession also. Moreover, option trading (one type of booking share) is something that makes the share market efficient. Pricing of the option has been a long standing academic issue. F. Black and M. Scholes were the first to, using intuition from Physics, namely diffusion equation of heat, give a reasonable formula [10] for the call (and hence put) option. In this letter we re-derive the Black-Scholes formula, visualising call option as one component of profit flow rather than temperature, as a particular case of more general class of feasible formulas. Unobservable factor market volatility, too gets split up.

We will take India and Indian currency, Rupee, as a background for our purpose of the paper. But the full content will be holding true, if India and Indian currency are replaced globally, in the letter, by any country and the corresponding currency.

Maxwell's equations. – We recall that the basic variables of electrodynamics are electric field, \vec{E} and magnetic field, \vec{B} . These two fields can exist without, can generate in a medium or, can be produced by electric charge density, ρ and electric current density, \vec{j} . The relations, whenever relevant, between electromagnetic fields and charge (current) in a vacuum (material medium) are fixed by permittivity constant, ϵ_0 , and permeability constant, μ_0 . These four variables have an interesting inter-relationship. Moreover, the charge density and current constrain each other through a constitutive relation. Let us describe along that line in the paragraph to follow

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The four equations of electrodynamics as completed by Maxwell are as [8]

$$\epsilon_0 \nabla \cdot \vec{E} = \rho \quad (1)$$

$$\nabla \times \vec{E} = -\frac{\partial}{\partial t} \vec{B} \quad (2)$$

$$\nabla \cdot \vec{B} = 0 \quad (3)$$

$$\nabla \times \vec{B} = \mu_0 \epsilon_0 \frac{\partial}{\partial t} \vec{E} + \mu_0 \vec{j} \quad (4)$$

The constitutive relation, called continuity equation, is

$$\nabla \cdot \vec{j} + \frac{\partial}{\partial t} \rho = 0 \quad (5)$$

The force acting on a charge distribution is given by Lorentz Force Law

$$\vec{F} = \rho(\vec{E} + \vec{v} \times \vec{B}) \quad (6)$$

Analogous economic variables. – We denote the main economic variables as follows:

- competition flow as \vec{c}
- profit flow as \vec{P}
- money flow as \vec{M}
- money density as n
- Ambition of a person as \vec{Am}
- Price index as Pi
- Choice flow as \vec{Ch}
- Economic power flow as \vec{E}_p
- Economic activity as E_a
- inverse of basic strength-scale of currency, at least for macro economy, as s_0
- basic technical knowhow+political power, at least for macro economy, as k_0
- human infrastructure as h

Correspondence. –

- $\vec{E} \longleftrightarrow \vec{c}$
- $\vec{B} \longleftrightarrow \vec{P}$
- $\vec{j} \longleftrightarrow -\vec{M}$
- $\rho \longleftrightarrow -n$

parameters.

- $\epsilon_0 \longleftrightarrow s_0$
- $\mu_0 \longleftrightarrow k_0$
- $\epsilon_0 \epsilon_r \longleftrightarrow s$
- $\mu_0 \mu_r \longleftrightarrow k$
- $\sigma \longleftrightarrow h$

functions.

- $\vec{v} \longleftrightarrow \vec{Am}$
- Scalar potential, $V \longleftrightarrow -Pi$
- Vector potential, $\vec{A} \longleftrightarrow -\vec{Ch}$
- Poynting vector, $\vec{S} = \frac{1}{\mu_0} \vec{E} \times \vec{B} \longleftrightarrow \vec{E}_p$
- energy density $\longleftrightarrow E_a$
- σ_{cross} multiplied by power, $P \longleftrightarrow \langle \text{employment} \rangle$, employment generation rate

Analogy brought inside out. –

Maxwell's equations.

- Excess liquidity stimulates economic activity i.e. generates competition. Faraway from mints, activity drops to zero, competition fizzles out.

To understand it better, let us consider the following simple situation, one has left a one rupee note on the road separating two parts of a market, it will lead to a competition among the onlookers to pick it up. Imagine, instead one lakh rupee note kept on the road. It will lead to fiercer competition among the onlookers. Not only that competition which is under way along the road or, along either part of the market, will get a component across the road. Hence money density in a place generates divergence in competition flow and proportional. This is proportional at least to the first approximation. Moreover, competition points towards the money.

