On the efficiency of very small refrigerators

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We investigate whether size imposes a fundamental constraint on the efficiency of small thermal machines. We analyse in detail a model of a small self-contained refrigerator consisting of three qubits. We show that this system can reach the Carnot efficiency, and thus demonstrate that there exists no complementarity between size and efficiency.

Thermodynamics lies at the very heart of physics as one of its most important and successful branches. Its foundations were laid by Sadi Carnot when he showed that nature imposes fundamental limitations on the efficiency of thermal machines. To arrive at this result Carnot did not consider a well chosen physical system but rather had the insight to work with abstract, model-independent machines. Thus the second law of thermodynamics, one of the cornerstones of physics, was derived by setting aside physics. However, can physics really be set aside for good, or do there exist other fundamental limitations which arise when we fully take it into account?

In a recent work [1] the question was raised of whether or not there exists a fundamental limitation on the size of thermal machines. We found out that no such limitation exists. Here we go one step further and ask whether the size of a thermal machine imposes fundamental limitations upon its efficiency?

In a Carnot machine the engine passes adiabatically through *very many* states; indeed through infinitely many of them. For example in the case of an ideal gas contained within a cylinder, a piston is slowly moved, with the volume of the gas changing continuously from its initial to final value. However, for the case of a quantum fridge with only a small number of distinct states it is no longer the case that we can transition through many states, a design constraint which appears at first sight rather drastic. The question therefore arises as to whether this imposes an additional bound which will prevent us from achieving an efficiency equal to, or even close to, the Carnot limit. It is conceivable then that this bound tends towards the Carnot efficiency as the number of states increases towards infinity. We therefore ask whether or not there may exist in nature a complementarity between size and efficiency? Is it the case that to be efficient you must be large, having access to many states, or can you be small and efficient also? In the present work surprisingly we show that there exists no such complementarity between size and efficiency – we demonstrate that machines with only a small number of states can reach the Carnot efficiency.

Quantum thermodynamics is by now a well developed field [2–4]. In particular, there has been significant interest in quantum heat engines [5–9] as well as refrigerators [10, 11]. Quantum analogues of Carnot engines have been studied extensively [12–15] as well as other cycles, such as Otto cycles [16–19] and Brownian motors [20]. There has also been an interest from the perspective of quantum information [21]. The focus to date however has been on thermal machines which contain quantum parts but which, either explicitly or implicitly, have macroscopic objects in the background which supply either work or some form of control, for example systems which are externally driven, or make use of sequences of unitary evolutions. Here we are interested in fully quantum machines, and hence study small *self contained* refrigerators.

THE MODEL

To start with let us introduce more precisely the model which we will focus on. As stated above, this is a model of small, self contained refrigerators. By small we mean that we consider quantum systems composed of very few states, and by self contained we mean that we consider systems whose internal evolution is governed by a time-independent Hamiltonian and whose supply of free energy comes solely through contact with thermal reservoirs at differing temperatures; therefore no external work is involved. We showed that it is possible to construct refrigerators meeting our requirements, hence demonstrating that there is no fundamental limit on the size of such thermal machines. Initially let us consider 3 non-interacting qubits. The free Hamiltonian for the three particles is given by

$$H_0 = H_1 + H_2 + H_3 = E_1\Pi_1 + E_2\Pi_2 + E_3\Pi_3 \quad (1)$$

where $\Pi_i = |1\rangle_i\langle 1|$ is the projector onto the excited state for each particle. We will constrain the energy levels such that $E_2 = E_1 + E_3$ for reasons which will become evident.

We take each qubit to be in contact with a thermal reservoir. The temperature of each reservoir will be taken to be different; we denote the temperatures of the reservoir of qubits 1, 2 and 3 as T_C, T_R and T_H respectively, which we will refer to as the "*cold*", "*room*" and "*hot*" reservoirs. To model the process of thermalisation of each qubit by the bath, we take a simple reset model, whereby with probability density p_i per time δt each qubit may be reset to a standard thermal state τ of its bath. Formally this amounts, in time δt, to the non-unitary process

$$\rho \mapsto \sum_i p_i \delta t \tau_i \otimes \mathrm{Tr}_i \rho + (1 - p_i \delta t)\rho \tag{2}$$

where, taking $k_B = 1$, $\tau_i = e^{-H_i/T_i}/Z \equiv r_i|0\rangle_i\langle 0| + \overline{r_i}|1\rangle_i\langle 1|$ is the Boltzmannian, $Z = \mathrm{Tr}e^{-H/T_i}$ is the partition function and r_i and $\overline{r_i}$ are the probabilities for the i^{th} qubit to be in the ground and excited state respectively, given by

$$r_i = 1/(1+e^{-E_i/T_i}) \quad \overline{r_i} = e^{-E_i/T_i}/(1+e^{-E_i/T_i}) \tag{3}$$

To turn this system into a refrigerator we introduce the interaction Hamiltonian H_{int},

$$H_{int} = g(|010\rangle\langle 101| + |101\rangle\langle 010|) \tag{4}$$

which couples the three spins. Given the imposed constraint, $E_2 = E_1 + E_3$, we see that this Hamiltonian couples only states degenerate in energy.

