

# CONVERGENCE OF HESTON TO SVI

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ABSTRACT. In this short note, we prove by an appropriate change of variables that the SVI implied volatility parameterization presented in [2] and the large-time asymptotic of the Heston implied volatility derived in [1] agree algebraically, thus confirming a conjecture from [2] as well as providing a simpler expression for the asymptotic implied volatility in the Heston model. We show how this result can help in interpreting SVI parameters.

## 1. INTRODUCTION

The *stochastic volatility inspired* or *SVI* parameterization of the implied volatility surface was originally devised at Merrill Lynch in 1999. This parameterization has two key properties that have led to its subsequent popularity with practitioners:

- For a fixed time to expiry  $T$ , the implied Black-Scholes variance  $\sigma_{BS}^2(k, T)$  is linear in the log-strike  $k$  as  $|k| \rightarrow \infty$  consistent with Roger Lee's moment formula [4].
- It is relatively easy to fit listed option prices whilst ensuring no calendar spread arbitrage.<sup>1</sup>

The result we prove in this note shows that SVI is an exact solution for the implied variance in the Heston model in the limit  $T \rightarrow \infty$  thus providing a direct interpretation of the SVI parameters in terms of the parameters of the Heston model.

In Section 2, we present our notation. In Section 3, we motivate the conjecture which we prove in Section 4. We conclude in Section 5 by showing how our result can help us interpret SVI parameters resulting from an SVI fit to an empirically observed volatility smile.

## 2. NOTATIONS

From [2], recall that the SVI parameterization for the implied variance reads

$$(1) \quad \sigma_{SVI}^2(x) = \frac{\omega_1}{2} \left( 1 + \omega_2 \rho x + \sqrt{(\omega_2 x + \rho)^2 + 1 - \rho^2} \right), \quad \text{for all } x \in \mathbb{R},$$

where  $x$  represents the time-scaled log-moneyness, and consider the Heston model where the stock price process  $(S_t)_{t \geq 0}$  satisfies the following stochastic differential equation:

$$\begin{aligned} dS_t &= \sqrt{v_t} S_t dW_t, \quad S_0 \in \mathbb{R}_+^* \\ dv_t &= \kappa (\theta - v_t) dt + \sigma \sqrt{v_t} dZ_t, \quad v_0 \in \mathbb{R}_+^* \\ d\langle W, Z \rangle_t &= \rho dt, \end{aligned}$$

with  $\rho \in [-1, 1]$ ,  $\kappa, \theta, \sigma$  and  $v_0$  are strictly positive real numbers satisfying  $2\kappa\theta \geq \sigma^2$  (this is the Feller condition ensuring that the process  $(v_t)_{t \geq 0}$  never reaches zero almost surely). We further make the following assumption as in [1], under which the Heston asymptotic implied volatility is derived.

**Assumption 1.**  $\kappa - \rho\sigma > 0$ .

Note that this assumption is usually assumed in the literature, either explicitly or implicitly when assuming a negative correlation  $\rho < 0$  between the spot and the volatility as observed in equity markets. When this condition is not satisfied, the stock price process is still a true martingale, but moments greater than one will cease to exist after a certain amount of time, as pointed out in [6], which refers to this special case as the

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<sup>1</sup>It is seemingly impossible to eliminate the possibility of butterfly arbitrage but this is rarely a problem in practice.

large correlation regime. Let us now consider the following choice of SVI parameters in terms of the Heston parameters,

$$(2) \quad \omega_1 := \frac{4\kappa\theta}{\sigma^2(1-\rho^2)} \left( \sqrt{(2\kappa-\rho\sigma)^2 + \sigma^2(1-\rho^2)} - (2\kappa-\rho\sigma) \right), \quad \text{and} \quad \omega_2 := \frac{\sigma}{\kappa\theta}.$$

