Sliding Mode Control of Two-Level Quantum Systems

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Abstract

This paper proposes a robust control method based on sliding mode design for two-level quantum systems with bounded uncertainties. An eigenstate of the two-level quantum system is identified as a sliding mode. The objective is to design a control law to steer the system's state into the sliding mode domain and then maintain it in that domain when bounded uncertainties exist in the system Hamiltonian. We propose a controller design method using the Lyapunov methodology and periodic projective measurements. In particular, we give conditions for designing such a control law, which can guarantee the desired robustness in the presence of the uncertainties. The sliding mode control method has potential applications to quantum information processing with uncertainties.

Index Terms

quantum control, sliding mode control, bounded uncertainty, periodic projective measurement, Lyapunov methodology.

I. INTRODUCTION

The manipulation and control of quantum systems is becoming an important task in many fields [1]-[3], such as atomic physics [4], molecular chemistry [5] and quantum information [6]. It is desirable to develop quantum control theory in a systematic way in order to adapt it to the development of quantum technology [7]. Several useful tools from classical control theory have

been introduced to the control analysis and design of quantum systems. For example, optimal control theory has been used to assist in control design for molecular systems and spin systems [8]-[12]. A learning control method has been presented for guiding the control of chemical reactions [5], [13]. Quantum feedback control approaches including measurement-based feedback and coherent feedback have been used to improve performance for several classes of tasks such as preparing quantum states, quantum error correction, controlling quantum entanglement [14]-[27]. Robust control tools have been introduced to enhance the robustness of quantum feedback networks and linear quantum stochastic systems [28], [29].

Although some progress has been made, more research effort is necessary in controlling quantum phenomena. In particular, the robustness of quantum control systems has been recognized as a key issue in developing practical quantum technology [30]-[32]. In this paper, we focus on the robustness problem for quantum control systems. In [29], James and co-workers have formulated and solved a quantum robust control problem using the H^∞ method for linear quantum stochastic systems. Here, we develop a variable structure control approach with sliding modes to enhance the robustness of quantum systems. The variable structure control strategy is a widely used design method in classical control theory and industrial applications where one can change the controller structure according to a specified switching logic in order to obtain desired closed-loop properties [33], [34]. In [35], Dong and Petersen have formulated and solved a variable structure control problem for the control of quantum systems. However, the results in [35] only involve open-loop control design using an idea of changing controller structures and do not consider the robustness which can be obtained through sliding mode control. Ref. [35] and Ref. [36] have briefly discussed the possible application of sliding mode control to quantum systems. In [37], two approaches based on sliding mode design have been proposed for the control of quantum systems and potential applications of sliding mode control to quantum information processing have been presented. Following these results, this paper formally presents a sliding mode control method for two-level quantum systems to deal with bounded uncertainties in the system Hamiltonian [38]. In particular, we propose two approaches of designing the measurement period for different situations which are dependent on the bound on the uncertainties and the allowed probability of failure.

Variable structure control design with sliding modes generally includes two main steps: selecting a sliding surface (sliding mode) and controlling the system to and maintaining it in this

sliding surface. Being in the sliding surface guarantees that the quantum system has the desired dynamics. We will select an eigenstate of the free Hamiltonian of the controlled quantum system as a sliding mode. In the second step, direct feedback control is not directly applicable since we generally cannot acquire state information without destroying the quantum system's state. Hence, we propose a new method to accomplish this task, which is based on the Lyapunov methodology and periodic projective measurements. The main feature of the proposed method is that the control law can guarantee control performance when bounded uncertainties exist in the system Hamiltonian.

This paper is organized as follows. Section II introduces a quantum control model, defines the sliding mode and formulates the control problem. In Section III, we present a sliding mode control method based on the Lyapunov methodology and periodic projective measurements for two-level quantum systems with bounded uncertainties. Using the known information about uncertainties (e.g., the uncertainty bound and type of uncertainties), we propose two approaches (i.e., Eqs. (13) and (14)) for designing the measurement period to guarantee the control performance. An illustrative example is presented to demonstrate the proposed method. The detailed proofs of the main theorems are presented in Section IV. Concluding remarks are given in Section V.

II. SLIDING MODES AND PROBLEM FORMULATION

In this section, we first introduce a two-level quantum control model. Then a sliding mode is defined using an eigenstate. Finally the control problem considered in this paper is formulated.

A. Quantum Control Model

In this paper, we focus on two-level pure-state quantum systems. The quantum state can be represented by a two-dimensional unit vector $|\psi\rangle$ in a Hilbert space $\mathscr{H}$. Since the global phase of a quantum state has no observable physical effect, we do not consider the effect of global phase. If we denote the Pauli matrices $\sigma = (\sigma_x, \sigma_y, \sigma_z)$ as follows:

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \tag{1}$$

we may select the free Hamiltonian of the two-level quantum system as $H_0 = I_z = \frac{1}{2}\sigma_z$. Its two eigenstates are denoted as $|0\rangle$ and $|1\rangle$. To control a quantum system, we introduce the following

control Hamiltonian $H_u = \sum_k u_k(t)H_k$, where $u_k(t) \in \mathbf{R}$ and $\{H_k\}$ is a set of time-independent Hamiltonians. For simplicity, the control Hamiltonian for two-level systems can be written as $H_u = u_x(t)I_x + u_y(t)I_y + u_z(t)I_z$, where

$$I_x = \frac{1}{2}\sigma_x = \frac{1}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad I_y = \frac{1}{2}\sigma_y = \frac{1}{2}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}. \tag{2}$$

The controlled dynamical equation can be described as (we have assumed $\hbar = 1$ by using atomic units in this paper)

$$i|\dot{\psi}(t)\rangle = H_0|\psi(t)\rangle + \sum_{k=x,y,z} u_k(t)I_k|\psi(t)\rangle,$$
$$|\psi(t=0)\rangle = |\psi_0\rangle. \tag{3}$$

This control problem is converted into the following problem: given an initial state and a target state, find a set of controls $\{u_k(t)\}$ in (3) to drive the controlled system from the initial state to the target state.

In practical applications, we often use the density operator (or density matrix) ρ to describe the quantum state of a quantum system. For a pure state $|\psi\rangle$, the corresponding density operator is $\rho \equiv |\psi\rangle\langle\psi|$. For a two-level quantum system, the state ρ can be represented in terms of the Bloch vector $\mathbf{r} = (x,y,z) = (\mathrm{tr}\{\rho\sigma_x\}, \mathrm{tr}\{\rho\sigma_y\}, \mathrm{tr}\{\rho\sigma_z\})$:

$$\rho = \frac{1}{2}(I + \mathbf{r}\cdot\sigma). \tag{4}$$

The evolution equation of ρ can be written as

$$\dot{\rho} = -i[H,\rho] \tag{5}$$

where $[A,B] = AB - BA$ and H is the total system Hamiltonian.

After we represent the state ρ with the Bloch vector, the pure states of a two-level quantum system correspond to the surface of the Bloch sphere, where $(x,y,z) = (\sin\theta\cos\varphi, \sin\theta\sin\varphi, \cos\theta)$, $\theta \in [0,\pi]$, $\varphi \in [0,2\pi]$. An arbitrary pure state $|\psi\rangle$ for a two-level quantum system can be represented as

$$|\psi\rangle = \cos\frac{\theta}{2}|0\rangle + e^{i\varphi}\sin\frac{\theta}{2}|1\rangle. \tag{6}$$

B. Sliding Modes

Sliding modes play an important role in variable structure control [33]. Usually, the sliding mode is constructed so that the system has desired dynamics in the sliding surface. For a quantum control problem, a sliding mode may be represented as a functional of the state $|\psi\rangle$ and the Hamiltonian H; i.e., $S(|\psi\rangle, H) = 0$. For example, an eigenstate $|\phi_j\rangle$ of the free Hamiltonian H_0 (i.e., $H_0|\phi_j\rangle = \lambda_j|\phi_j\rangle$ where λ_j is one eigenvalue of H_0) can be selected as a sliding mode. We can define $S(|\psi\rangle, H) = 1 - |\langle\psi|\phi_j\rangle|^2 = 0$. If the initial state $|\psi_0\rangle$ is in the sliding mode; i.e., $S(|\psi_0\rangle, H) = 1 - |\langle\psi_0|\phi_j\rangle|^2 = 0$, we can easily prove that the quantum system will maintain its state in this surface under only the action of the free Hamiltonian H_0. In fact, $|\psi(t)\rangle = e^{-iH_0 t}|\psi_0\rangle$, and we have

$$
\begin{aligned}
S(|\psi(t)\rangle, H) &= 1 - |\langle\psi(t)|\phi_j\rangle|^2 = 1 - |\langle\psi_0|e^{iH_0 t}|\phi_j\rangle|^2 \\
&= 1 - |\langle\psi_0|\phi_j\rangle e^{i\lambda_j t}|^2 = 1 - |\langle\psi_0|\phi_j\rangle|^2 |e^{i\lambda_j t}|^2 \\
&= 0.
\end{aligned}
\tag{7}
$$

That is, an eigenstate of H_0 can be identified as a sliding mode. For two-level quantum systems, we may select either $|0\rangle$ or $|1\rangle$ as a sliding mode. Without loss of generality, we identify the eigenstate $|0\rangle$ of a two-level quantum system as the sliding mode in this paper.