Furthermore we consider only the scenario in which this interaction is *weak*, that is we take $g \ll E_i$. In this regime the interaction Hamiltonian does not appreciable alter the energy eigenvalues or eigenvectors of the system, which remain governed by H_0. We are therefore justified in our definition of the thermal state for the qubits, which depends only upon H_0. Note also that in general the addition of H_{int} between the particles requires a modification of the dissipative dynamics, (2), if it is to remain consistent [22]. However, we are interested only in the limit where g and p_i vanish such that g/p_i remains constant. Since corrections to the dissipative dynamics are of order pg or higher, in this limit they vanish, and hence (2) remains a consistent dynamics.

In [1] we gave a detailed analysis of how we arrived at this model and why it behaves as a refrigerator. Here, to briefly understand the basic idea behind its functioning, we note that the Hamiltonian simply interchanges the population of the states $|010\rangle$ and $|101\rangle$. In the absence of any interaction with the environment the transitions in either direction are equiprobable and therefore we achieve nothing. However, by introducing environments at different temperatures, $T_R < T_H$, we are able to 'bias' the interaction and significantly alter the final occupation probabilities of the states relative to their values at thermal equilibrium and thus we are able to construct a refrigerator; the end result is that we are able to achieve a stationary temperature for qubit one lower than that of its environment, $T_1^S < T_C$.

Since we have qubits interacting with an environment the dynamics is described by a master equation. The master equation governing the dynamics of our refrigerator is given by

$$\frac{\partial \rho}{\partial t} = -i[H_0 + H_{int}, \rho] + \sum_{i=1}^{3} p_i(\tau_i \otimes \mathrm{Tr}_i \rho - \rho). \tag{5}$$

We are interested in the stationary (or long term) behaviour of the system, and thus wish to find ρ^S satisfying

$$0 = -i[H_0 + H_{int}, \rho^S] + \sum_{i=1}^{3} p_i(\tau_i \otimes \mathrm{Tr}_i \rho^S - \rho^S). \tag{6}$$

This equation can be solved exactly and analytically. It can be checked straightforwardly that the solution is given by

$$\rho^S = \tau_1\tau_2\tau_3 + \gamma\Big(Q_1 Z_1 \tau_2\tau_3 + Q_2 \tau_1 Z_2 \tau_3 + Q_3 \tau_1\tau_2 Z_3 + q_1\tau_1 Z_{23} + q_2\tau_2 Z_{13} + q_3 Z_{12}\tau_3 + Z_{123} + \frac{q}{2g}Y_{123}\Big) \tag{7}$$

where Y_{123} and Z_{123} are Pauli-like operators given by

$$\begin{aligned} Y_{123} &= i|101\rangle\langle 010| - i|010\rangle\langle 101| \\ Z_{123} &= |010\rangle\langle 010| - |101\rangle\langle 101| \end{aligned} \tag{8}$$

and operators such as Z_{12} and Z_3 are given by $Z_{12} = \mathrm{Tr}_3 Z_{123}$ and $Z_3 = \mathrm{Tr}_{12} Z_{123}$. Furthermore the parameters Q_i and q_j, depending only upon the thermalisation rates p_k, are given by

$$\begin{aligned} Q_1 &= \frac{p_2p_3}{p_1}\left(\frac{1}{p_1+p_2} + \frac{1}{p_1+p_3}\right), \quad q_1 = \frac{p_1}{p_2+p_3}, \\ Q_2 &= \frac{p_1p_3}{p_2}\left(\frac{1}{p_1+p_2} + \frac{1}{p_2+p_3}\right), \quad q_2 = \frac{p_2}{p_1+p_3}, \\ Q_3 &= \frac{p_1p_2}{p_3}\left(\frac{1}{p_1+p_3} + \frac{1}{p_2+p_3}\right), \quad q_3 = \frac{p_3}{p_1+p_2}, \end{aligned} \tag{9}$$

and $q = p_1 + p_2 + p_3$. Finally, the parameter γ is given by

$$\gamma = \frac{-\Delta}{2 + \frac{q^2}{2g^2} + Q_1\Omega_{23} + Q_2\Omega_{13} + Q_3\Omega_{12} + q_1 + q_2 + q_3} \tag{10}$$

where

$$\begin{aligned} \Delta &= r_1\overline{r_2}r_3 - \overline{r_1}r_2\overline{r_3}, \qquad \Omega_{12} = r_1\overline{r_2} + \overline{r_1}r_2, \\ \Omega_{13} &= r_1r_3 + \overline{r_1}\overline{r_3}, \qquad \Omega_{23} = r_2\overline{r_3} + \overline{r_2}r_3. \end{aligned} \tag{11}$$

The first notable features of the solution is that all single-party and two-party reduced density matrices are diagonal. Second, and of most importance, is the form of the single-party states which is given by

$$\rho_i^S = \tau_i + \tfrac{q\gamma}{p_i} Z_i \tag{12}$$

thus the occupation probability of the ground state for each qubit is shifted from its value at equilibrium by an amount proportional to the parameter γ.