Now we know from [1] that the implied variance in the Heston model in the large time limit  $T \rightarrow \infty$  takes the following form:

$$(3) \quad \sigma_\infty^2(x) = 2 \left( 2V^*(x) - x + 2 \left( \mathbf{1}_{x \in (-\theta/2, \bar{\theta}/2)} - \mathbf{1}_{x \in \mathbb{R} \setminus (-\theta/2, \bar{\theta}/2)} \right) \sqrt{V^*(x)^2 - xV^*(x)} \right), \quad \text{for all } x \in \mathbb{R},$$

where  $\bar{\theta} := \kappa\theta / (\kappa - \rho\sigma)$ , and the function  $V^* : \mathbb{R} \rightarrow \mathbb{R}_+$  is defined by

$$(4) \quad V^*(x) := p^*(x)x - V(p^*(x)), \quad \text{for all } x \in \mathbb{R},$$

where

$$V(p) := \frac{\kappa\theta}{\sigma^2} \left( \kappa - \rho\sigma p - d(p) \right), \quad \text{for all } p \in (p_-, p_+),$$

$$d(p) := \sqrt{(\kappa - \rho\sigma p)^2 + \sigma^2 p(1-p^2)}, \quad \text{for all } p \in (p_-, p_+),$$

$$p^*(x) := \frac{\sigma - 2\kappa\rho + (\kappa\theta\rho + x\sigma)\eta \left( x^2\sigma^2 + 2x\kappa\theta\rho\sigma + \kappa^2\theta^2 \right)^{-1/2}}{2\sigma\bar{\rho}^2}, \quad \text{for all } x \in \mathbb{R},$$

$$\eta := \sqrt{4\kappa^2 + \sigma^2 - 4\kappa\rho\sigma}, \quad p_\pm := \left( -2\kappa\rho + \sigma \pm \sqrt{\sigma^2 + 4\kappa^2 - 4\kappa\rho\sigma} \right) / (2\sigma\bar{\rho}^2), \quad \text{and} \quad \bar{\rho} := \sqrt{1 - \rho^2}.$$

Note that in this asymptotic Heston form for the implied volatility,  $x$  corresponds to a time-scaled log-moneyness, *i.e.* the implied volatility corresponds to call/put options with strike  $S_0 \exp(xT)$ , where  $T \geq 0$  represents the maturity of the option.

### 3. THE SADDLE-POINT CONDITION

In this section, we give a non-rigorous motivation for the conjecture in [2] that the  $T \rightarrow \infty$  limit of the Heston volatility smile should be SVI.

Consider equation (5.7) on page 60 of [3] which relates the implied volatility  $\sigma_{BS}(k, T)$  at log-strike  $k$  and expiration  $T$  to the characteristic function  $\phi_T(\cdot)$  of the log-stock price. We rewrite this equation in the form

$$(5) \quad \int_{-\infty}^{\infty} \frac{du}{u^2 + \frac{1}{4}} e^{-iuk} \phi_T(u - i/2) = \int_{-\infty}^{\infty} \frac{du}{u^2 + \frac{1}{4}} e^{-iuk} e^{-\frac{1}{2}(u^2 + \frac{1}{4})\sigma_{BS}^2(k, T)T}.$$

In the limit  $T \rightarrow \infty$ , the Heston characteristic function has the form

$$\phi_T(u - i/2) \sim e^{-\psi(u)T}.$$

Then, as pointed out on page 186 of [5], we may apply the saddle-point method to both sides in equation (5) to obtain

$$(6) \quad e^{-ik\tilde{u}} \frac{e^{-\psi(\tilde{u})T}}{\tilde{u}^2 + \frac{1}{4}} \sqrt{\frac{2\pi}{\psi''(\tilde{u})T}} \sim 4 \exp \left\{ -\frac{vT}{8} - \frac{k^2}{2vT} \right\} \sqrt{\frac{2\pi}{vT}},$$

where  $v$  is short-form notation for  $\sigma_{BS}^2(k, T)T$  and  $\tilde{u}$  is such that

$$\psi'(\tilde{u}) = -i \frac{k}{T},$$

so that  $\tilde{u}$  (which is in general a function of  $k$ ) is a saddle-point, which in the Heston model at least, may be computed explicitly as in Lemma 5.3 of [1].