C. Problem Formulation

In Section II.B, we have identified an eigenstate $|0\rangle$ as a sliding mode. This means that if a quantum system is driven into the sliding mode, its state will be maintained in the sliding surface under the action of the free Hamiltonian. However, in practical applications, it is inevitable that there exist noises and uncertainties. In this paper, we suppose that the uncertainties can be approximately described as perturbations in the Hamiltonian. That is, the uncertainties can be denoted as $H_\Delta = \varepsilon_x(t)I_x + \varepsilon_y(t)I_y + \varepsilon_z(t)I_z$. The unitary errors in [30] belong to this class of uncertainties and uncertainties in one-qubit (one quantum bit) gate also correspond to this class of uncertainties [37]. Further, we assume the uncertainties are bounded; i.e.,

$$
\sqrt{\varepsilon_x^2(t) + \varepsilon_y^2(t) + \varepsilon_z^2(t)} \leq \varepsilon \quad (\varepsilon \geq 0).
\tag{8}
$$

When $\varepsilon = 0$, $H_\Delta = 0$. That is, there exist no uncertainties, which is trivial for our problem. Hence, in the following we assume $\varepsilon > 0$. An important advantage of classical sliding mode control is its robustness. Our main motivation for introducing sliding mode control to quantum

systems is to deal with uncertainties. We further suppose that the corresponding system without uncertainties is completely controllable and arbitrary unitary control operations can be generated. This assumption can be guaranteed for a two-level quantum system if we can realize arbitrary rotations along the z-axis and ζ-axis ($\zeta = x$ or y) (e.g., see [39] for details).

The control problem under consideration is stated as follows: design a control law to drive and then maintain the quantum system's state in a sliding mode domain even when bounded uncertainties exist in the system Hamiltonian. Here a sliding mode domain may be defined as $\mathscr{D} = \{|\psi\rangle : |\langle 0|\psi\rangle|^2 \geq 1 - p_0, 0 < p_0 < 1\}$, where p_0 is a given constant. Here we assume $p_0 \neq 0, 1$ since the case $p_0 = 0$ only occurs in the sliding mode surface and the case $p_0 = 1$ is always true. Hence, the two cases with $p_0 = 0$ and $p_0 = 1$ are trivial for our problem. The definition of the sliding mode domain implies that the system has a probability of at most p_0 (which we call the probability of failure) to collapse out of $\mathscr{D}$ when making a measurement. This behavior is quite different from that which occurs in traditional sliding mode control. Hence, we expect that our control laws will guarantee that the system's state remains in $\mathscr{D}$ except that a measurement operation may take it away from $\mathscr{D}$ with a small probability (not greater than p_0). The control problem considered in this paper includes three main subtasks: (i) for any initial state (assumed to be known), design a control law to drive the system's state into a defined sliding mode domain $\mathscr{D}$; (ii) design a control law to maintain the system's state in $\mathscr{D}$; (iii) design a control law to drive the system's state back to $\mathscr{D}$ if a measurement operation takes it away from $\mathscr{D}$. For simplicity, we suppose that there exist no uncertainties during the control processes (i) and (iii).

III. SLIDING MODE CONTROL BASED ON LYAPUNOV METHODS AND PROJECTIVE MEASUREMENTS

A. General Method

The first task is to design a control law to drive the controlled system to the chosen sliding mode domain $\mathscr{D}$. Lyapunov-based methods are widely used to accomplish this task in traditional sliding mode control. If the gradient of a Lyapunov function is negative in the neighborhood of the sliding surface, then the controlled system's state will be attracted to and maintained in $\mathscr{D}$. The Lyapunov methodology has also been used to design control laws for quantum systems [40]-[43]. However, these existing results do not consider the issue of robustness against uncertainties. Since the measurement of a quantum system will inevitably destroy the measured state, most

existing results on Lyapunov-based control for quantum systems in fact use a feedback design to construct an open-loop control. That is, Lyapunov-based control can be used to first design a feedback law which is then used to find the open-loop control by simulating the closed-loop system. Then the control can be applied to the quantum system in an open-loop way. Hence, the traditional sliding mode control methods using Lyapunov control cannot be directly applied to our problem.

Although quantum measurement often has deleterious effects in quantum control tasks, recent results have shown that it can be combined with unitary transformations to complete some quantum manipulation tasks and enhance the capability of quantum control [36], [44]-[48]. For example, Vilela Mendes and Man'ko [36] showed nonunitarily controllable systems can be made controllable by using "measurement plus evolution". Quantum measurement can be used as a control tool as well as a method of information acquisition. It is worth mentioning that the effect of measurement on a quantum system as a control tool can be achieved through the interaction between the system and measurement apparatus. In this paper, we will combine the Lyapunov methodology and projective measurements (with the measurement operator σ_z) to accomplish the sliding mode control task for two-level quantum systems.

The steps in the control algorithm are as follows (see Fig. 1):

1) Select an eigenstate $|0\rangle$ of H_0 as a sliding mode $S(|\psi\rangle, H) = 0$, and define the sliding mode domain as $\mathscr{D} = \{|\psi\rangle : |\langle 0|\psi\rangle|^2 \geq 1 - p_0\}$.

2) For a known initial state $|\psi_0\rangle$, construct a Lyapunov function $V(|\psi_0\rangle, S)$ to find the control law that can drive $|\psi_0\rangle$ into the sliding mode S.

3) For a specified probability of failure p_0 and $V(|\psi_0\rangle, S)$, construct the control period T_0 so that the control law can drive the system's state into $\mathscr{D}$ in a time period T_0.

4) For an initial condition which is another eigenstate $|1\rangle$, design a Lyapunov function $V(|1\rangle, S)$ and construct the period T_1 by using a similar method to that in 3).

5) According to p_0 and ε, design the period T for periodic projective measurements.

6) Use the designed control law to drive the system's state into $\mathscr{D}$ in T_0. Then repeat the following operations: make periodic projective measurements with the period T to maintain the system's state in $\mathscr{D}$; if the measurement result corresponds to $|1\rangle$, we use the corresponding control law to drive the state back into $\mathscr{D}$.

From the above control algorithm, we see that the design of Lyapunov functions and the
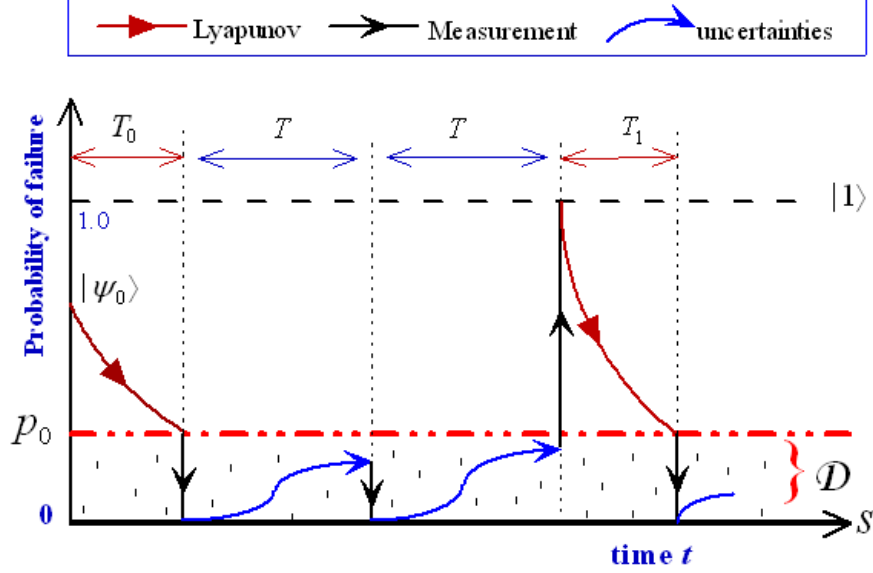
Fig. 1. The sliding mode control scheme for a two-level quantum system based on Lyapunov methods and periodic projective measurements. In this figure, "Lyapunov", "Measurement" and "uncertainties" mean the evolution process of the quantum system under the Lyapunov control law, the projective measurement and uncertainties in the system Hamiltonian, respectively.