For our model to act as a refrigerator we need that stationary temperature of qubit 1 to be colder than its bath temperature, i.e. $T_1^S < T_c$. This happens whenever the occupation probability of the ground state for particle 1 is increased compared to its thermal population. This happens whenever $\gamma > 0$. From (10) it can be checked that the denominator is a positive quantity and therefore the sign of γ depends only upon the numerator, $-\Delta$. Using the definitions (11) and (3) it can be shown that the condition $-\Delta > 0$ is equivalent to

$$e^{-E_1/T_C} e^{-E_3/T_H} > e^{-E_2/T_R} \tag{13}$$

which, upon further manipulation, can be re-expressed as

$$\frac{E_1}{E_3} < \frac{1-\frac{T_R}{T_H}}{\frac{T_R}{T_C}-1} \tag{14}$$

This is the fundamental design constraint on our refrigerator; as long as this condition is satisfied our model works as a refrigerator. As the ratio E_1/E_3 approaches the above limit, the temperature of the cold qubit approaches from below the temperature of its bath; everything else being held constant, this implies that it will take longer for the refrigerator to draw heat from the cold bath, similarly to what happens to a classical refrigerator as we approach the reversible limit, as its functioning becomes adiabatically slow. The above fundamental design constraint will play the central role in analysing the efficiency.

THE QUANTUM EFFICIENCY

To connect to the refrigerator we must first derive an expression for the amount of heat that our quantum machine is able to exchange with the thermal reservoirs in which it is in contact. To do this let us consider the change of one of the particles in a small time δt induced by the resevoir. From (2) we find that

$$\begin{aligned} \delta\rho_i(t) &= \rho_i(t+\delta t) - \rho_i(t) = p_i \delta t \tau_i + (1 - p_i\delta t)\rho_i(t), \\ &= p_i \delta t(\tau_i - \rho_i(t)). \end{aligned} \tag{15}$$

To this change of state corresponds a change in energy, $\delta\mathcal{E}_i$, given by

$$\delta\mathcal{E}_i = \mathrm{Tr}(H_i \delta\rho_i(t)) = p_i \delta t \mathrm{Tr}(H_i(\tau_i - \rho_i(t)) \tag{16}$$

thus, taking the limit $\delta t \to 0$ gives us the rate of change of energy of the particle due to the interaction with the reservoir

$$\frac{d\mathcal{E}_i}{dt} = p_i \mathrm{Tr}(H_i(\tau_i - \rho_i(t)) \tag{17}$$

which in other words it is the amount of energy supplied to the particle from the bath and is therefore the rate of heat flow, which we shall denote Q_i.

Using the explicit form previously obtained for ρ_i, (12) along with the definition of the Hamiltonian (1) we find that this can be re-written as

$$\frac{d\mathcal{E}_i}{dt} = p_i \mathrm{Tr}(E_i \Pi_i(-\tfrac{q\gamma}{p_i} Z_i)) = (-1)^{i+1} q\gamma E_i, \tag{18}$$

where the factor $(-1)^{i+1}$ arises due to the fact that $Z_1 = -Z_2 = Z_3 = Z$, the standard Pauli operator. Thus we see that the rate of heat flow between each bath and particle is given by

$$Q_C = q\gamma E_1, \quad Q_R = -q\gamma E_2, \quad Q_H = q\gamma E_3, \tag{19}$$

and thus the efficiency of our quantum refrigerator is given by

$$\eta^Q = \frac{Q_C}{Q_H} = \frac{E_1}{E_3}. \tag{20}$$

We arrive at the interesting result that although the individual heat currents have a rather complicated dependence upon all of the parameters in the problem, through q and γ, the efficiency of the fridge is in fact independent on all parameters except the ratio of energy levels. This result, although at first sight contradictory, is consistent with the results found in [7] and can be understood qualitatively: It is the interaction Hamiltonian which takes the particles away from their thermal equilibrium states, and since the Hamiltonian only acts on particles 1 and 3 simultaneously its clear that the rates at which they exchange heat with their reservoirs must be proportional to each other – hence the dependence in each case cancels when looking at the ratio.