Defining  $k := xT$  and equating the arguments of the exponentials in equation (6), the dependence on  $T$  cancels and we obtain

$$(7) \quad \frac{v(x)}{8} + \frac{x^2}{2v(x)} = \psi(\tilde{u}(x)) + ix\tilde{u}(x),$$

where we have reinstated explicit dependence on  $x$  for emphasis.

With the help of *e.g.* Mathematica, one can verify that in the  $T \rightarrow \infty$  limit of the Heston model and with the choice (2) of SVI parameters, expression (1) exactly solves the saddle-point condition (7):

$$\frac{\sigma_{SVI}^2(x)}{8} + \frac{x^2}{2\sigma_{SVI}^2(x)} = \psi(\tilde{u}(x)) + \mathbf{i} x \tilde{u}(x).$$

We are thus led to conjecture that  $\sigma_{SVI}^2(x) = \sigma_\infty^2(x)$  so that the  $T \rightarrow \infty$  limit of implied variance in the Heston model is SVI.

#### 4. MAIN RESULT AND PROOF

We now state and prove the main result of this note,

**Proposition 1.** *Under Assumption 1 and the choice of SVI parameters (2),  $\sigma_{SVI}^2(x) = \sigma_\infty^2(x)$  for all  $x \in \mathbb{R}$ .*

*Proof.* Let us now introduce the following notations:  $\Delta(x) := \sqrt{\sigma^2 x^2 + 2\kappa\theta\rho\sigma x + \kappa^2\theta^2}$ , where  $\eta$  and  $\bar{\rho}$  are defined in Section 2. Under the change of variables (2), the SVI implied variance takes the form

$$(8) \quad \sigma_{SVI}^2(x) = \frac{2}{\sigma^2 \bar{\rho}^2} \left( \eta - (2\kappa - \rho\sigma) \right) \left( \kappa\theta + \rho\sigma x + \Delta(x) \right), \quad \text{for all } x \in \mathbb{R}.$$

We now move on to simplify the expression for  $\sigma_\infty^2$  as written in (3). We first start by the expression for  $V^*(x)$  appearing in (3). We have

$$V^*(x) = \frac{A(x)\Delta(x) + B(x)\eta}{2\sigma^2 \bar{\rho}^2 \Delta(x)},$$

with

$$A(x) := x\sigma^2 - 2x\kappa\rho\sigma - 2\kappa^2\theta + \kappa\theta\rho\sigma, \quad \text{and} \quad B(x) := 2x\sigma\kappa\theta\rho + x^2\sigma^2 + \kappa^2\theta^2\rho^2 + \kappa^2\theta^2\bar{\rho}^2.$$

Note that  $B(x) = \Delta^2(x)$ , so that  $V^*(x) = (A(x) + \Delta(x)\eta) / (2\sigma^2 \bar{\rho}^2)$ . We further have

$$(9) \quad 2V^*(x) - x = \frac{A(x) + \Delta(x)\eta - x\sigma^2\bar{\rho}^2}{\sigma^2 \bar{\rho}^2} = \frac{\Delta(x)\eta - (2\kappa - \rho\sigma)(\kappa\theta + x\rho\sigma)}{\sigma^2 \bar{\rho}^2},$$

where we use the factorisation  $A(x) - x\sigma^2\bar{\rho}^2 = -(2\kappa - \rho\sigma)(\kappa\theta + x\rho\sigma)$ .

