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Pricing options in illiquid markets: optimal systems, symmetry reductions and exact solutions

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Abstract
We study a class of nonlinear pricing models which involves the feedback effect from the dynamic hedging strategies on the price of asset introduced by Sircar and Papanicolaou. We are first to study the case of a nonlinear demand function involved in the model. Using a Lie group analysis we investigate the symmetry properties of these nonlinear diffusion equations. We provide the optimal systems of subalgebras and the complete set of non-equivalent reductions of studied PDEs to ODEs. In most cases we obtain families of exact solutions or derive particular solutions to the equations.

Keywords: illiquid market, nonlinearity, explicit solutions, Lie group analysis

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1 Introduction

One of the important assumptions of the classical Black-Scholes theory is the assumptions that any trading strategy of any trader on the market do not affect asset prices. This assumption is failed in the presence of large traders whose orders involve a significant part of the available shares. Their trading strategy has a strong feedback effect on the price of the asset, and from there back onto the price of derivative products. The continuously increasing volumes of financial markets as well as a significant amount of large traders acting on these markets force us to develop and to study new option pricing models.
There are a number of suggestions on how to incorporate in a mathematical model the feedback effects which correspond to different types of frictions on the market like illiquidity or transaction costs. Most financial market models are characterized by nonlinear partial differential equations (PDEs) of the parabolic type. They contain usually a small perturbation parameter ρ which vanishes if the feedback effect is removed. If ρ tends to zero then the corresponding nonlinear PDE tends to the Black-Scholes equation.

Some of the option pricing models in illiquid markets possess complicated analytical and algebraic structures which are singular perturbed. We deal with singular perturbed PDEs if one of the nonlinear terms in the studied equation incorporates the highest derivative multiplied by the small parameter ρ. It is a demanding task to study such models. Solutions to a singular perturbed equation may blow up in the case $\rho = 0$ and may not have any pendants in the linear case. An example of a singular perturbed model is the continuous-time model developed by Frey [8]. He derived a PDE for perfect replication trading strategies and option pricing for the large traders. An option price $u(S,t)$ in this case is a solution to the nonlinear PDE

$$u_t + \frac{1}{2}\frac{\sigma^2 S^2 u_{SS}}{(1-\rho\lambda(S)Su_{SS})^2} = 0\,, \tag{1.1}$$

where t is time, S denotes the price and σ the volatility of the underlying asset. The continuous function $\lambda(S)$ included in the adjusted diffusion coefficient depends on the payoff of the derivative product. The Lie group analysis and properties of the invariant solutions to Eq. (1.1) for different types of the function $\lambda(S)$ were studied in [4]-[6]. The analytic form of the invariant solutions to this model allow us to follow up the behavior of these solutions.

Under the similar assumptions Cetin, Jarrow and Protter [7] developed a model which includes liquidity risk for a large trader. Liquidity risk is the additional risk due to the timing and size of trade. The value $u(S,t)$ of a self financing trading strategy for the large trader in this setting is a solution of the following nonlinear PDE

$$u_t + \frac{1}{2}\sigma^2 S^2 u_{SS}(1-\rho S u_{SS})^2 = 0. \tag{1.2}$$

This equation seems to be simpler as the previous one but it is still a singular perturbed PDE. The Lie group analysis and the symmetry algebra admitted by this equation were studied in [2].

Sircar and Papanicolaou in [14] present a class of nonlinear pricing models that account for the feedback effect from the dynamic hedging strategies on the price of asset using the idea of a demand function of the reference traders relative to the supply. They obtain a nonlinear PDE of the following type

$$u_t + \frac{1}{2}\left[\frac{U^{-1}(1-\rho u_S)U'(U^{-1}(1-\rho u_S))}{U^{-1}(1-\rho u_S)U'(U^{-1}(1-\rho u_S)) - \rho S u_{SS}}\right]^2 \sigma^2 S^2 u_{SS} + r(Su_S - u) = 0, \tag{1.3}$$

where t is time, S and σ is the price and the volatility of the underlying asset respectively, and the parameter r is the risk-free interest rate. The value $u(S,t)$ is the price of the derivative security and depends on the form of the demand function $U(\cdot)$. The expression $U^{-1}(\cdot)$ denotes the correspondingly inverse function, because of the strong monotonicity of the demand function the existence of the inverse function $U^{-1}(\cdot)$ is guaranteed. In the bulk of their paper [14] authors studied the particular model arising from taking $U(\cdot)$ as linear, i.e.