selection of the period T for the projective measurements are the two most important tasks. To design a control law for quantum systems, several Lyapunov functions have been constructed, such as state distance-based and average value-based approaches [40]-[43]. Here we select a function based on the Hilbert-Schmidt distance between a state $|\psi\rangle$ and the sliding mode state $|\phi_j\rangle$ as a Lyapunov function [41], [43]; i.e.,

$$V(|\psi\rangle, S) = \frac{1}{2}(1 - |\langle\phi_j|\psi\rangle|^2).$$

It is clear that $V \geq 0$. The first-order time derivative of V is

$$\begin{aligned}
\dot{V} &= -\sum_{k=x,y,z} u_k \Im[\langle\psi|\phi_j\rangle\langle\phi_j|I_k|\psi\rangle] \\
&= -\sum_{k=x,y,z} u_k |\langle\psi|\phi_j\rangle| \Im[e^{i\angle\langle\psi|\phi_j\rangle}\langle\phi_j|I_k|\psi\rangle],
\end{aligned} \tag{9}$$

where $\Im[a+bi] = b$ $(a, b \in \mathbf{R})$ and $\angle c$ denotes the argument of a complex number c. To ensure $\dot{V} \leq 0$, we choose the control laws as in [43]:

$$u_k = K_k f_k(\Im[e^{i\angle\langle\psi|\phi_j\rangle}\langle\phi_j|I_k|\psi\rangle]), \quad (k = x, y, z) \tag{10}$$

where $K_k > 0$ may be used to adjust the control amplitude and $f(\cdot)$ satisfies $xf(x) \geq 0$. Define $\angle\langle\psi|\phi_j\rangle = 0°$ when $\langle\psi|\phi_j\rangle = 0$. LaSalle's invariance principle guarantees that the state of the

two-level quantum system converges to the sliding mode $|0\rangle$ under the control law constructed using the Lyapunov methodology (for details, see [43]).

Another important task is to design the measurement period T. We can estimate a bound according to the bound ε on the uncertainties and the allowed probability of failure p_0. Then, we construct a period T to guarantee control performance according to the estimated bound. An extreme case is $T \to 0$. That is, after the quantum system's state is driven into the sliding mode, we make frequent measurements. This corresponds to the quantum Zeno effect [49] and can guarantee that the state is maintained in the sliding mode in spite of the presence of uncertainties. However, it is usually a difficult task to make such frequent measurements. We may conclude that the smaller T is, the bigger the cost of accomplishing the periodic measurements becomes. Hence, we wish to design a measurement period T as large as possible. In the following subsection, we will propose two approaches of designing T for different situations.

B. The Design of the Measurement Period T

We select the sliding mode as $S(|\psi\rangle, H) = 1 - |\langle\psi|0\rangle|^2 = 0$. If there exist no uncertainties and we have driven the system's state to the sliding mode at time t_0, it will be maintained in the sliding mode using only the free Hamiltonian H_0; i.e., $S(|\psi_{(t \geq t_0)}\rangle, H_0) = 0$. That is, if the quantum system's state is driven into the sliding mode, it will evolve in the sliding surface. However, in practical applications, some uncertainties are unavoidable, which may drive the system's state away from the sliding mode. We wish to design a control law to ensure the desired robustness in the presence of uncertainties. Assume that the state at time t is ρ_t. If we make measurements on this system, the probability p that it will collapse into $|1\rangle$ (the probability of failure) is

$$p = \langle 1|\rho_t|1\rangle = \frac{1 - z_t}{2}. \tag{11}$$

In this paper, we have assumed that the possible uncertainties can be described by $H_\Delta = \varepsilon_x(t)I_x + \varepsilon_y(t)I_y + \varepsilon_z(t)I_z$, where unknown $\varepsilon_x(t)$, $\varepsilon_y(t)$ and $\varepsilon_z(t)$ satisfy $\sqrt{\varepsilon_x^2(t) + \varepsilon_y^2(t) + \varepsilon_z^2(t)} \leq \varepsilon$. We now give detailed discussions to design the measurement period T for possible uncertainties.

First we consider a special case $H_\Delta = \varepsilon(t)I_z$ ($|\varepsilon(t)| \leq \varepsilon$). This case corresponds to phase-flip

type bounded uncertainties. For any $H_\Delta = \varepsilon_z I_z$ (where $|\varepsilon_z| \le \varepsilon$), if $S(|\psi_0\rangle, H) = 0$, we have

$$
\begin{aligned}
S(|\psi(t)\rangle, H) &= 1 - |\langle\psi(t)|0\rangle|^2 \\
&= 1 - |\langle\psi_0|e^{i(H_0 + \varepsilon_z I_z)t}|0\rangle|^2 \\
&= 1 - |\langle\psi_0|0\rangle|^2 |e^{\frac{i(1+\varepsilon_z)}{2}t}|^2 \\
&= 0.
\end{aligned}
\tag{12}
$$

This type of uncertainty does not drive the system's state away from the sliding mode. Hence we ignore this type of uncertainty in our method.

Now we consider the unknown uncertainties $H_\Delta = \varepsilon_x(t)I_x + \varepsilon_y(t)I_y$ (where $\sqrt{\varepsilon_x^2(t) + \varepsilon_y^2(t)} \le \varepsilon$) and have the following theorem.

Theorem 1: For a two-level quantum system with the initial state $|\psi(0)\rangle = |0\rangle$ at the time $t = 0$, the system evolves to $|\psi(t)\rangle$ under the action of $H(t) = I_z + \varepsilon_x(t)I_x + \varepsilon_y(t)I_y$ (where $\sqrt{\varepsilon_x^2(t) + \varepsilon_y^2(t)} \le \varepsilon$ and $\varepsilon > 0$). If $t \in [0, T^{(1)}]$, where

$$
T^{(1)} = \frac{\arccos(1 - 2p_0)}{\varepsilon},
\tag{13}
$$

the system's state will remain in $\mathscr{D} = \{|\psi\rangle : |\langle 0|\psi\rangle|^2 \ge 1 - p_0\}$ (where $0 < p_0 < 1$). When one makes a projective measurement with the measurement operator σ_z at the time t, the probability of failure $p = |\langle 1|\psi(t)\rangle|^2$ is not greater than p_0.

Using Theorem 1, we may try to maintain the system's state in $\mathscr{D}$ (i.e., the subtask (ii)) by implementing periodic projective measurements with the measurement period $T = T^{(1)}$. If we have more knowledge about the uncertainties, it is possible to improve the measurement period $T^{(1)}$. Now assume that the uncertainty is $H_\Delta = \varepsilon(t)I_\zeta$ ($\zeta = x$ or y) and $p_0 \in (0, \frac{\varepsilon^2}{1+\varepsilon^2}]$. We have the following theorem.

Theorem 2: For a two-level quantum system with the initial state $|\psi(0)\rangle = |0\rangle$ at the time $t = 0$, the system evolves to $|\psi(t)\rangle$ under the action of $H(t) = I_z + \varepsilon(t)I_\zeta$ (where $\zeta = x$ or y, $|\varepsilon(t)| \le \varepsilon$ and $\varepsilon > 0$). If $p_0 \in (0, \frac{\varepsilon^2}{1+\varepsilon^2}]$ and $t \in [0, T^{(2)}]$, where

$$
T^{(2)} = \frac{\arccos[1 - 2(1 + \frac{1}{\varepsilon^2})p_0]}{\sqrt{1 + \varepsilon^2}},
\tag{14}
$$

the system's state will remain in $\mathscr{D} = \{|\psi\rangle : |\langle 0|\psi\rangle|^2 \ge 1 - p_0\}$ (where $0 < p_0 < 1$). When one makes a projective measurement with the measurement operator σ_z at the time t, the probability of failure $p = |\langle 1|\psi(t)\rangle|^2$ is not greater than p_0.

Remark 1: The proofs of Theorem 1 and Theorem 2 will be presented in Section IV. The two theorems mean the following fact. For a two-level quantum system with unknown uncertainties $H_\Delta = \varepsilon_x(t)I_x + \varepsilon_y(t)I_y$ (where $\sqrt{\varepsilon_x^2(t) + \varepsilon_y^2(t)} \leq \varepsilon$), if its initial state is in the sliding mode $|0\rangle$, we can ensure that the probability of failure is not greater than a given constant p_0 ($0 < p_0 < 1$) through implementing periodic projective measurements with the measurement period $T = T^{(1)}$ using (13). Further, if we know that p_0 and ε satisfy the relationship $0 < p_0 \leq \frac{\varepsilon^2}{1+\varepsilon^2}$ and there exists only one type of uncertainty (i.e., $H_\Delta = \varepsilon(t)I_x$ or $H_\Delta = \varepsilon(t)I_y$, where $|\varepsilon(t)| \leq \varepsilon$), we can design a measurement period $T = T^{(2)}$ using (14) which is larger than $T^{(1)}$. This measurement period will guarantee the required robustness. It is easy to prove the relationship $T^{(2)} \geq T^{(1)}$ for arbitrary $p_0 \in (0, \frac{\varepsilon^2}{1+\varepsilon^2}]$. The detailed proof will be presented in the Appendix. Based on the above analysis, the selection rule for T is summarized in Table I. Moreover, from (13) and (14), it is clear that for a constant ε, $T^{(1)} \to 0$ and $T^{(2)} \to 0$ when $p_0 \to 0$. That is, for a given bound ε on the uncertainties, if we expect to guarantee the probability of failure $p_0 \to 0$, it requires us to implement frequent measurements such that the measurement period $T \to 0$. This special case corresponds to the quantum Zeno effect [49]. Another special case is $\varepsilon \to +\infty$, which leads to $T^{(1)} \to 0$ and $T^{(2)} \to 0$. That is, to deal with very large uncertainties, we need to make frequent measurements ($T \to 0$) to guarantee the desired robustness. From (13), we also know that for a given p_0, $T^{(1)}$ monotonically decreases with increasing ε. This means that we need to employ a smaller measurement period to deal with uncertainties with a larger bound ε.