Now, back to (3), where we denote  $\Phi(x) := V^*(x)^2 - xV^*(x)$ . We have

$$\Phi(x) = \left( \frac{\Delta(x)\eta}{2\sigma^2 \bar{\rho}^2} \right)^2 + \alpha(x)\Delta(x) + \beta(x),$$

where

$$\alpha(x) := -\frac{\eta(2\kappa - \rho\sigma)(\kappa\theta + x\rho\sigma)}{2\sigma^4 \bar{\rho}^4}, \quad \text{and} \quad \beta(x) := \frac{1}{4\sigma^4 \bar{\rho}^4} \left\{ (2\kappa - \rho\sigma)^2 (\kappa\theta + x\rho\sigma)^2 - x^2 \sigma^4 \bar{\rho}^4 \right\}.$$

We now use the following factorisations:

$$(10) \quad \Delta^2(x) = (\kappa\theta + x\rho\sigma)^2 + x^2\sigma^2\bar{\rho}^2, \quad \text{and} \quad \eta^2 = (2\kappa - \rho\sigma)^2 + \sigma^2\bar{\rho}^2,$$

so that we can write  $\beta(x) = (4\sigma^4 \bar{\rho}^4)^{-1} \left( (2\kappa - \rho\sigma)^2 \Delta^2(x) - x^2 \sigma^2 \bar{\rho}^2 \eta^2 \right)$  and hence

$$(11) \quad \begin{aligned} \Phi(x) &= \frac{1}{4\sigma^4 \bar{\rho}^4} \left\{ \left[ (2\kappa - \rho\sigma)^2 + \sigma^2 \bar{\rho}^2 \right] \Delta^2(x) + a(x)\Delta(x) + (\eta^2 - \sigma^2 \bar{\rho}^2) (\Delta^2(x) - x^2 \sigma^2 \bar{\rho}^2) - x^2 \sigma^4 \bar{\rho}^4 \right\} \\ &= \frac{1}{4\sigma^4 \bar{\rho}^4} \left\{ (2\kappa - \rho\sigma)^2 \Delta^2(x) + a(x)\Delta(x) + \eta^2 (\kappa\theta + x\rho\sigma)^2 \right\} \\ &= \frac{1}{4\sigma^4 \bar{\rho}^4} \left\{ \eta (\kappa\theta + x\rho\sigma) - (2\kappa - \rho\sigma) \Delta(x) \right\}^2, \end{aligned}$$

where, for convenience, we denote  $a(x) := 4\sigma^4 \bar{\rho}^4 \alpha(x)$ . To complete the proof, we need to take the square root of  $\Phi(x)$ , i.e. we need to study the sign of the expression under the square in (11). Using again (10), we have

$$\begin{aligned} &\eta (\kappa\theta + x\rho\sigma) - (2\kappa - \rho\sigma) \Delta(x) \\ &= (\kappa\theta + x\rho\sigma) \sqrt{(2\kappa - \rho\sigma)^2 + \sigma^2 \bar{\rho}^2} - (2\kappa - \rho\sigma) \sqrt{(\kappa\theta + x\rho\sigma)^2 + x^2 \sigma^2 \bar{\rho}^2} \\ &= \sqrt{\gamma(x) + \sigma^2 \bar{\rho}^2 (\kappa\theta + x\rho\sigma)^2} - \sqrt{\gamma(x) + x^2 \sigma^2 \bar{\rho}^2 (2\kappa - \rho\sigma)^2}, \end{aligned}$$

where  $\gamma(x) := (2\kappa - \rho\sigma)^2 (\kappa\theta + x\rho\sigma)^2$ . Now, because  $\gamma(x) \geq 0$  for all  $x \in \mathbb{R}$ , then the sign of this whole expression is simply given by the sign of the difference  $\psi(x) := \sigma^2 \bar{\rho}^2 (\kappa\theta + x\rho\sigma)^2 - x^2 \sigma^2 \bar{\rho}^2 (2\kappa - \rho\sigma)^2$ . Note further that we actually have  $\psi(x) = \kappa\sigma^2 \bar{\rho}^2 (2x + \theta)(2x\rho\sigma + \kappa\theta - 2\kappa x)$ , that this polynomial has exactly two real roots  $-\theta/2$  and  $\bar{\theta}/2$ , and that its second-order coefficient reads  $-4\kappa\sigma^2 \bar{\rho}^2 (\kappa - \rho\sigma) < 0$  under Assumption 1. So, plugging (9) and (11) into (3), we exactly obtain (8) and the proposition follows.  $\square$