$U(z) = \beta z,\ \beta > 0$. The authors mainly focused on the numerical solution and discuss the difference to the classical Black-Scholes option pricing theory.

In the present paper we study a more general case in which the demand function of the type $U(z) = \beta z^{\alpha},\ \alpha, \beta \neq 0$ is incorporated. Consistency of (1.3) with the Black-Scholes model characterizes the class of the admitted demand functions and leads to the condition $U'(z) = \beta\alpha z^{\alpha-1} > 0$. In this case the model (1.3) takes the form

$$\frac{\partial u}{\partial t} + \frac{1}{2}\left[\frac{1-\rho\frac{\partial u}{\partial S}}{1-\rho\frac{\partial u}{\partial S} - \frac{\rho}{\alpha}S\frac{\partial^2 u}{\partial S^2}}\right]^2 \sigma^2 S^2 \frac{\partial^2 u}{\partial S^2} + r\left(S\frac{\partial u}{\partial S} - u\right) = 0. \qquad (1.4)$$

The diffusion coefficient in Eq. (1.4) depends on both, u_S and u_{SS} multiplied by the small perturbation parameter ρ. It depends also on the parameter α which characterizes the type of the demand function and on the interest rate r.

We give a short overview of analytical properties of Eq. (1.4) in the next Section 2. In Section 3 we provide the Lie group analysis of this equation. Depending on whether the interest rate $r = 0$ or $r \neq 0$, we obtain different Lie algebras admitted by the respectively equation. Then we provide optimal systems of subalgebras in the both cases. The optimal systems of subalgebras give us the possibility to describe the set of independent reductions of these nonlinear PDEs to different ordinary differential equations (ODEs). In some cases we found the explicit solutions to these equations. We discuss the properties of invariant solutions in Section 4.

2 Basic analytical properties of Eq. (1.4)

In the model (1.4) introduced by Sircar and Papanicolaou in [14] the diffusion coefficient has a very complicated analytical and non trivial algebraic structure. In particular the diffusion term is represented by a fraction which contains derivatives $u_S(S,t), u_{SS}(S,t)$. The authors analyzed some analytical properties of this equation in the case $\alpha = 1$ in the vicinity of the Black-Scholes equation and considered a valuation of a European derivative security with a convex payoff using their model. They give asymptotic results for when the volume of assets traded by the large traders is small compared to the total number of unites of the asset. In this Section we discuss some global analytical properties of Eq. (1.4) for $\alpha \neq 1$.

We pay the main attention to the second term in (1.4). We assume that the space variable $S \in \Omega \cup \{0\}$, where $\Omega = \mathbb{R}^+$ and the time variable t lie in $\mathcal{T} \cup \{0\}$, where $\mathcal{T} = \mathbb{R}^+$. This term can vanish for some values of the variable S or on some set of smooth functions and then the equation may change the type from the parabolic one to another one. Other hand the fraction in the second term may became meaningless because of vanishing of the denominator on some set of smooth functions. We should exclude such functions from the domain of definition of our model.

The classical linear diffusion equation of type $u_t = u_{SS}$ is well defined on the space $D = C^{2,1}(\Omega \times \mathcal{T}) \bigcap C(\{\Omega \cup \{0\}\} \times \{\mathcal{T} \cup \{0\}\})$ and $u(S,t)$ map the space D to a space of continuous functions $M = C(\{\Omega \cup \{0\}\} \times \{\mathcal{T} \cup \{0\}\})$.

Let us check whether the expression for the diffusion coefficient in Eq. (1.4) vanishes or has singularities.
Fist we study the case that the denominator of the fraction in (1.4) is equal to zero, i.e. we have to solve the equation

$$1 - \rho u_S - \frac{\rho}{\alpha} S u_{SS} = 0. \tag{2.1}$$

It is easy to see that this equation has the following solution

$$\begin{aligned} u_{sing}(S,t) &= \frac{S}{\rho} + c_1(t)\frac{\alpha}{\alpha-1}S^{\frac{\alpha-1}{\alpha}} + c_2(t),\ \alpha \neq 1, \\ u_{sing}(S,t) &= \frac{S}{\rho} + c_1(t)\ln(S) + c_2(t),\ \alpha = 1, \end{aligned} \tag{2.2}$$

where $c_1(t)$ and $c_2(t)$ are arbitrary functions of t. We can rewrite the expressions in (2.2) as one expression which includes the case $\alpha = 1$ as a limit case. Then we obtain

$$u_{sing}(S,t) = \frac{S}{\rho} + c_1(t)\frac{\alpha}{\alpha-1}\left(S^{\frac{\alpha-1}{\alpha}} - 1\right) + c_2(t). \tag{2.3}$$