Type of uncertainties	$H_\Delta = \varepsilon_x(t)I_x + \varepsilon_y(t)I_y$	$H_\Delta = \varepsilon(t)I_\zeta$ ($\zeta = x$ or y)	
Allowed probability of failure p_0	$0 < p_0 < 1$	$0 < p_0 \leq \frac{\varepsilon^2}{1+\varepsilon^2}$	$\frac{\varepsilon^2}{1+\varepsilon^2} < p_0 < 1$
The measurement period T	$T = T^{(1)}$	$T = T^{(2)}$	$T = T^{(1)}$

TABLE I

A SUMMARY ON THE SELECTION RULE OF THE MEASUREMENT PERIOD T

C. An Illustrative Example

Now we present an illustrative example to demonstrate the proposed method. Assume $p_0 = 0.01$. Consider two cases: (a) $\varepsilon = 0.02$; (b) $\varepsilon = 0.2$. For simplicity, we assume $|\psi_0\rangle = |1\rangle$. Hence, $T_0 = T_1$. We first design the control and T_1 using (10). Here, we consider control only
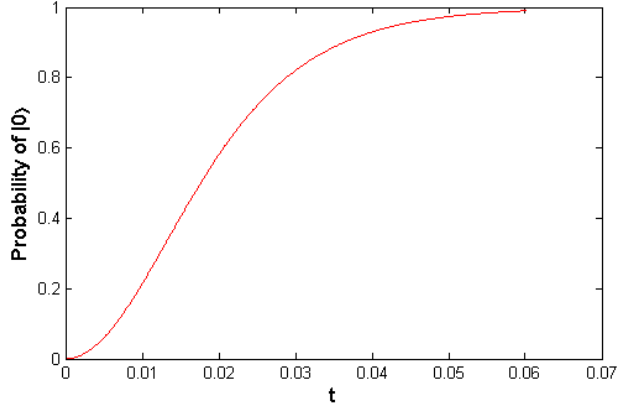
Fig. 2.   The probability of $|0\rangle$ under Lyapunov control.

using $H_u = \frac{1}{2}u(t)\sigma_y$. Using (10), we select $u(t) = K(\Im[e^{i\angle\langle\psi(t)|0\rangle}\langle0|\sigma_y|\psi(t)\rangle])$ and $K = 100$. Let the time stepsize be given by $\delta t = 10^{-4}$. We can obtain the probability curve of $|0\rangle$ shown in Fig. 2, the control value shown in Fig. 3 and $T_1 = 0.060$. For $\varepsilon = 0.02$, we can design the measurement period $T = T^{(1)} = 10.017$ using (13). For $\varepsilon = 0.2$, we can design the measurement period $T = T^{(1)} = 1.002$ using (13). Since $p' = \frac{\varepsilon^2}{1+\varepsilon^2} = 3.8 \times 10^{-2} > p_0$ when $\varepsilon = 0.2$, if the uncertainties take the form of $H_\Delta = \varepsilon(t)I_\zeta$ ($\zeta = x$ or y), we can improve the measurement period to $T = T^{(2)} = 1.049$ using (14). It is clear that $T \gg T_1$ in these two cases, which makes the assumption of no uncertainties in the control process reasonable. Moreover, the fact that the measurement period T is much greater than the control time required to go to $|0\rangle$ from $|1\rangle$ indicates the possibility of realizing such a periodic measurement on a practical quantum system.

Remark 2: In the process of designing the control law for driving the system's state from $|1\rangle$ to $|0\rangle$, we employ an approach based on the Lyapunov methodology. An advantage of such an approach is that it is relatively easy to find a control law by simulation. It is worth noting that most existing applications of the Lyapunov methodology to quantum systems do not involve measurement. Here, we combine the Lyapunov-based control and projective measurements for controlling quantum systems, which in some applications make our method more useful than the Lyapunov-based control for quantum systems proposed in previous papers. In [37], an approach based on time-optimal control design has also been proposed to complete this task. The advantage of such an approach is that we take the shortest time to complete the control task. However,
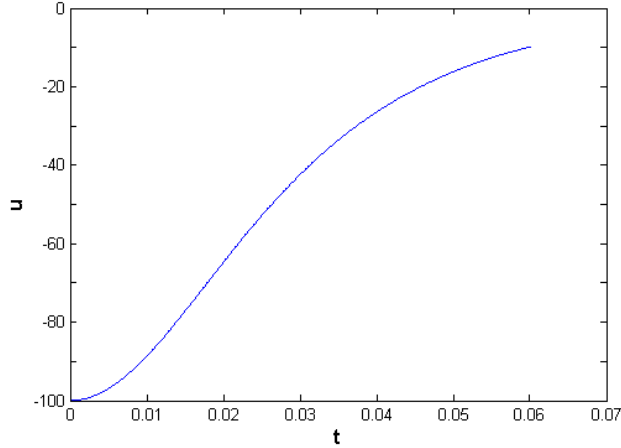
Fig. 3.   The control value $u(t)$.

it is generally difficult to find a complete time-optimal solution for high-dimensional quantum systems. For the above simple task, it has been proven that the time-optimal control employs a bang-bang control strategy [12]. Using the method in [12], we should take $u = -100$ in $t \in [0, 0.016]$ and then use $u = 100$ in $t \in (0.016, 0.030]$. In this case, the total time required is $T_1' = 0.030 \ (< T_1 = 0.060)$.

IV. PROOF OF THEOREMS

This section will present the detailed proofs of Theorem 1 and Theorem 2. Considering that the arguments in the proof of Theorem 2 are useful for the proof of Theorem 1, we will first present the proof of Theorem 2 and then prove Theorem 1.

A. Proof of Theorem 2

Proof: For $H^A = I_z + \varepsilon(t)I_x$, using $\dot{\rho} = -i[H^A, \rho]$ and (4), we have

$$\begin{pmatrix} \dot{z}_t & \dot{x}_t - i\dot{y}_t \\ \dot{x}_t + i\dot{y}_t & -\dot{z}_t \end{pmatrix} = \begin{pmatrix} \varepsilon(t)y_t & -y_t - ix_t + i\varepsilon(t)z_t \\ -y_t + ix_t - i\varepsilon(t)z_t & -\varepsilon(t)y_t \end{pmatrix}. \tag{15}$$

That is,

$$\begin{pmatrix} \dot{x}_t \\ \dot{y}_t \\ \dot{z}_t \end{pmatrix} = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & -\varepsilon(t) \\ 0 & \varepsilon(t) & 0 \end{pmatrix} \begin{pmatrix} x_t \\ y_t \\ z_t \end{pmatrix}, \tag{16}$$

where $(x_0, y_0, z_0) = (0, 0, 1)$.