Let us think the exactly same situation happening twenty five years back. Then, one rupee note would have given the same divergence in the competition flow as ten thousand rupees give today. Within past twenty five years, rupee has gotten devalued by huge amount. Hence, the proportionality factor s_0 stands for the inverse of strength-scale of the currency.

This sequence of arguments follow even if we consider not this kind of free notes but constrained notes. We mean, the same kind of situation will arise with the salary of an advertised job also. We will be concerned in this paper with competition associated with the constrained notes.

Hence we deduce the first law analogous to the eq. (1)

$$s_0 \nabla \cdot \vec{c} = -n \quad (7)$$

In this sense, money density is analogue of negative charge density. Scarcity is analogue of positive charge. Scarcity density is more like hole density than free positive charge density. Note and scarcity, in equal magnitude form dipole. An arbitrary distribution of note (scarcity) over space can be cast into the form of multipole expansion.

In an organisation, when money is not flowing or, notes are stationary there is no competition. This is like $\vec{E} = 0$ in a conductor.

- In general profit is a composite object composed of money, labor etc. In the simplest cases profit is quantified as money gain. In any exchange, positive profit of one is equal to, in magnitude, the negative profit of the other. Hence, in any exchange, net change in profit is zero. If there is no exchange, there is no change in profit, either way. Hence, we have

$$\nabla \cdot \vec{P} = 0. \quad (8)$$

- Profit flow coming from retail chain sector leads local businessmen to get united and protest. Protest is a form of competition flow. We may note that this is what experienced in pure diamagnetic phenomenon or, when a bar magnet is pushed orthogonally towards a wire loop. Initial reactions to software coming to India were also similar. This motivates us to write

$$\nabla \times \vec{c} = -\frac{\partial}{\partial t} \vec{P}. \quad (9)$$

This also indicates that Faraday's law boils down to Ricardo's principle in economics.

- Like magnetic field profit is also non-conservative field. If there is no money, there is no profit. Circulation of notes gives rise to profit. As money starts incoming more and more to a place, profit also increases, say in a place, to some people more and more. As money comes more, differences in money contents from person to person, say, increase more. Rich becomes richer, poor becomes poorer.

Let us think of the opposite limit, where there is no money flow into a place. But if competition flow, say promotional competition in a company, changes with time, like in some months of the year, this leads to more spending, hence more profit circulation in the local economy or, micro-economy. Product differentiation too leads to circulation of profit in a local economy. These considerations lead us to the relation

$$\nabla \times \vec{P} = s_0 k_0 \frac{\partial}{\partial t} \vec{c} - k_0 \vec{M} \quad (10)$$

continuity equation. We know that no one creates (destroys) money, unless one is crazy. The amount of money that enters (goes out) from one's pocket, or, from one ATM, or, from one bank, in unit time is just equal to the rate of change of money in that pocket or, ATM or, the bank. This is just the continuity eq. (5).

But there is an exception. Notes are destroyed or, generated at the mint(s), leading to appreciation or, depreciation w.r.t. a standard currency.

So the relation(5) gets modified, in case of economics, to

$$\nabla \cdot \vec{M} + \frac{\partial}{\partial t} n = \frac{\partial}{\partial t} n_p \quad (11)$$

where, n_p is the amount of money being printed or, destroyed in a mint.

Lorentz Force Law. Let us imagine, competition has started flowing in a place, buy a house or, buy sports goods or, buy a ticket for a show. A person will respond or, not and if responds to what extent, depends on how much money is there in his pocket. Whether a locality around an ATM will respond or, not or, to what extent will depend on how much notes are there at the ATM. Response varies directly also with the appeal or, magnitude of the competition flow. So the force along the competition flow on a person or, a local society around an ATM is proportional to the competition flow, to the first approximation and the proportionality factor is money density. The same thing occurs for a nation about a Federal bank, in response to an oncoming competition flow. Here, we are meaning by competition flow, social competition flow. Let us consider an opposite situation. Reality sector boom is coming onto a place, along the "third dimension". A person will respond provided he has business ambition. The response will be proportional to the money he owes. Once he responds this will give sidewise pushes to the people around him, who might be harbouring academic ambition only, on-setting competition along the direction perpendicular to the person's ambition direction and the profit flow direction.