The numerator of the second term in (1.4) is equal to zero if one of the equations is satisfied

$$\begin{aligned} S^2 u_{SS} &= 0, \\ 1 - \rho u_S &= 0. \end{aligned} \tag{2.4}$$

The first equation is satisfied on all linear functions of S and in the point $S = 0$, the second equation has the following solution

$$u_0(S,t) = \frac{S}{\rho} + c_2(t). \tag{2.5}$$

We notice that in the case $c_1(t) = 0$ the functions $u_{sing}(S,t)$ and $u_0(S,t)$ coincide. It means in this case the numerator and the denominator of the fraction in the equation (1.4) are simultaneously equal to zero.

In the second step we should define a limiting procedure to explain what we means if we say that (2.5) is a solution to (1.4).

We chose in the space D a one-parametric family of functions $u_\epsilon(S,t)$ of the following type

$$u_\epsilon(S,t) = d_1(t)S + d_2(t) + \epsilon v(S,t), \tag{2.6}$$

where $\epsilon \in \mathbb{R}$ is a parameter, the functions $d_1(t), d_2(t)$ are arbitrary functions of time and $v(S,t) \in D$. If now the parameter $\epsilon \to 0$ then the family of functions of the type (2.6) converges in the norm of the space D to a linear function of S, i.e. to $u_0(S,t) = d_1(t)S + d_2(t)$. We apply to this family the differential operator defined by (1.4)

$$d_1^{'}(t)S + d_2^{'}(t) + \epsilon v_t(S,t) + \frac{\sigma^2}{2}\frac{\epsilon S^2 v_{SS}(1-\rho d_1(t) - \epsilon\rho v_S)^2}{(1-\rho d_1(t) - \epsilon\rho v_S - \epsilon\beta\rho S v_{SS})^2} = 0\,, \qquad (2.7)$$

here $d_1^{'}(t), d_2^{'}(t)$ denotes the first derivatives of the corresponding functions. From (2.7) follows that any linear function of S with constant coefficients will be solution to equation (1.4) if the last term in (2.7) is a bounded function in the norm of the space M. If we replace the linear part in (2.7) by $u_0(S,t)$ we see that this function (2.5) is a solution to (1.4) if $c_2(t) = const.$. The fraction in (2.7) is not well defined just in one case if $v(S,t)$ coincide with the second term in (2.3). So far we use as the domain of definition for our model the space D the functions of type (2.3) with $c_1(t) \neq 0$ are excluded because they or their derivatives have singularities in the point $S = 0$ and consequently they do not belong to the space D.

In the classical case of a linear parabolic diffusion equation solutions of type (2.3) which do not belongs to the set of classical solutions are called viscosity solutions [1], [9] and these solutions are well studied.

We proved that the functions of type (2.3) should be excluded from the further investigation because the model (1.4) is not well defined on them. The linear function (2.5) with $c_2(t) = const.$ is a solution to (1.4) because any one parametric family of functions $u_\epsilon(S,t)$ in the norm of the space D convergent to u_0 is mapped by the differential operator defined by (1.4) to a zero-convergent family of functions in the norm of the space M.

3 Symmetry properties of the model

We provide in this Section the Lie group analysis of Eq. (1.4) first for the case $r = 0$ then for $r \neq 0$. In both cases it is possible to find the non-trivial Lie algebras admitted by the equation. We use the standard method to obtain the symmetry group suggested by Sophus Lie and developed further in [12], [11] and [10]. In the case $r = 0$ we obtain a four dimensional Lie algebra L_4 and by $r \neq 0$ Eq. (1.4) admits a three dimensional algebra L_3 defined in the subsection 3.2.

All three and four dimensional real Lie algebras and their subalgebras were classified by Pattera and Winternitzs in [13]. The authors looked for classifications of the subalgebras into equivalence classes under their group of inner automorphisms. They used also the idea of normalization which guarantees that the constructed optimal system of subalgebras is unique up to the isomorphisms.

The symmetry group G_4 related to the symmetry algebra L_4 is generated by a usual exponential map. We use the similar procedure to obtain to each subalgebra h_i from the optimal system of subalgebras the correspondingly subgroup H_i.