We now consider $\varepsilon(t)$ as a control input and select the performance measure as

$$J(\varepsilon) = z_f. \tag{17}$$

Also, we introduce the Lagrange multiplier vector $\lambda(t) = (\lambda_1(t), \lambda_2(t), \lambda_3(t))^T$ and obtain the corresponding Hamiltonian function as follows:

$$\mathbb{H}(\mathbf{r}(t), \varepsilon(t), \lambda(t), t) \equiv \lambda^T(t) \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & -\varepsilon(t) \\ 0 & \varepsilon(t) & 0 \end{pmatrix} \begin{pmatrix} x_t \\ y_t \\ z_t \end{pmatrix}, \tag{18}$$

where $\mathbf{r}(t) = (x_t, y_t, z_t)$. That is

$$\mathbb{H}(\mathbf{r}(t), \varepsilon(t), \lambda(t), t) = -\lambda_1(t) y_t + \lambda_2(t) x_t + \varepsilon(t)(\lambda_3(t) y_t - \lambda_2(t) z_t). \tag{19}$$

According to Pontryagin's minimum principle [50], a necessary condition for $\varepsilon^*(t)$ to minimize $J(\varepsilon)$ is

$$\mathbb{H}(\mathbf{r}^*(t), \varepsilon^*(t), \lambda^*(t), t) \leq \mathbb{H}(\mathbf{r}^*(t), \varepsilon(t), \lambda^*(t), t). \tag{20}$$

Hence, if we do not consider singular cases (i.e., $\lambda_3(t) y_t - \lambda_2(t) z_t \equiv 0$), the optimal control $\varepsilon^*(t)$ should be chosen as follows:

$$\varepsilon^*(t) = -\bar{\varepsilon} \mathrm{sgn}(\lambda_3(t) y_t - \lambda_2(t) z_t). \tag{21}$$

That is, the optimal control strategy for $\varepsilon(t)$ is bang-bang control; i.e., $\varepsilon^*(t) = \bar{\varepsilon} = +\varepsilon$ or $-\varepsilon$. Now we consider $H^B = I_z + \bar{\varepsilon} I_x$ which leads to the state equation

$$\begin{pmatrix} \dot{x}_t \\ \dot{y}_t \\ \dot{z}_t \end{pmatrix} = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & -\bar{\varepsilon} \\ 0 & \bar{\varepsilon} & 0 \end{pmatrix} \begin{pmatrix} x_t \\ y_t \\ z_t \end{pmatrix}, \tag{22}$$

where $(x_0, y_0, z_0) = (0, 0, 1)$. The corresponding solution is

$$\begin{pmatrix} x_t \\ y_t \\ z_t \end{pmatrix} = \begin{pmatrix} -\frac{\bar{\varepsilon}}{1+\varepsilon^2} \cos \omega t + \frac{\bar{\varepsilon}}{1+\varepsilon^2} \\ -\frac{\bar{\varepsilon}}{\sqrt{1+\varepsilon^2}} \sin \omega t \\ \frac{\varepsilon^2}{1+\varepsilon^2} \cos \omega t + \frac{1}{1+\varepsilon^2} \end{pmatrix} \tag{23}$$

where $\omega = \sqrt{1+\varepsilon^2}$. From (23), we know that z_t is a monotonically decreasing function in t when $t \in [0, \frac{\pi}{\sqrt{1+\varepsilon^2}}]$. Hence, we only consider the case $t \in [0, t_f]$ where $t_f \in [0, \frac{\pi}{\sqrt{1+\varepsilon^2}}]$.

Now consider the optimal control problem with a fixed final time t_f and a free final state $\mathbf{r}_f = (x_f, y_f, z_f)$. According to Pontryagin's minimum principle, $\lambda^*(t_f) = \frac{\partial}{\partial \mathbf{r}} r^*(t_f)$. From this, it is straightforward to verify that $(\lambda_1(t_f), \lambda_2(t_f), \lambda_3(t_f)) = (0, 0, 1)$. Now let us consider another necessary condition $\dot{\lambda}(t) = -\frac{\partial \mathbb{H}(\mathbf{r}(t), \varepsilon(t), \lambda(t), t)}{\partial \mathbf{r}}$ which leads to the following relationships:

$$\dot{\lambda}(t) = \begin{pmatrix} \dot{\lambda}_1(t) \\ \dot{\lambda}_2(t) \\ \dot{\lambda}_3(t) \end{pmatrix} = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & -\bar{\varepsilon} \\ 0 & \bar{\varepsilon} & 0 \end{pmatrix} \begin{pmatrix} \lambda_1(t) \\ \lambda_2(t) \\ \lambda_3(t) \end{pmatrix}, \tag{24}$$

where $(\lambda_1(t_f), \lambda_2(t_f), \lambda_3(t_f)) = (0, 0, 1)$. The corresponding solution is

$$\begin{pmatrix} \lambda_1(t) \\ \lambda_2(t) \\ \lambda_3(t) \end{pmatrix} = \begin{pmatrix} -\frac{\bar{\varepsilon}}{1+\varepsilon^2} \cos \omega(t_f - t) + \frac{\bar{\varepsilon}}{1+\varepsilon^2} \\ \frac{\bar{\varepsilon}}{\sqrt{1+\varepsilon^2}} \sin \omega(t_f - t) \\ \frac{\varepsilon^2}{1+\varepsilon^2} \cos \omega(t_f - t) + \frac{1}{1+\varepsilon^2} \end{pmatrix}. \tag{25}$$

We obtain

$$\lambda_3(t) y_t - \lambda_2(t) z_t = \frac{-\bar{\varepsilon}}{\omega^3}[\sin \omega t + \varepsilon^2 \sin \omega t_f + \sin \omega(t_f - t)]. \tag{26}$$

It is easy to show that the quantity $(\lambda_3(t) y_t - \lambda_2(t) z_t)$ occurring in (21) does not change its sign when $t_f \in [0, \frac{\pi}{\sqrt{1+\varepsilon^2}}]$ and $t \in [0, t_f]$.

Now we further exclude the possibility that there exists a singular case. Suppose that there exists a singular interval $[t_0, t_1]$ (where $t_0 \geq 0$) such that when $t \in [t_0, t_1]$

$$h(t) = \lambda_3(t) y_t - \lambda_2(t) z_t \equiv 0. \tag{27}$$

We also have the following relationship

$$\dot{h}(t) = \lambda_3(t) x_t - \lambda_1(t) z_t \equiv 0 \tag{28}$$

where we have used (16) and the following costate equation

$$\dot{\lambda}(t) = \begin{pmatrix} \dot{\lambda}_1(t) \\ \dot{\lambda}_2(t) \\ \dot{\lambda}_3(t) \end{pmatrix} = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & -\varepsilon(t) \\ 0 & \varepsilon(t) & 0 \end{pmatrix} \begin{pmatrix} \lambda_1(t) \\ \lambda_2(t) \\ \lambda_3(t) \end{pmatrix}. \tag{29}$$

If $t_0 = 0$, we have $(x_0, y_0, z_0) = (0, 0, 1)$. By the principle of optimality [50], we may consider the case $t_f = t_1$. Using (27), (28) and $(\lambda_1(t_1), \lambda_2(t_1), \lambda_3(t_1)) = (0, 0, 1)$, we have $x_{t_1} = 0$ and $y_{t_1} = 0$. Using the relationship of $x_t^2 + y_t^2 + z_t^2 = 1$, we obtain $z_{t_1} = 1$ or -1. If $z_{t_1} = 1$, the initial

state and the final state are the same state $|0\rangle$. However, if we use the control $\varepsilon(t) = \bar{\varepsilon}$, from (23) we have $z_{t_1}(\bar{\varepsilon}) = \frac{\varepsilon^2}{1+\varepsilon^2}\cos\omega t_1 + \frac{1}{1+\varepsilon^2} < z_{t_1} = 1$. Hence, this contradicts the fact that we are considering the optimal case $\min z_f$. If $z_{t_1} = -1$, there exists $0 < \tilde{t}_1 < t_1$ such that $z_{\tilde{t}_1} = 0$. By the principle of optimality [50], we may consider the case $t_f = \tilde{t}_1$. From the two equations (27) and (28), we know that $z_{\tilde{t}_1}^2 = 1$ which contradicts $z_{\tilde{t}_1} = 0$. Hence, no singular condition can exist if $t_0 = 0$.

If $t_0 > 0$, using (21) we must select $\varepsilon(t) = \bar{\varepsilon}$ when $t \in [0, t_0]$. From (26), we know that there exist no $t_0 \in (0, t_f)$ satisfying $\lambda_3(t_0)y_{t_0} - \lambda_2(t_0)z_{t_0} = 0$. Hence, there exist no singular cases for our problem.

From the above analysis, $\varepsilon(t) = \bar{\varepsilon}$ is the optimal control when $t \in [0, \frac{\pi}{\sqrt{1+\varepsilon^2}}]$. Hence $z_t^A = z_t(\varepsilon(t)) \geq z_t(\bar{\varepsilon}) = z_t^B$. From (11), it is clear that the probabilities of failure satisfy $p_t^A = \frac{1-z_t^A}{2} \leq p_t^B = \frac{1-z_t^B}{2}$. That is, the probability of failure p_t^A is not greater than p_t^B for $t \in [0, \frac{\pi}{\sqrt{1+\varepsilon^2}}]$. When $t \in [0, \frac{\pi}{\sqrt{1+\varepsilon^2}}]$, z_t^B is monotonically decreasing and p_t^B is monotonically increasing. When $t = \frac{\pi}{\sqrt{1+\varepsilon^2}}$, using (23) we have $z_t^B = \frac{1-\varepsilon^2}{1+\varepsilon^2}$. That is, the probability of failure $p' = \frac{\varepsilon^2}{1+\varepsilon^2}$. Hence, we can design the measurement period T using the case of H^B when $0 < p_0 \leq \frac{\varepsilon^2}{1+\varepsilon^2}$.