## 5. INTERPRETATION OF SVI PARAMETERS

In this section, we use our result in Proposition 1 to help interpret the SVI parameters. From [2], the standard SVI parameterization in terms of the log-strike  $k$  reads

$$(12) \quad \sigma_{SVI}^2(k) = a + b \left\{ \tilde{\rho}(k - m) + \sqrt{(k - m)^2 + \tilde{\sigma}^2} \right\}.$$

Equating (12) with (1) and with the parameter choice (2):

$$\omega_1 := \frac{4\kappa\theta}{\sigma^2(1 - \rho^2)} \left( \sqrt{(2\kappa - \rho\sigma)^2 + \sigma^2(1 - \rho^2)} - (2\kappa - \rho\sigma) \right), \quad \text{and} \quad \omega_2 := \frac{\sigma}{\kappa\theta},$$

we find the following correspondence between SVI parameters and Heston parameters;

$$(13) \quad \begin{aligned} a &= \frac{\omega_1}{2} (1 - \rho^2), \\ b &= \frac{\omega_1 \omega_2}{2T}, \\ \tilde{\rho} &= \rho, \\ m &= -\frac{\rho T}{\omega_2}, \\ \tilde{\sigma} &= \frac{\sqrt{1 - \rho^2} T}{\omega_2}. \end{aligned}$$

For concreteness, imagine that we are given an SVI fit to the implied volatility smile generated from the Heston model with  $T$  very large so that we have the SVI parameters  $a$ ,  $b$ ,  $\tilde{\rho}$ ,  $m$  and  $\tilde{\sigma}$ . Our first observation is that the SVI parameter  $\tilde{\rho}$  is exactly the correlation  $\rho$  between changes in instantaneous variance  $v$  and changes in the underlying  $S$  in the Heston process. That is, we can read off correlation directly from the orientation of the volatility smile. In particular, the smile is symmetric when  $\rho = 0$ .

The parameter  $b$  gives the angle between the asymptotes of the implied variance smile. We see from (12) and (13) that the angle between the asymptotes of the total variance smile  $\sigma_{SVI}^2(k)T$  is constant for large  $T$  but that the overall level increases with  $T$ .

From equation (1),  $\omega_1$  is the at-the-money implied variance  $\sigma_{SVI}^2(0, T)$ . From (2), in the limit  $\sigma \ll \kappa$ , we have

$$\omega_1 = \theta \left\{ 1 + \frac{\rho\sigma}{\kappa} + O\left(\left(\frac{\sigma}{\kappa}\right)^2\right) \right\},$$

so that the at-the-money volatility is given directly by  $\theta$  when the volatility of volatility is small. In the limit  $\sigma \gg \kappa$ , we have

$$\omega_1 = \frac{4\kappa\theta}{\sigma(1 - \rho)} \left\{ 1 - 2\frac{\kappa}{\sigma} + O\left(\left(\frac{\kappa}{\sigma}\right)^2\right) \right\},$$

showing that at-the-money volatility decreases as the volatility-of-volatility increases and as the volatility becomes less correlated with the underlying.

Finally, the minimum of the variance smile is attained at  $x = -2\rho/\omega_2$ , providing a simple interpretation of the parameter  $\omega_2$ . In particular, if  $\rho = 0$ , the minimum is exactly the at-the-money point. The minimum shifts to the upside  $x > 0$  if  $\rho < 0$  and to the downside  $x < 0$  if  $\rho > 0$ .

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