The optimal system of subalgebras allows us to divide the invariant solutions into non-intersecting equivalence classes. In this way it is possible to find the complete set of essential different invariant solutions to the equation under consideration.

Using the invariants of these subgroups we reduce the studied PDE to different ODEs. Solutions to these ODEs give us the invariant solutions to the nonlinear PDE (1.4) in an analytical form. In the both cases whether by $r = 0$ or by $r \neq 0$ we skip the study of invariant reductions to the two- and three- dimensional subgroups because of they give trivial results for the studied equation.

3.1 Symmetry reductions in the case $r = 0$

In the first step we solve the Lie determining equations for the equation

$$u_t + \frac{1}{2}\frac{\sigma^2(1-\rho u_S)^2}{\left(1-\rho u_S - \frac{\rho}{\alpha}Su_{SS}\right)^2}S^2 u_{SS} = 0\,, \alpha \neq 0, \tag{3.1}$$

and obtain the Lie algebra admitted by this equations. We formulate the results in the following theorem.

Theorem 3.1. Eq. (3.1) admits a four dimensional Lie algebra L_4 with the following infinitesimal generators

$$e_1 = -\frac{S}{2}\frac{\partial}{\partial S} + \left(\frac{S}{2\rho} - u\right)\frac{\partial}{\partial u},\ e_2 = \frac{\partial}{\partial u},\ e_3 = \frac{\partial}{\partial t},\ e_4 = \rho S\frac{\partial}{\partial S} + S\frac{\partial}{\partial u}. \tag{3.2}$$

The commutator relations are

$$[e_1, e_3] = [e_1, e_4] = [e_2, e_3] = [e_2, e_4] = [e_3, e_4] = 0,\ [e_1, e_2] = e_2, \tag{3.3}$$

Remark 3.1. In the very short letter [3] there is a misprint in the theorem formulation. We apologize by readers for the inconvenience.

The Lie algebra L_4 has a two-dimensional subalgebra $L_2 =< e_1, e_2 >$ spanned by the generators e_1, e_2. The algebra L_4 is a decomposable Lie algebra and can be represented as a semi-direct sum $L_4 = L_2 \bigoplus e_3 \bigoplus e_4$. The optimal system of subalgebras for L_4 were provided in [13] and presented in Table 1.

The optimal system of the one-dimensional subalgebras involves four subalgebras $h_i^0,\ i = 1, \ldots, 4$. We take step-by-step each of these subalgebras h_i^0 and the corresponding symmetry subgroup H_i^0 and study which invariant reductions of the studied PDE are possible.

Case H_1^0. This one-dimensional subgroup $H_1^0 \subset G_4$ is generated by the subalgebra $h_1^0 =< e_2 >=< \frac{\partial}{\partial u} >$. It means that we deal with a subgroup of translations in the u - direction. Hence, to each solution to Eq. (3.1) we can add an arbitrary constant without destroying the property of the function to be solution. This subgroup does not provide any reduction.

Dimension	Subalgebras
1	$h_1 =< e_2 >,\ h_2 =< e_3 \cos(\phi) + e_4 \sin(\phi) >,$ $h_3 =< e_1 + x(e_3 \cos(\phi) + e_4 \sin(\phi)) >,$ $h_4 =< e_2 + \epsilon(e_3 \cos(\phi) + e_4 \sin(\phi)) >$
2	$h_5 =< e_1 + x(e_3 \cos(\phi) + e_4 \sin(\phi)), e_2 >,\ h_6 =< e_3, e_4 >,$ $h_7 =< e_1 + x(e_3 \cos(\phi) + e_4 \sin(\phi)), e_3 \sin(\phi) - e_4 \cos(\phi) >,$ $h_8 =< e_2 + \epsilon(e_3 \cos(\phi) + e_4 \sin(\phi)), e_3 \sin(\phi) - e_4 \cos(\phi) >,$ $h_9 =< e_2, e_3 \sin(\phi) - e_4 \cos(\phi) >$
3	$h_{10} =< e_1, e_3, e_4 >,\ h_{11} =< e_2, e_3, e_4 >,$ $h_{12} =< e_1 + x(e_3 \cos(\phi) + e_4 \sin(\phi)), e_3 \sin(\phi) - e_4 \cos(\phi), e_2 >$

Table 1: [13] The optimal system of subalgebras h_i of the algebra L_4 where $x \in \mathbb{R},\ \epsilon = \pm 1,\ \phi \in [0, \pi]$.