Using (11) and (23), for $t \in [0, \frac{\pi}{\sqrt{1+\varepsilon^2}}]$ we obtain the probability of failure

$$p_t^B = \frac{\varepsilon^2}{1+\varepsilon^2} \frac{1-\cos\omega t}{2}. \tag{30}$$

Hence, we can design the maximum measurement period as follows

$$T^{(2)} = \frac{\arccos[1 - 2(1 + \frac{1}{\varepsilon^2})p_0]}{\sqrt{1+\varepsilon^2}}, \tag{31}$$

For $H_\Delta = \varepsilon(t)I_y$ (where $|\varepsilon(t)| \leq \varepsilon$), we can obtain the same conclusion as that in the case $H_\Delta = \varepsilon(t)I_x$ (where $|\varepsilon(t)| \leq \varepsilon$). ∎

B. Proof of Theorem 1

To prove Theorem 1, we first prove two lemmas (Lemma 3 and Lemma 5).

Lemma 3: For a two-level quantum system with the initial state $(x_0, y_0, z_0) = (0, 0, 1)$ (i.e., $|0\rangle$), the system evolves to (x_t^A, y_t^A, z_t^A) and (x_t^B, y_t^B, z_t^B) under the action of $H^A = I_z + \varepsilon_0\cos\gamma_0 I_x + \varepsilon_0\sin\gamma_0 I_y$ (ε_0 is a nonzero constant) and $H^B = \varepsilon_0\cos\gamma_0 I_x + \varepsilon_0\sin\gamma_0 I_y$, respectively. For arbitrary $t \in [0, \frac{\pi}{|\varepsilon_0|}]$, $z_t^A \geq z_t^B$.

Proof: For the system with Hamiltonian $H^A = I_z + \varepsilon_0 \cos \gamma_0 I_x + \varepsilon_0 \sin \gamma_0 I_y$, using $\dot{\rho} = -i[H, \rho]$ and (4), we obtain the following state equations

$$
\begin{pmatrix} \dot{x}_t^A \\ \dot{y}_t^A \\ \dot{z}_t^A \end{pmatrix} = \begin{pmatrix} 0 & -1 & \varepsilon_0 \sin \gamma_0 \\ 1 & 0 & -\varepsilon_0 \cos \gamma_0 \\ -\varepsilon_0 \sin \gamma_0 & \varepsilon_0 \cos \gamma_0 & 0 \end{pmatrix} \begin{pmatrix} x_t^A \\ y_t^A \\ z_t^A \end{pmatrix},
\tag{32}
$$

where $(x_0^A, y_0^A, z_0^A) = (0, 0, 1)$. The corresponding solution is as follows

$$
\begin{pmatrix} x_t^A \\ y_t^A \\ z_t^A \end{pmatrix} = \begin{pmatrix} \frac{\varepsilon_0 \sin \gamma_0}{\sqrt{1+\varepsilon_0^2}} \sin \omega_0 t - \frac{\varepsilon_0 \cos \gamma_0}{1+\varepsilon_0^2} \cos \omega_0 t + \frac{\varepsilon_0 \cos \gamma_0}{1+\varepsilon_0^2} \\ -\frac{\varepsilon_0 \cos \gamma_0}{\sqrt{1+\varepsilon_0^2}} \sin \omega_0 t - \frac{\varepsilon_0 \sin \gamma_0}{1+\varepsilon_0^2} \cos \omega_0 t + \frac{\varepsilon_0 \sin \gamma_0}{1+\varepsilon_0^2} \\ \frac{\varepsilon_0^2}{1+\varepsilon_0^2} \cos \omega_0 t + \frac{1}{1+\varepsilon_0^2} \end{pmatrix}.
\tag{33}
$$

where $\omega_0 = \sqrt{1+\varepsilon_0^2}$.

For the system with Hamiltonian $H^B = \varepsilon_0 \cos \gamma_0 I_x + \varepsilon_0 \sin \gamma_0 I_y$, using $\dot{\rho} = -i[H, \rho]$ and (4), we obtain the following state equations

$$
\begin{pmatrix} \dot{x}_t^B \\ \dot{y}_t^B \\ \dot{z}_t^B \end{pmatrix} = \begin{pmatrix} 0 & 0 & \varepsilon_0 \sin \gamma_0 \\ 0 & 0 & -\varepsilon_0 \cos \gamma_0 \\ -\varepsilon_0 \sin \gamma_0 & \varepsilon_0 \cos \gamma_0 & 0 \end{pmatrix} \begin{pmatrix} x_t^B \\ y_t^B \\ z_t^B \end{pmatrix},
\tag{34}
$$

where $(x_0^B, y_0^B, z_0^B) = (0, 0, 1)$. We can obtain the corresponding solution as

$$
\begin{pmatrix} x_t^B \\ y_t^B \\ z_t^B \end{pmatrix} = \begin{pmatrix} \sin \gamma_0 \sin \varepsilon_0 t \\ -\cos \gamma_0 \sin \varepsilon_0 t \\ \cos \varepsilon_0 t \end{pmatrix}.
\tag{35}
$$

Since $\cos \varepsilon_0 t = \cos(-\varepsilon_0 t)$, we may first assume $\varepsilon_0 > 0$. We define $F(t)$ and $f(t)$ as follows.

$$
F(t) = z_t^A - z_t^B,
\tag{36}
$$

$$
f(t) = F'(t) = \varepsilon_0 \sin \varepsilon_0 t - \frac{\varepsilon_0^2}{\sqrt{1+\varepsilon_0^2}} \sin \omega_0 t.
\tag{37}
$$

Now we consider $t \in [0, \frac{\pi}{\sqrt{1+\varepsilon_0^2}}]$, and obtain

$$
f'(t) = \varepsilon_0^2 (\cos \varepsilon_0 t - \cos \omega_0 t) = 2\varepsilon_0^2 \sin \frac{\omega_0 + \varepsilon_0}{2} t \sin \frac{\omega_0 - \varepsilon_0}{2} t \geq 0
\tag{38}
$$

It is clear that $f'(t) = 0$ only when $t = 0$. Hence $f(t)$ is a monotonically increasing function and

$$\min_t f(t) = f(0) = 0.$$

Hence, we have

$$f(t) \geq 0. \tag{39}$$

From this, it is clear that $F(t)$ is a monotonically increasing function and

$$\min_t F(t) = F(0) = 0.$$

Hence $F(t) \geq 0$ when $t \in [0, \frac{\pi}{\sqrt{1+\varepsilon_0^2}}]$. Moreover, it is clear that $\min_t z_t^A = z_t^A |_{t=\frac{\pi}{\sqrt{1+\varepsilon_0^2}}}$ and $z_t^B = \cos\varepsilon_0 t$ is a monotonically decreasing function when $t \in [0, \frac{\pi}{|\varepsilon_0|}]$. It is easy to obtain the following relationship

$$z_t^B |_{t\in[\frac{\pi}{\sqrt{1+\varepsilon_0^2}}, \frac{\pi}{|\varepsilon_0|}]} \leq z_t^B |_{t=\frac{\pi}{\sqrt{1+\varepsilon_0^2}}} < z_t^A |_{t=\frac{\pi}{\sqrt{1+\varepsilon_0^2}}}.$$

Hence we can conclude that $z_t^A \geq z_t^B$ for arbitrary $t \in [0, \frac{\pi}{|\varepsilon_0|}]$.

∎

Let $\gamma_0 = 0$, and we have the following corollary.

Corollary 4: For a two-level quantum system with the initial state $(x_0, y_0, z_0) = (0, 0, 1)$ (i.e., $|0\rangle$), the system evolves to (x_t^A, y_t^A, z_t^A) and (x_t^B, y_t^B, z_t^B) under the action of $H^A = I_z + \varepsilon_0 I_x$ (where ε_0 is a nonzero constant) and $H^B = \varepsilon_0 I_x$, respectively. For arbitrary $t \in [0, \frac{\pi}{|\varepsilon_0|}]$, $z_t^A \geq z_t^B$.

We now present another lemma.

Lemma 5: For a two-level quantum system with the initial state $(x_0, y_0, z_0) = (0, 0, 1)$ (i.e., $|0\rangle$), the system evolves to (x_t^A, y_t^A, z_t^A) and (x_t^B, y_t^B, z_t^B) under the action of $H^A = I_z + \varepsilon(t)I_x$ (where $|\varepsilon(t)| \leq \varepsilon$) and $H^B = \varepsilon I_x$, respectively. For arbitrary $t \in [0, \frac{\pi}{\varepsilon}]$, $z_t^A \geq z_t^B$.