Hence we heuristically come down to an equation of economic force, which is exactly the same form as Lorentz force law

$$\vec{F} = -n(\vec{c} + \vec{Am} \times \vec{P}) \quad (12)$$

Here, we observe that only competition flows cannot give a man having scarcity, equilibrium but profit flows can. This is like Earnshaw's theorem. Second part of the statement is like magnetic confinement of charge.

Here, we also notice that two twins having the same money, same ambition and subjected to the same competition and profit flows, will feel the same force. But depending on their accumulated entrepreneurial skills their venture accelerations will be different. For example, one will set-up a cyber cafe much earlier than other, if the first one has software and little bit management training whereas the second one does not have that skill set. Hence

economic inertial mass of a person is reciprocal of the number of his entrepreneurial skills. We denote from hereon,

- economic inertial mass= M_e
- Number of skills= N_{es}

The same story will follow for two twin companies or, two twin countries. Hence we have the following identification

- $M_e = \frac{1}{N_{es}}$

Material. – Let us think that competition flow is oncoming to a place. This will create money accumulation among some and scarcity among others, giving rise to something like polarisation, bound money density at the surface of the society and at the volume. As a consequence, net competition flow will be different from the external competition flow. For weakly responsive society, polarisation vector will be equal to $s_0 R_c \vec{c}$. R_c is the measure of the response of the society. \vec{c} refers to the net competition flow in the society. The equation (7) will get modified to

$$\nabla \cdot s \vec{c} = -n. \quad (13)$$

n refers to external money density. $s = s_r s_0 = s_0(1 + R_c)$. Similarly, profit flow leads to bound surface and volume circulation of notes. This results in the net profit flow differing from the external profit flow vector. This leads to a relation modified from the equation (8)

$$\nabla \cdot k \vec{P} = 0 \quad (14)$$

where, $k = k_0 k_r = k_0(1 + R_p)$.

Probably, s, k span a two dimensional plane. Presumably, existence of black market is an example of s, k being both negative [11].

Profit and competition flows both polarize.

conductivity. Sometimes economy is conducive. Competition vector is proportional to money flow vector or, liquidity just like in conductor,

$$\vec{j} = \sigma \vec{E} \quad (15)$$

Proportionality factor, h , in economic system, like conductivity, is a measure of the quality of the human infrastructure of the company. So we have here the following rule

$$\vec{M} = -h \vec{c} \quad (16)$$

In highly efficient ($h \rightarrow \infty$) organisation, internal competition is zero always, which is like in metal ($\sigma \rightarrow \infty$). h can stand for *HumanCapital*.

Potential Formulation. – To show the form of the scalar potential, let us notice the following,

$$\vec{c} = -\nabla(-Pi) \quad (17)$$

implies

$$\nabla^2 Pi = -n \frac{1}{s_0} \quad (18)$$

As money density increases, Price-index also increases, we see inflation.

Price index over space and time is determined by two considerations

- Prices and consumption ratios of various items at a place at a given time.
- Prices and consumption ratios of items at another time and/or at another place, compared to the base prices and consumption ratios.

The prices and consumption ratios of items change continuously over the space and time.

Hence, Price index, Pi , change continuously over space and time. So, Price index, Pi , is analogous to scalar potential, V . The first consideration sets a fixed reference value to the price-index for all other places at that time as well as for all other times. A relevant fact worth mentioning in this context is that gas index in U.S. is based on the price of gas at a point where majority of the gas pipelines intersect.

To show the form of the vector potential, let us notice the following,

$$\nabla^2 \vec{Ch} = -k_0 \vec{M} \quad (19)$$

wherever, choice flow is divergence less. This continues to be as long as there is no will.

Hence, \vec{Ch} is in the same direction as \vec{M} , which is our experience.

Moreover, (Pi, \vec{Ch}) can be combined into a four vector. Ambition, \vec{Am} , multiplied by Price index can be choice. Maximum Ambition is determined by the velocity of light and in fact, may be taken as velocity of light. We would like to move in any direction with the magnitude of velocity of light, c , given chance. Therefore it's quite plausible to write

$$\vec{Ch} = \frac{\vec{Ch} - \vec{Am} Pi}{\sqrt{1 - \frac{Am^2}{c^2}}} \quad (20)$$