Case H_2^0. The subalgebra h_2^0 is spanned by the generator $e_3 cos(\phi) + e_4 sin(\phi)$. In terms of the variables S, t, u it takes the form

$$h_2^0 =< \frac{\partial}{\partial t} \cos(\phi) + \left(\rho S \frac{\partial}{\partial S} + S \frac{\partial}{\partial u} \right) \sin(\phi) > . \tag{3.4}$$

The invariants z, w of the corresponding subgroup H_2^0 are equal to

$$z = S \exp(-t\rho \tan(\phi)),\ w = u - \frac{1}{\rho} S \tag{3.5}$$

and we take them as the new dependent and independent variables, respectively. Then the PDE (3.1) is reduced to the ordinary differential equation of the following form

$$-\rho\delta z w_z + \frac{1}{2}\sigma^2 w_{zz} z^2 \left(\frac{\alpha w_z}{\alpha w_z + z w_{zz}} \right)^2 = 0,\ \delta = \tan(\phi), \phi \in [0, \pi], \phi \neq \frac{\pi}{2}. \tag{3.6}$$

This second order ODE is reduced to the first order equation by the substitution $w_z = v(z)$ which takes the form

$$zv \left(-\rho\delta + \frac{\sigma^2}{2} z (\ln v)_z \left(\frac{\alpha}{z (\ln v)_z + \alpha} \right)^2 \right) = 0. \tag{3.7}$$

Eq. (3.7) has two trivial solutions $z = 0$ and $v(z) = w_z = 0$ which are not very interesting for applications. The non trivial solutions we obtain if we set the last factor in Eq. (3.7) equal to zero. We obtain the solution to (3.7) in the form

$$w(z) = c_1 z^p,\ p = 1 + \alpha - a \pm \sqrt{a(a - 2\alpha)},\ a = \frac{\sigma^2}{4\rho\delta}, \tag{3.8}$$

where c_1 is an arbitrary constant. In terms of the variables S, t, u the solution (3.8) is equivalent to the following solution to Eq.(3.1)

$$u(S, t) = c_1 S^p \exp(-p\rho \tan(\phi)\ t),\ p = 1 + \alpha - a \pm \sqrt{a(a - 2\alpha)},\ a = \frac{\sigma^2}{4\rho\delta}, \tag{3.9}$$

where $\delta = \tan(\phi), \phi \in [0, \pi], \phi \neq \frac{\pi}{2}$.

Case H_3^0. The subalgebra h_3^0 is spanned by

$$h_3^0 =< x\cos(\phi)\frac{\partial}{\partial t} + \left(x\rho\sin(\phi) - \frac{1}{2}\right) S\frac{\partial}{\partial S} + \left(\left(\frac{1}{2\rho} + x\sin(\phi)\right) S - u\right)\frac{\partial}{\partial u} > .$$

The invariants z, w of the corresponding subgroup H_3^0 are given by the expressions

$$z = Se^{-ct}, \ w = S^b u - \frac{1}{\rho}S^{1+b}, \tag{3.10}$$

where $b = (x\rho\sin(\phi) - \frac{1}{2})^{-1}$, $c = (bx\cos(\phi))^{-1}$, $x \neq 0$, $\phi \in [0, \pi], \phi \neq \frac{\pi}{2}$ and $x\rho\sin(\phi) - \frac{1}{2} \neq 0$.
We use z, w as the new invariant variables and reduce Eq.(3.1) to the ODE

$$-czw_z + \frac{\sigma^2\alpha^2}{2}\frac{\left(z^2w_{zz} - 2bzw_z + b\,(1+b)\,w\right)(zw_z - bw)^2}{\left(\alpha\,(zw_z - bw) + (z^2w_{zz} - 2bzw_z + b\,(1+b)\,w)\right)^2} = 0. \tag{3.11}$$

Eq. (3.11) admits a solution of the type

$$w(z) = c_1 z^q, \tag{3.12}$$

where q is a real root of the polynomial of the degree 5

$$\begin{aligned} &-cq(q(q-1) + (\alpha - 2b)\,q + b\,(1+b-\alpha))^2 \\ &+\frac{\sigma^2\alpha^2}{2}(q(q-1) - 2bq + b\,(1+b))\,(q-b)^2 = 0. \end{aligned}$$