Proof: First, we take an arbitrary evolution state (except $|1\rangle$) starting from $|0\rangle$ as a new initial state. For $H^B = \varepsilon I_x$, the initial state can be represented as $(x_0', y_0', z_0) = (0, -\sin\theta_0, \cos\theta_0)$, where $\theta_0 \in [0, \pi)$. We have

$$\begin{pmatrix} \dot{x}_t^B \\ \dot{y}_t^B \\ \dot{z}_t^B \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -\varepsilon \\ 0 & \varepsilon & 0 \end{pmatrix} \begin{pmatrix} x_t^B \\ y_t^B \\ z_t^B \end{pmatrix}. \tag{40}$$

The corresponding solution is as follows:

$$\begin{pmatrix} x_t^B \\ y_t^B \\ z_t^B \end{pmatrix} = \begin{pmatrix} 0 \\ -z_0 \sin \varepsilon t + y_0' \cos \varepsilon t \\ z_0 \cos \varepsilon t + y_0' \sin \varepsilon t \end{pmatrix}. \tag{41}$$

For $H^A = I_z + \varepsilon(t)I_x$, we have

$$\begin{pmatrix} \dot{x}_t^A \\ \dot{y}_t^A \\ \dot{z}_t^A \end{pmatrix} = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & -\varepsilon(t) \\ 0 & \varepsilon(t) & 0 \end{pmatrix} \begin{pmatrix} x_t^A \\ y_t^A \\ z_t^A \end{pmatrix}, \tag{42}$$

where $(x_0^A, y_0^A, z_0^A) = (x_0, y_0, z_0) = (\sin \theta_0 \cos \varphi_0, \sin \theta_0 \sin \varphi_0, \cos \theta_0)$ and $\varphi_0 \in [0, 2\pi]$. From (42), we have

$$\dot{z}_t|_{t=0} = \lim_{\Delta t \to 0} \frac{z_{\Delta t}^A - z_0}{\Delta t} = \varepsilon(0)y_0 = \varepsilon(0) \sin \theta_0 \sin \varphi_0. \tag{43}$$

From (43) and (41), it is easy to obtain the following relationship:

$$\begin{aligned} z_{\Delta t}^A - z_{\Delta t}^B &= z_0 + \varepsilon(0) \sin \theta_0 \sin \varphi_0 \Delta t - z_0 [1 - \frac{\varepsilon^2 (\Delta t)^2}{2}] + \sin \theta_0 \varepsilon \Delta t + O((\Delta t)^2) \\ &= \Delta t \sin \theta_0 (\varepsilon + \varepsilon(0) \sin \varphi_0) + O((\Delta t)^2). \end{aligned} \tag{44}$$

When $\theta_0 \in [0, \pi)$, $\sin \theta_0 \geq 0$. Moreover, it is always true that $\varepsilon + \varepsilon(0) \sin \varphi_0 \geq 0$. If $\theta_0 \neq 0$ and $\varepsilon + \varepsilon(0) \sin \varphi_0 \neq 0$, we have

$$z_{\Delta t}^A > z_{\Delta t}^B. \tag{45}$$

If $\theta_0 = 0$, we have $(x_0', y_0', z_0) = (x_0, y_0, z_0) = (0, 0, 1)$. According to Corollary 4 and the proof of Theorem 2, we have

$$z_{\Delta t}^A \geq z_{\Delta t}^B. \tag{46}$$

For $\varepsilon + \varepsilon(0) \sin \varphi_0 = 0$, it corresponds to two cases: (a) $\varepsilon(0) = \varepsilon$ and $\varphi_0 = \frac{3\pi}{2}$; (b) $\varepsilon(0) = -\varepsilon$ and $\varphi_0 = \frac{\pi}{2}$. In the following, we consider the case (a) (for the case (b) we have the same conclusion as the case (a)). Since $\varphi_0 = \frac{3\pi}{2}$, $(x_0, y_0, z_0) = (0, -\sin \theta_0, \cos \theta_0)$. Using a similar argument to the proof of Theorem 2, we know that for $H^A = I_z + \varepsilon(t)I_z$ with $(x_0, y_0, z_0) = (0, -\sin \theta_0, \cos \theta_0)$, the optimal control for the performance index $J(\varepsilon) = z_f$ takes a form of bang-bang control $\varepsilon(t) = \bar{\varepsilon} = \varepsilon$ or $-\varepsilon$. So we only need to consider a bang-bang strategy.

For such a bang-bang strategy as $H^A = I_z + \bar{\varepsilon} I_x$, we have

$$\begin{pmatrix} \dot{x}_t^A \\ \dot{y}_t^A \\ \dot{z}_t^A \end{pmatrix} = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & -\bar{\varepsilon} \\ 0 & \bar{\varepsilon} & 0 \end{pmatrix} \begin{pmatrix} x_t^A \\ y_t^A \\ z_t^A \end{pmatrix}, \tag{47}$$

where $(x_0^A, y_0^A, z_0^A) = (x_0, y_0, z_0) = (0, -\sin\theta_0, \cos\theta_0)$ and $\varphi_0 \in [0, 2\pi]$. We can obtain the corresponding solution as

$$\begin{pmatrix} x_t^A \\ y_t^A \\ z_t^A \end{pmatrix} = \begin{pmatrix} \frac{-\bar{\varepsilon}\cos\theta_0}{1+\varepsilon^2}\cos\omega t + \frac{\sin\theta_0}{\sqrt{1+\varepsilon^2}}\sin\omega t + \frac{\bar{\varepsilon}\cos\theta_0}{1+\varepsilon^2} \\ \frac{-\bar{\varepsilon}\cos\theta_0}{\sqrt{1+\varepsilon^2}}\sin\omega t - \sin\theta_0\cos\omega t \\ \frac{\varepsilon^2\cos\theta_0}{1+\varepsilon^2}\cos\omega t - \frac{\bar{\varepsilon}\sin\theta_0}{\sqrt{1+\varepsilon^2}}\sin\omega t + \frac{\cos\theta_0}{1+\varepsilon^2} \end{pmatrix}. \tag{48}$$

Now, we consider the limit as $\Delta t \to 0$ and obtain

$$\begin{aligned}
z_{\Delta t}^A - z_{\Delta t}^B &= \frac{\varepsilon^2\cos\theta_0}{1+\varepsilon^2}[1 - \frac{(1+\varepsilon^2)(\Delta t)^2}{2}] - \frac{\bar{\varepsilon}\sin\theta_0}{\sqrt{1+\varepsilon^2}}\sqrt{1+\varepsilon^2}\Delta t \\
&\quad + \frac{\cos\theta_0}{1+\varepsilon^2} - \cos\theta_0[1 - \frac{\varepsilon^2(\Delta t)^2}{2}] + \sin\theta_0\varepsilon\Delta t - \frac{\varepsilon^3}{6}(\Delta t)^3\sin\theta_0 + O((\Delta t)^4) \\
&= \Delta t\sin\theta_0(\varepsilon - \bar{\varepsilon}) + \sin\theta_0(\Delta t)^3(\bar{\varepsilon} + \bar{\varepsilon}\varepsilon^2 - \varepsilon^3) + O((\Delta t)^4) \\
&> 0.
\end{aligned} \tag{49}$$

Hence, for arbitrary $z_0 = \cos\theta_0$ ($\theta_0 \in [0, \pi)$), we have

$$z_{\Delta t}^A \geq z_{\Delta t}^B. \tag{50}$$

For $t = \frac{\pi}{\varepsilon}$, $z_t^B = -1$. Hence the relationship $z_t^A \geq z_t^B$ is always true.

We now define $g(t) = z_t^A - z_t^B$ and assume that there exist $t = t_1 \in [0, \frac{\pi}{\varepsilon})$ such that $z_{t_1}^A < z_{t_1}^B$. That is, $g(t_1) < 0$. Since $g(t)$ is continuous in t and $g(0) = 0$, there exists a time $t^* = \sup\{t | 0 \leq t < t_1, g(t) = 0\}$ satisfying $g(t) < 0$ for $t \in (t^*, t_1]$. However, we have proven that for any $z_t^A = z_t^B$ and $\Delta t \to 0$, $z_{t+\Delta t}^A \geq z_{t+\Delta t}^B$, which contradicts $g(t) < 0$ for $t \in (t^*, t_1]$. Hence, we have the following relationship for $t \in [0, \frac{\pi}{\varepsilon}]$

$$z_t^A \geq z_t^B. \tag{51}$$

∎

Now we can prove Theorem 1 using Lemma 3 and Lemma 5.

Proof: For $H^A = I_z + \varepsilon_x(t)I_x + \varepsilon_y(t)I_y$, using $\dot{\rho} = -i[H, \rho]$ and (4), we can obtain the following state equations

$$\begin{pmatrix} \dot{x}_t^A \\ \dot{y}_t^A \\ \dot{z}_t^A \end{pmatrix} = \begin{pmatrix} 0 & -1 & \varepsilon_y(t) \\ 1 & 0 & -\varepsilon_x(t) \\ -\varepsilon_y(t) & \varepsilon_x(t) & 0 \end{pmatrix} \begin{pmatrix} x_t^A \\ y_t^A \\ z_t^A \end{pmatrix}, \tag{52}$$

where $(x_0^A, y_0^A, z_0^A) = (0,0,1)$.