Equation (20) however must be taken in conjunction with the basic design constraint, equation (14), which then yields an upper bound on the quantum efficiency:

$$\eta^Q < \frac{1-\frac{T_R}{T_H}}{\frac{T_R}{T_C}-1}. \tag{21}$$

It is important to note that since the refrigerator works as long as the condition (14) is satisfied that this is indeed an achievable bound on the efficiency of the refrigerator. In other words, we can get as close as we like to the following quantum efficiency

$$\eta^Q_{\max} = \frac{1-\frac{T_R}{T_H}}{\frac{T_R}{T_C}-1}. \tag{22}$$

THE CARNOT EFFICIENCY

In order to see the significance of the above derived maximum quantum efficiency for our particular model, we need to compare it with the Carnot efficiency derived from a 'standard' model.

In the standard analysis of the efficiency of a heat engine or refrigerator the efficiency is defined in terms of the *work*; we are interested in how much work we can extract from a given amount of heat, or how much heat we can extract for a given amount of work. However, in the current scenario we have avoided the explicit notion of work – the only free energy we allow ourselves access to is in the form of two baths at differing temperatures. We must therefore analyse the efficiency of such a device. Diagrammatically the machine we need to consider is depicted in Fig. 1 (a).

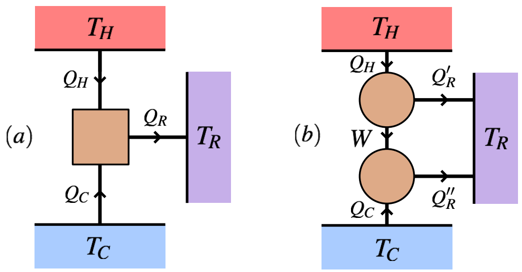

FIG. 1: (a) Diagrammatic representation of a thermal machine which uses a supply of heat Q_H extracted from a reservoir at T_H to extract an amount of heat Q_C from a reservoir at T_C, i.e. a refrigerator whose source of work is supplied by a thermal bath. (b) An explicit construction of such a device composed of two Carnot machines – the top functioning as a heat engine, the bottom as a heat pump.

That is, by extracting heat Q_H from a hot reservoir at temperature T_H, we are able to extract an amount of heat Q_C from a cold reservoir at temperature T_C whilst 'dumping' an amount of heat Q_R into a reservoir at some intermediate temperature T_R. It follows that the appropriate measure of efficiency for such a machine is given by

$$\eta = \frac{Q_C}{Q_H} \tag{23}$$

that is, for a given supply of heat from a hot bath, how much heat can we possibly extract from the cold bath. The two important points to note are first that the most efficient such machine will be a *reversible* machine, just as in the case of all thermodynamic machines. The second point to note is that all reversible machines – however they are constructed – must run at the same efficiency, and therefore we can focus on a specific reversible model without loss of generality. The model we will focus on is comprised of a Carnot heat engine supplying an amount of work W into a Carnot heat pump, as depicted in Fig. 1 (b).

To calculate the efficiency of this machine, we first apply the first law of thermodynamics to the heat engine and heat pump separately to obtain

$$Q_H = Q'_R + W, \qquad Q_C + W = Q''_R, \tag{24}$$

followed by the second law, telling us that entropy is conserved in a Carnot machine,

$$\frac{Q_H}{T_H} = \frac{Q'_R}{T_R}, \qquad \frac{Q_C}{T_C} = \frac{Q''_R}{T_R}. \tag{25}$$

Equations (24) together imply that $Q'_R + Q''_R = Q_H + Q_C$, which, when combined with (25) leads to the Carnot efficiency for this machine,

$$\eta^c = \frac{Q_C}{Q_H} = \frac{1 - \frac{T_R}{T_H}}{\frac{T_R}{T_C} - 1} \tag{26}$$

and is thus an upper bound on the efficiency of any such engine which we run between three reservoirs and which extracts heat from the bath at T_C using a supply of heat from the reservoir at T_H. We note that when $T_R = T_H$ then we have an efficiency of zero; in this case we are unable to extract any work with the heat engine and thus are unable to power the heat pump. Conversely, when $T_R \to T_C$ we see that η^c diverges; in this limit we can effectively move heat between the two reservoirs for 'free'.

CONCLUSIONS

By comparing the maximum quantum efficiency of our model (22) and the Carnot efficiency, (26) we see a remarkable result: they coincide. In other words, despite the fact that our refrigerator has a discrete and very small number of states, which one could have assumed to lead to stringent limitations on its efficiency, we see that it can actually achieve the maximum efficiency possible in nature.

Finally, we note that there have been many discussions of whether or not the second law of thermodynamics is valid in the context of quantum mechanics. We haven't address this particular question, what we have done is to study a particular model of a thermal machine – the smallest possible refrigerator. It is tantalising though that we have found its maximum efficiency to be exactly the Carnot efficiency. This suggests that indeed the second law of thermodynamics is valid in the context of quantum mechanics.

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