Eq. (3.11) has a complicate structure and is hardly possible to solve it in the general form. But by a special values of involved parameters we can simplify the equation and obtain some particular classes of solutions.
We take the special case of Eq. (3.1) with $\alpha = 1$.
Under the special choice of the parameters $\phi = 0, \pi$ and $b = -2$ in Eq. (3.10) and by using the invariants z, w in the form

$$z = S\exp\left(\frac{1}{2x}bt\right), x \neq 0, \ w = S^{-2}u - (\rho S)^{-1},$$

we reduce Eq.(3.1) to the ODE

$$zw_z((z(z^2w)_z)_z)^2 + \sigma^2 x \ (z^2w)_{zz}((z^2w)_z)^2 = 0. \tag{3.13}$$

The substitution (3.12) in this case leads to the second order algebraic equation on the value of the parameter q

$$q^2 + q\left(2 + \sigma^2 x\cos(\phi)\right) + \sigma^2 x\cos\phi = 0, \ q \neq -2, \phi = 0, \pi, \tag{3.14}$$

which has two roots $q_1 = -\sigma^2 x\cos(\phi)$ and $q_2 = -2$. For the future study we take just the first value q_1. The value $q_2 = -2$ leads to the know solution $u_0(S,t)$ (2.5). Since by $q_2 = -2$ both, the numerator and the denominator in the fraction in (3.1) vanish, and the solution (3.12) coincide then with $u_0(S,t)$ by $c_2(t) = const.$ which we discussed in the previous section.

We notice that the solution (3.12) differs from the function $u_{sing}(S,t)$ (2.2) which involves the logarithmic term in the case $\alpha = 1$.

The solution to (3.13) or respectively to Eq.(3.1) in the form (3.12) in terms of S, t, u variables is equal to

$$u(S,t) = \frac{S}{\rho} + C_1 S^{2-\sigma^2 x} e^{-\frac{\sigma^2}{2}t} + C_2, \;\; \alpha \neq -2, \tag{3.15}$$

where C_1, C_2 are arbitrary constants, $x \in \mathbb{R}$ and the first term is the only term which contains the dependency on the parameter ρ. We skip the factor $\cos(\phi)$ by x in this expression because $\cos(\phi) = \pm 1$ in the case $\phi = 0, \pi$ and $x \in \mathbb{R}$.

It is remarkable that the reduced equation (3.13) does not contain any more the parameter ρ. Hence all invariant solutions of this class can be represented as a sum of two terms: the first one is equal to S/ρ and the second one depends on z only but not on the parameter ρ.

If we left the values of parameters like in the previous case, but take the invariants in another form $z = \ln S + tb/4$ and $w(z) = (u/S - 1/\rho)S^\gamma$ than we obtain the different form of the reduced ODE

$$\begin{aligned} w_{zz}^2 w_z + w_{zz}(w_z^2(4(1-\gamma)+\kappa) + w_z w(2(1-\gamma)^2 + 2\kappa(1-\gamma)) \\ + w^2\kappa(1-\gamma)^2) + w_z^3(4(1-\gamma)^2 + \kappa(1-2\gamma)) + w_z^2 w(1-\gamma)(4(1-\gamma)^2 \\ + \kappa(2-5\gamma)) + w_z w^2(1-\gamma)^2((1-\gamma)^2 + \kappa(1-4\gamma)) - \kappa\gamma(1-\gamma)^3 w^3 = 0, \end{aligned} \tag{3.16}$$

where $\kappa = 2\sigma^2/b$. This second order equation can be reduced in the case $w_z \neq 0$ to a first order ODE. We substitute $p(w) = w_z(z(w))$ and correspondingly $w_{zz} = p_w p$, i.e., w is the independent variable and p is the dependent variable in this case. Then we obtain the first order ODE

$$\begin{aligned} p_w^2 p^3 + p_w p(p^2(4(1-\gamma)+\kappa) + pw(2(1-\gamma)^2 + 2\kappa(1-\gamma)) + w^2\kappa(1-\gamma)^2) \\ + p^3(4(1-\gamma)^2 + \kappa(1-2\gamma)) + p^2 w(1-\gamma)(4(1-\gamma)^2 + \kappa(2-5\gamma)) \\ + pw^2(1-\gamma)^2((1-\gamma)^2 + \kappa(1-4\gamma)) - \kappa\gamma(1-\gamma)^3 w^3 = 0. \end{aligned} \tag{3.17}$$

This equation is quadratic in the first derivative p_w and it is equivalent to the system of two first order equations. For some values of the constants γ and κ it can be explicitly solved. The simplest case we obtain if we chose $\gamma = 1$ then the solution coincide with (3.15). In other cases the equation can be studied using qualitative methods.

