

Dynamic risk measures

Beatrice Acciaio*

Irina Penner†

first version: November 13, 2009; this version: February 19, 2010

Abstract

This paper gives an overview of the theory of dynamic convex risk measures for random variables in discrete time setting. We summarize robust representation results of conditional convex risk measures, and we characterize various time consistency properties of dynamic risk measures in terms of acceptance sets, penalty functions, and by supermartingale properties of risk processes and penalty functions.

1 Introduction

Risk measures are quantitative tools developed to determine minimum capital reserves, which are required to be maintained by financial institutions in order to ensure their financial stability. An axiomatic analysis of risk assessment in terms of capital requirements was initiated by Artzner, Delbaen, Eber, and Heath [2, 3], who introduced coherent risk measures. Föllmer and Schied [21] and Frittelli and Rosazza Gianin [23] replaced positive homogeneity by convexity in the set of axioms and established the more general concept of a convex risk measure. Since then, convex and coherent risk measures and their applications have attracted a growing interest both in mathematical finance research and among practitioners.

One of the most appealing properties of a convex risk measure is its robustness against model uncertainty. Under some regularity condition, it can be represented as a suitably modified worst expected loss over a whole class of probabilistic models. This was initially observed in [3, 21, 23] in the static setting, where financial positions are described by random variables on some probability space and a risk measure is a real-valued functional. For a comprehensive presentation of the theory of static coherent and convex risk measures we refer to Delbaen [14] and Föllmer and Schied [22, Chapter 4].

*Department of Economy, Finance and Statistics, University of Perugia, Via A. Pascoli 20, 06123 Perugia, Italy. Email: beatrice.acciaio@stat.unipg.it. Financial support from the European Science Foundation (ESF) “Advanced Mathematical Methods for Finance” (AMaMeF) under the exchange grant 2281, and hospitality of Vienna University of Technology are gratefully acknowledged.

†Humboldt-Universität zu Berlin, Institut für Mathematik, Unter den Linden 6, 10099 Berlin, Germany. Email: penner@math.hu-berlin.de. Supported by the DFG Research Center MATHEON “Mathematics for key technologies”. Financial support from the European Science Foundation (ESF) “Advanced Mathematical Methods for Finance” (AMaMeF) under the short visit grant 2854 is gratefully acknowledged.

A natural extension of a static risk measure is given by a conditional risk measure, which takes into account the information available at the time of risk assessment. As its static counterpart, a conditional convex risk measure can be represented as the worst conditional expected loss over a class of suitably penalized probability measures; see [31, 30, 17, 6, 26, 12]. In the dynamical setting described by some filtered probability space, risk assessment is updated over the time in accordance with the new information. This leads to the notion of dynamic risk measure, which is a sequence of conditional risk measures adapted to the underlying filtration.

A crucial question in the dynamical framework is how risk evaluations at different times are interrelated. Several notions of time consistency were introduced and studied in the literature. One of today's most used notions is strong time consistency, which corresponds to the dynamic programming principle; see [4, 15, 17, 26, 12, 7, 20, 13, 16] and references therein. As shown in [15, 7, 20], strong time consistency can be characterized by additivity of the acceptance sets and penalty functions, and also by a supermartingale property of the risk process and the penalty function process. Similar characterizations of the weaker notions of time consistency, so called rejection and acceptance consistency, were given in [18, 29]. Rejection consistency, also called prudence in [29], seems to be a particularly suitable property from the point of view of a regulator, since it ensures that one always stays on the safe side when updating risk assessment. The weakest notions of time consistency considered in the literature are weak acceptance and weak rejection consistency, which require that if some position is accepted (or rejected) for any scenario tomorrow, it should be already accepted (or rejected) today; see [4, 35, 33, 9, 32].

As pointed out in [25, 19], risk assessment in the multi-period setting should also account for uncertainty about the time value of money. This requires to consider entire cash flow processes rather than total amounts at terminal dates as risky objects, and it leads to a further extension of the notion of risk measure. Risk measures for processes were studied in [4, 30, 10, 11, 12, 13, 24, 25, 1]. The new feature in this framework is that not only the amounts but also the timing of payments matters; cf. [12, 13, 25, 1]. However, as shown in [4] in the static and in [1] in the dynamical setting, risk measures for processes can be identified with risk measures for random variables on an appropriate product space. This allows a natural translation of results obtained in the framework of risk measures for random variables to the framework of processes; see [1].