Application. –

Phillip's curve. We know, in economics, Inflation rate, Π , is defined as

$$\Pi = \frac{d}{dt} \ln Pi. \quad (21)$$

Since,

$$V \leftrightarrow Pi, \quad (22)$$

$$\frac{d}{dt} \ln V \leftrightarrow \Pi \quad (23)$$

or, time derivative of logarithm of scalar potential is expected to show features of economic inflation. To proceed along that line, we note from the theory of radiation in electrodynamics,

$$\frac{d}{dt} \ln V = \omega, \quad (24)$$

for electric dipole radiation, whereas, the total power radiated by the dipole is given by

$$\langle P \rangle = \text{constant } \omega^4 \quad (25)$$

Hence,

$$\frac{d}{dt} \ln V \sim \langle P \rangle^{\frac{1}{4}} \quad (26)$$

Here we recall that when an electromagnetic radiation falls on a medium, three processes occur. For low energy, photoelectric effect is the dominant process. As the energy increases of the infalling radiation, Compton scattering starts becoming important. At still higher energy, pair production takes over. For the photoelectric effect, cross-section, σ_{cross} , or, probability for the process to occur

$$\sigma_{cross} \sim \frac{1}{\omega^{\frac{7}{2}}} \quad (27)$$

Photoelectric effect is producing free electrons at the cost of work-function. This phenomenon is exactly similar to employment generation from the pool of unemployed youth at the cost of lump sum money. In India, this is like giving one-time small money/loan to buy say an auto/a cab to an unemployed young man and making him self-employed. Compton scattering is pumping money in risky assets. Pair production is like bringing an woman to work place at the cost of a vacancy at the household cores.

Again we know, product of employment generation rate and unemployment generation rate is constant, because the two processes occur in mutually exclusive sectors, influencing each other in extreme cases, viz. percolation of software jobs to mechanical and clerical sectors.

As a result we come down to the following conclusion for the low scale economic activity inflow,

$$\Pi \sim \frac{1}{\langle \text{unemployment} \rangle^2}. \quad (28)$$

This is nothing but Phillip's curve, qualitatively.

On the other hand, in the domain where Compton scattering becomes important [12]

$$\sigma_{cross} \sim \frac{1}{\omega} \ln \omega. \quad (29)$$

Then

$$\Pi \sim \frac{1}{\langle \text{unemployment} \rangle^{\frac{1}{3}}}. \quad (30)$$

apart from the slowly varying scale-dependent logarithmic part.

Hence, in the scale of economic activity inflow, $|\vec{E}_p|$ where, Compton scattering-type of phenomenon becomes important compared to photoelectric type, we get sudden increase of inflation with unemployment. This is stagflation. This is stagflation with scale-dependence setting in.

If one is interested in total absorption cross-section, one can look in [13] as well as in [14] and surmise about the details of the ensuing Inflation vs unemployment curve.

Recession. A Recessing phase corresponds to one inertial frame for a macro-economy. The recessing inertial frame has lower ambition, $|\vec{A}_m|$, with respect to that of an almost contemporary macro-economy. Going to the

recessing frame occurs due to saturations of collective biological activities of the society attached with the macro-economy.

The inertial frame's ambition corresponding to the macro-economy, can be thought as group ambition of the society. As a result we see in the recessing phase, lower price index, lower choice flow, hence lower consumption. This gets manifest through deflation, unemployment.

Since $\nabla \cdot \vec{C}\hbar$ is not Lorentz invariant, $\nabla \cdot \vec{C}\hbar \neq 0$ in the recessing phase. This is like at mint $\nabla \cdot \vec{M} \neq 0$. That implies number of choice lines striking a populace from one side is less than the number of lines leaving the populace in the other side. That means human will is setting in and populace is not spending to the brim. That is change in consumption pattern of commodities as well as that of prices at each place with time. This in turn will lead to lesser and lesser production and more and more unemployment.

Black-Scholes formula. Let us suppose that we have gone to the stock-market armed with the set of equations we have heuristically gotten and embark on analysing the share trading. Moreover, let us focus on profit attached with call option. Then the instantaneous profit is call option value for someone having a share and writing a call option for that share. Now let us try to find the value. Let us guide ourselves by the thread of physical considerations of Black and Scholes as appears in the first few pages of the reference [10].