Case H_4^0. We consider subalgebra h_4^0 spanned by

$$h_4^0 =< \epsilon\cos(\phi)\frac{\partial}{\partial t} + \epsilon\rho S\sin(\phi)\frac{\partial}{\partial S} + (1+\epsilon S\sin(\phi))\frac{\partial}{\partial u} > .$$

The invariants z, w are given by the expressions

$$\begin{aligned} z &= S\exp(-t\rho\tan(\phi)),\ \ \phi \in (0,\pi), \phi \neq \frac{\pi}{2}, \\ w &= \frac{\epsilon}{\rho\sin(\phi)}\ln S + \frac{1}{\rho}S - V,\ \ \epsilon = \pm 1. \end{aligned}$$

Using these expressions as the independent and dependent variables we reduce the original equation to the ODE

$$-\rho\tan(\phi)w_z z + \frac{1}{2}\sigma^2(w_{zz}z^2 + a)\left[\frac{\alpha(w_z z - a)}{\alpha(w_z z - a) + a + w_{zz}z^2}\right]^2 = 0, \qquad (3.18)$$

where $a = (\epsilon\rho\sin(\phi))^{-1}$ and $\phi \in (0,\pi), (\phi) \neq \frac{\pi}{2}$. Eq. (3.18) is possible to reduce to the first order ODE

$$(v + a)\left(zv_z - (1-\alpha)v + a\right)^2 - \frac{\sigma^2\alpha^2}{2\rho\tan(\phi)}v^2(zv_z - v + a) = 0 \qquad (3.19)$$

after substitution $v(z) = zw_z - a$. It is a quadratic algebraic equation on the value zv_z which roots depends on v only. If we denote the roots as $f_{\pm}(v)$ we represent the solutions to Eq. (3.19) in the parametric form

$$\int \frac{\mathrm{d}v}{f_{\pm}(v)} = \ln z + c_1,\ \ c_1 \in \mathbb{R}. \qquad (3.20)$$

3.2 Symmetry reductions in the case $r \neq 0$

Theorem 3.2. Eq. (1.4) admits a three dimensional Lie algebra L_3 with the following operators

$$e_1 = \frac{\partial}{\partial t},\ \ e_2 = (S - \rho)\frac{\partial}{\partial u},\ \ e_3 = S\frac{\partial}{\partial S} + u\frac{\partial}{\partial u}. \qquad (3.21)$$

The algebra L_3 is abelian.

Similar to the previous investigation for $r = 0$ first we find an optimal systems of subalgebras for the algebra L_3. Because we are interested just in the one-dimensional subalgebras we provide the optimal system for these subalgebras only

$$h_1 =< e_3 >,\ \ h_2 =< e_2 + xe_3 >,\ \ h_3 =< e_1 + xe_2 + ye_3 > . \qquad (3.22)$$

First we provide for each of these three one-dimensional subalgebras $h_i,\ i = 1,2,3$ and the corresponding subgroup $H_i,\ i = 1,2,3$ a set of invariants. Then we use the invariants as the new independent and independent variables and reduce Eq. (1.4) to some ODEs. In the cases where it is possible we solve the ODEs.

Case H_1. The algebra h_1 is spanned by

$$h_1 =< S\frac{\partial}{\partial S} + u\frac{\partial}{\partial u} > . \tag{3.23}$$

It describes scaling symmetry of Eq. (1.4) and means that if we multiply the variable S and u with one and the same non vanishing constant the equation will be unaltered. Respectively the invariants of this transformations are

$$z = t,\ w = \frac{u}{S}. \tag{3.24}$$

If we use these expressions as the new invariant variables we obtain a rather simple reduction of the original equation

$$w_z = 0, \tag{3.25}$$

with the trivial solution $w = c_1 = const$. It describes all solutions to Eq. (1.4) of the type

$$u(S,t) = Sw(z) = Sw(t) = Sc_1. \tag{3.26}$$

Case H_2. The second subalgebra h_2 in the optimal system of subalgebras (3.22) is given by

$$h_2 =< x\ S\frac{\partial}{\partial S} + (S + u(x-\rho))\frac{\partial}{\partial u} >,\ x \in \mathbb{R}. \tag{3.27}$$