Define $\varepsilon(t) = \sqrt{\varepsilon_x^2(t) + \varepsilon_y^2(t)}$ and $\varepsilon_x(t) = \varepsilon(t)\cos\gamma_t$, $\varepsilon_y(t) = \varepsilon(t)\sin\gamma_t$. This leads to the following equation

$$
\begin{pmatrix} \dot{x}_t^A \\ \dot{y}_t^A \\ \dot{z}_t^A \end{pmatrix} = \begin{pmatrix} 0 & -1 & \varepsilon(t)\sin\gamma_t \\ 1 & 0 & -\varepsilon(t)\cos\gamma_t \\ -\varepsilon(t)\sin\gamma_t & \varepsilon(t)\cos\gamma_t & 0 \end{pmatrix} \begin{pmatrix} x_t^A \\ y_t^A \\ z_t^A \end{pmatrix} \tag{53}
$$

where $(x_0^A, y_0^A, z_0^A) = (\sin\theta_0\cos\varphi_0, \sin\theta_0\sin\varphi_0, \cos\theta_0)$ and $\varphi_0 \in [0, 2\pi]$. From (53), we have

$$
\dot{z}_t|_{t=0} = \lim_{\Delta t \to 0} \frac{z_{\Delta t}^A - z_0}{\Delta t} = \varepsilon(0)\cos\gamma_0 y_0 - \varepsilon(0)\sin\gamma_0 x_0 \tag{54}
$$

From (54) and (41), it is easy to obtain the following relationship:

$$
\begin{aligned}
z_{\Delta t}^A - z_{\Delta t}^B &= z_0 + \varepsilon(0)\cos\gamma_0 y_0\Delta t - \varepsilon(0)\sin\gamma_0 x_0\Delta t \\
&\quad -z_0[1 - \tfrac{\varepsilon^2(\Delta t)^2}{2}] + \sin\theta_0\varepsilon\Delta t + O((\Delta t)^2) \\
&= \Delta t\sin\theta_0(\varepsilon + \varepsilon(0)\sin(\varphi_0 - \gamma_0)) + O((\Delta t)^2).
\end{aligned} \tag{55}
$$

When $\theta_0 \in [0, \pi)$, $\sin\theta_0 \geq 0$. Moreover, it is always true that $\varepsilon + \varepsilon(0)\sin(\varphi_0 - \gamma_0) \geq 0$. If $\theta_0 \neq 0$ and $\varepsilon + \varepsilon(0)\sin(\varphi_0 - \gamma_0) \neq 0$, we have

$$
z_{\Delta t}^A > z_{\Delta t}^B. \tag{56}
$$

If $\theta_0 = 0$, we have $(x_0', y_0', z_0) = (x_0, y_0, z_0) = (0,0,1)$. According to Lemma 3 and Pontryagin's minimum principle, we have

$$
z_{\Delta t}^A \geq z_{\Delta t}^B. \tag{57}
$$

For $\varepsilon + \varepsilon(0)\sin(\varphi_0 - \gamma_0) = 0$, it must be true that $\varepsilon(0) = \varepsilon$ or $-\varepsilon$. We consider $\varepsilon(0) = \varepsilon$ (for $\varepsilon(0) = -\varepsilon$ we have the same conclusion as $\varepsilon(0) = \varepsilon$). Moreover, we have $\gamma_0 = \varphi_0 + \frac{\pi}{2}$ or $\gamma_0 = \varphi_0 - \frac{3\pi}{2}$.

For $\gamma_0 = \varphi_0 + \frac{\pi}{2}$, it is easy to prove the relationship $z_{\Delta t}^A \geq z_{\Delta t}^B$ using Lemma 5 and Pontryagin's minimum principle. For $\gamma_0 = \varphi_0 - \frac{3\pi}{2}$, we have the same conclusion as in the case $\gamma_0 = \varphi_0 + \frac{\pi}{2}$. Thus, we obtain

$$
z_{\Delta t}^A \geq z_{\Delta t}^B. \tag{58}
$$

Now using a similar argument to that in the proof of Lemma 5, for arbitrary $t \in [0, \frac{\pi}{\varepsilon})$, we have

$$
z_t^A \geq z_t^B. \tag{59}
$$

From (11), it is clear that the probabilities of failure satisfy $p_t^A = \frac{1-z_t^A}{2} \leq p_t^B = \frac{1-z_t^B}{2}$. That is, the probability of failure p_t^A is not greater than p_t^B for $t \in [0, \frac{\pi}{\varepsilon})$. Hence, we can design the measurement period T using the case of H^B.

Using (11) and (41), for $t \in [0, \frac{\pi}{\varepsilon})$, we obtain the probability of failure

$$p_t^B = \frac{1 - \cos \varepsilon t}{2}. \tag{60}$$

Hence, we can design the maximum measurement period as follows

$$T^{(1)} = \frac{\arccos(1 - 2p_0)}{\varepsilon}. \tag{61}$$

∎

V. CONCLUSIONS

This paper proposes a variable structure control scheme with sliding modes for the robust control of two-level quantum systems where an eigenstate is identified as a sliding mode. We present a design method for the control laws based on a Lyapunov methodology and periodic projective measurements to drive and maintain this system's state in the sliding mode domain. The key task of the control problem is converted into a problem of designing the Lyapunov functions and the measurement period. The Lyapunov function can be constructed to define a control law. By using simulation, we obtain an open-loop control to drive the controlled quantum system's state into the sliding mode domain. For different situations of the uncertainties in the system Hamiltonian, we give two approaches to design the measurement period, which guarantees control performance in the presence of the uncertainties. This sliding mode control scheme provides a robust quantum engineering strategy for controller design for quantum systems and has potential applications in state preparation, decoherence control, quantum error correction [37], etc. Future work which can be carried out in this area includes the physical implementation of the proposed method on specific quantum systems, the extension from two-level systems to more complex quantum systems, and the exploration on practical applications of the proposed approaches.

APPENDIX: PROOF OF $T^{(2)} \geq T^{(1)}$

Proof: Take p_0 as the variable and define

$$F(p_0) = T^{(2)} - T^{(1)} \tag{62}$$

For $p_0 \in (0, \frac{\varepsilon^2}{1+\varepsilon^2})$, we have the following relationship

$$f(p_0) = F'(p_0) = \frac{1}{\sqrt{\varepsilon^2 p_0 - (1+\varepsilon^2)p_0^2}} - \frac{1}{\sqrt{\varepsilon^2 p_0 - \varepsilon^2 p_0^2}} \tag{63}$$

It is clear from (62) and (63) that $f(p_0) > 0$ for $p_0 \in (0, \frac{\varepsilon^2}{1+\varepsilon^2})$ and $F(p_0 \to 0^+) = 0$. Hence $F(p_0) \geq 0$ for $p_0 \in (0, \frac{\varepsilon^2}{1+\varepsilon^2})$.

When $p_0 = p' = \frac{\varepsilon^2}{1+\varepsilon^2}$,

$$T^{(1)}(p') = \frac{\arccos\left(\frac{1-\varepsilon^2}{1+\varepsilon^2}\right)}{\varepsilon}, \tag{64}$$

$$T^{(2)}(p') = \frac{\pi}{\sqrt{1+\varepsilon^2}}. \tag{65}$$

Let $x = \frac{1-\varepsilon^2}{1+\varepsilon^2}$ and

$$G(\varepsilon) = \frac{\varepsilon\pi}{\sqrt{1+\varepsilon^2}} - \arccos\left(\frac{1-\varepsilon^2}{1+\varepsilon^2}\right). \tag{66}$$

We have

$$\widetilde{G}(x) = \frac{\pi}{\sqrt{2}}\sqrt{1-x} - \arccos x. \tag{67}$$

For $x \in [-1, \ 1]$, $\widetilde{G}(x)$ is continuous in x and we also know that $\widetilde{G}(x) = 0$ only when $x = \pm 1$. It is easy to check $\widetilde{G}(x = 0) > 0$. Hence, we know that for $x \in [-1, \ 1]$, $\widetilde{G}(x) \geq 0$. That is, for $\varepsilon > 0$, $G(\varepsilon) \geq 0$. From the relationship $G(\varepsilon) \geq 0$, we know $T^{(2)}(p') \geq T^{(1)}(p')$ for $\varepsilon > 0$.

Hence we concluded that for arbitrary $p_0 \in (0, \frac{\varepsilon^2}{1+\varepsilon^2}]$, $T^{(2)} \geq T^{(1)}$. ∎

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