As long as \vec{E} which is analogue of competition flow, \vec{n} , is constant or, slowly changing with time, Maxwell's last two equations with the Ohm's law yields

$$\nabla^2 \vec{B} = \mu_0 \sigma \frac{\partial \vec{B}}{\partial t} \quad (31)$$

In terms of dimensionless length variables, this equation (31) appears as

$$\frac{\partial \vec{B}}{\partial t} = \mu_0 \sigma v^2 \nabla^2 \vec{B}, \quad (32)$$

where, $|v|$ is the drift speed in the medium. Translating to economic system by our dictionary and restricting us to the variation of \vec{P} along the third dimension, x , say in the stock market, we get

$$\frac{\partial P_i(x, t)}{\partial t} = k_0 h |\vec{A}_m| \frac{\partial^2 P_i(x, t)}{\partial x^2}. \quad (33)$$

where, for $i = 1, 2, 3$, P_i means P_x, P_y, P_z . Writing, $\tau = T - t$ and further doing the identification

- implied volatility, $\sigma = \sqrt{2k_0 h |\vec{A}_m|}$
- $P_i = C(S, t) e^{r\tau} = u$ is the profit at time T, corresponding to option trading at time t. $C(S, t)$ is the value of the option when it is traded at time t. $C(S, T) = \max(S - K, 0)$

we get from the equation (33) Black-Scholes differential equation as given in the reference [15],

$$\frac{\partial u(x, \tau)}{\partial \tau} = \frac{\sigma^2}{2} \frac{\partial^2 u(x, \tau)}{\partial x^2}. \quad (34)$$

At this point let us do some more dimensional considerations: in Option trading, relevant independent variables are

- Current stock price at time $t = S$
- Strike price or, agreed upon price of the stock at the expiry i.e. at time T is K
- Risk less interest rate is r (per year)
- Implied volatility in the stock price at time T is σ where, σ^2 has the dimension of time inverse (per year).

One way to combine these variables to get a dimensionless variable x is to write $x = \ln \frac{S}{K} + (r - \frac{\sigma^2}{2})\tau$. Once this is done, the straightforward solution of the equation (34) yields the price of the call option [10, 15],

$$C(S, t) = SN(d_1) - Ke^{-r(T-t)}N(d_2) \quad (35)$$

where,

$$d_1 = \frac{\ln(\frac{S}{K}) + (r + \frac{\sigma^2}{2})(T-t)}{\sigma\sqrt{T-t}},$$

$$d_2 = d_1 - \sigma\sqrt{T-t},$$

$$N(d) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^d dx e^{-\frac{x^2}{2}}.$$

Points. – Here we touch on some delicate issues. Competition flow in this letter is separate from "pure arbitrage flow" just like profit is more than money gain. We can think of three dimensional vector spaces, locally composed of two dimensional plane and a "third dimension". For a company, the "third dimension" is hierarchy. In the stock market, the "third dimension" is the "share" direction as we have explained in the previous subsection. Normally, the "third dimension" is the third dimension, communication is being made along that electrically or, electromagnetically i.e. by land line or, satellite.

Outlook. – Naively, one tends to wonder whether the topological considerations in mathematical economics can be related to magnetic topologies. Similarly many topics in economics, elementary as well as advanced, say, utility, supply-demand line, production, IS-LM model, are expected to be described by electrodynamics using the dictionary introduced in this paper.

One can take a straightforward route also. Consider the eqn.s (7-12) as the rules of economics, measure the variables and the parameters discussed, say, knowing *HumanCapital* ala economists, one can try to measure

competition flow using eq.(16), and therefrom try to explain as many economic empirical relations as possible. The unexplained empirical relations and the parameters put by hand, may give us hints how to get economic analogues of gravitation and other non-abelian models, as well as ways to generalise Maxwell's equations.

Conclusion. – We have given an alternative formulation of economics. The rules of the formulation are equations (7-12). The variables are as mentioned, e.g. profit flow, competition flow, money flow, constrained note density etc. These rules are analogue of Maxwell's equations. Moreover, we have obtained continuity equation, force rule, inertial mass for an economic system and an operational definition of *HumanCapital*. We have constructed a 4-potential formulation. Using the model we get Phillip's curve, describe stagflation, recession. Dwelling on stock-market we recover Call option function. We have gotten a scenario where, unobservable market volatility can be made observable if we can measure the drift ambition of sort-sellers. We have pointed to few avenues, amidst many, along which this approach can be explored further.

* * *

To the best of our knowledge, the topic covered in this manuscript was not dealt with anywhere else.

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