The invariants of the subgroup H_2 are

$$z = t,\ w = -\frac{S^{\frac{\rho}{x}}}{\rho} + S^{\frac{\rho}{x}-1}u,\ x \neq 0. \tag{3.28}$$

If we use the invariants (3.28) as the new dependent and independent variables then the reduced Eq. (1.4) takes the form

$$w_z - \rho\left(\frac{\sigma^2\alpha^2(x-\rho)}{2(\alpha x-\rho)^2} + \frac{r}{x}\right)w = 0. \tag{3.29}$$

It has the solution

$$w(z) = c_1 \exp\left(\rho\left(\frac{\sigma^2\alpha^2(x-\rho)}{2(\alpha x-\rho)^2} + \frac{r}{x}\right)\right)t,\ c_1, x \in \mathbb{R}, x \neq 0. \tag{3.30}$$

The corresponding solution to Eq. (1.4) in terms of variables S, t, u is given by

$$u(S,t) = c_1 S^{1-\frac{\rho}{x}} e^{\rho\gamma t} + \frac{1}{\rho}S,\ \gamma = \frac{\sigma^2\alpha^2(x-\rho)}{2(\alpha x-\rho)^2} + \frac{r}{x}, \tag{3.31}$$

where $c_1, x \in \mathbb{R}, x \neq 0$ are arbitrary constants.

Case H_3. The last subalgebra h_3 from the optimal system (3.22) is spanned by

$$h_3 =< \frac{\partial}{\partial t} + y\ S\frac{\partial}{\partial S} + (x\ S - x\rho u + y\ u)\ \frac{\partial}{\partial u} >,\ x, y \in \mathbb{R}. \tag{3.32}$$

We take the two invariants z, w

$$z = Se^{-ty},\ w = uS^{\kappa\rho - 1} - \frac{1}{\rho}S^{\kappa\rho},\ y \neq 0,\ \kappa = \frac{x}{y}.$$

as the new independent and dependent variables reduce Eq. (1.4) to the ODE

$$\begin{aligned} &(r-y)zw_z - r\kappa\rho w \qquad (3.33)\\ +\quad &\frac{\alpha^2\sigma^2}{2}\frac{(z^2w_{zz} + 2(1-\kappa\rho)zw_z - \kappa\rho(1-\kappa\rho)w)(zw_z + (1-\kappa\rho)w)^2}{(z^2w_{zz} + (\alpha + 2(1-\kappa\rho)zw_z + (1-\kappa\rho)(\alpha - \kappa\rho)w)^2} = 0. \end{aligned}$$

Eq. (3.33) possesses a solution of the type

$$w(z) = c_1 z^q,\ c_1 \in \mathbb{R}, \tag{3.34}$$

where q is a real root of the fifth order algebraic equation

$$\begin{aligned} &((r-y)q - r\kappa\rho)(q(q-1) + (\alpha + 2(1-\kappa\rho)q + (1-\kappa\rho)(\alpha - \kappa\rho))^2 \\ &+\frac{\alpha^2\sigma^2}{2}(q(q-1) + 2(1-\kappa\rho)q - \kappa\rho(1-\kappa\rho))(q + (1-\kappa\rho))^2 = 0. \end{aligned} \tag{3.35}$$

Respectively the solution to Eq. (1.4) takes in this case the form

$$w(z) = c_1 S^{q+1-\kappa\rho}e^{-yqt} + \frac{1}{\rho}S,\ c_1, y, \kappa \in \mathbb{R}, y \neq 0. \tag{3.36}$$

4 Conclusions

In the previous sections we studied the Sircar-Papanicolaou model (1.3) in the case of the nonlinear demand function $U(z) = \beta z,\ \beta > 0$. The model which includes the linear demand function was studied in [14] with numerical methods. In this paper we use the methods of Lie group analysis which gives us a general point of view on the structure of this equation. We found the symmetry algebra admitted by the nonlinear PDE (1.4) for $r \neq 0$ and by Eq. (3.1) in the case $r = 0$. We present in the both cases the optimal systems of subalgebras. Using the optimal systems of subalgebras we provide the complete set of non-equivalent reductions. In most cases we solve the ODEs or present particular solutions to them and respectively to Eqs. (1.4) and (3.1). The explicit and parametric solutions can be used as benchmarks for numerical methods.

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