The aim of this paper is to give an overview of the current theory of dynamic convex risk measures for random variables in discrete time setting; the corresponding results for risk measures for processes are given in [1]. The paper is organized as follows. Section 2 recalls the definition of a conditional convex risk measure and its interpretation as the minimal capital requirement from [17]. Section 3 summarizes robust representation results from [17, 20, 8]. In Section 4 we first give an overview of different time consistency properties based on [33]. Then we focus on the strong notion of time consistency, in Subsection 4.1, and we characterize it by supermartingale properties of risk processes and penalty functions. The results of this subsection are mainly based on [20], with the difference that here we give characterizations of time consistency also in terms of absolutely continuous probability measures, similar to [8]. In addition, we relate the martingale property of a risk process with the worst case measure, and we provide the explicit form of the Doob- and the Riesz-decomposition of the penalty function process. Subsection 4.2 generalizes [29, Sections 2.4, 2.5] and characterizes rejection and acceptance consistency in terms of acceptance sets,

penalty functions, and, in case of rejection consistency, by a supermartingale property of risk processes and one-step penalty functions. Subsection 4.3 recalls characterizations of weak time consistency from [33, 9], and Subsection 4.4 characterizes the recursive construction of time consistent risk measures suggested in [12, 13]. Finally, the dynamic entropic risk measure with a non-constant risk aversion parameter is studied in Section 5.

2 Setup and notation

Let $T \in \mathbb{N} \cup \{\infty\}$ be the time horizon, $\mathbb{T} := \{0, \dots, T\}$ for $T < \infty$, and $\mathbb{T} := \mathbb{N}_0$ for $T = \infty$. We consider a discrete-time setting given by a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \in \mathbb{T}}, P)$ with $\mathcal{F}_0 = \{\emptyset, \Omega\}$, $\mathcal{F} = \mathcal{F}_T$ for $T < \infty$, and $\mathcal{F} = \sigma(\cup_{t \geq 0} \mathcal{F}_t)$ for $T = \infty$. For $t \in \mathbb{T}$, $L_t^\infty := L^\infty(\Omega, \mathcal{F}_t, P)$ is the space of all essentially bounded \mathcal{F}_t -measurable random variables, and $L^\infty := L^\infty(\Omega, \mathcal{F}_T, P)$. All equalities and inequalities between random variables and between sets are understood to hold P -almost surely, unless stated otherwise. We denote by $\mathcal{M}_1(P)$ (resp. by $\mathcal{M}^e(P)$) the set of all probability measures on (Ω, \mathcal{F}) which are absolutely continuous with respect to P (resp. equivalent to P).

In this work we consider risk measures defined on the set L^∞ , which is understood as the set of discounted terminal values of financial positions. In the dynamical setting, a conditional risk measure ρ_t assigns to each terminal payoff X an \mathcal{F}_t -measurable random variable $\rho_t(X)$, that quantifies the risk of the position X given the information \mathcal{F}_t . A rigorous definition of a conditional convex risk measure was given in [17, Definition 2].

Definition 1. *A map $\rho_t : L^\infty \rightarrow L_t^\infty$ is called a conditional convex risk measure if it satisfies the following properties for all $X, Y \in L^\infty$:*

(i) *Conditional cash invariance: For all $m_t \in L_t^\infty$*

$$\rho_t(X + m_t) = \rho_t(X) - m_t;$$

(ii) *Monotonicity: $X \leq Y \Rightarrow \rho_t(X) \geq \rho_t(Y)$;*

(iii) *Conditional convexity: for all $\lambda \in L_t^\infty$, $0 \leq \lambda \leq 1$:*

$$\rho_t(\lambda X + (1 - \lambda)Y) \leq \lambda \rho_t(X) + (1 - \lambda) \rho_t(Y);$$

(iv) *Normalization: $\rho_t(0) = 0$.*

A conditional convex risk measure is called a conditional coherent risk measure if it has in addition the following property:

(iv) *Conditional positive homogeneity: for all $\lambda \in L_t^\infty$, $\lambda \geq 0$:*

$$\rho_t(\lambda X) = \lambda \rho_t(X).$$

In the dynamical framework one can also analyze risk assessment for cumulated cash flow *processes* rather than just for terminal pay-offs, i.e. one can consider a risk measure that accounts not only for the amounts but also for the timing of payments. Such risk measures were studied in [10, 11, 12, 13, 24, 25, 1]. As shown in [4] in the static and in [1] in the dynamical setting, convex risk measures for processes can be identified with convex risk measures for random variables on an appropriate product space. This allows to extend results obtained in our present setting to the framework of processes; cf. [1].

If ρ_t is a conditional convex risk measure, the function $\phi_t := -\rho_t$ defines a conditional monetary utility function in the sense of [12, 13]. The term “monetary” refers to conditional cash invariance of the utility function, the only property in Definition 1 that does not come from the classical utility theory. Conditional cash invariance is a natural request in view of the interpretation of ρ_t as a conditional capital requirement. In order to formalize this aspect we first recall the notion of the *acceptance set* of a conditional convex risk measure ρ_t :

$$\mathcal{A}_t := \{ X \in L^\infty \mid \rho_t(X) \leq 0 \}.$$

The following properties of the acceptance set were given in [17, Proposition 3].

Proposition 2. *The acceptance set \mathcal{A}_t of a conditional convex risk measure ρ_t is*

1. *conditionally convex, i.e. $\alpha X + (1 - \alpha)Y \in \mathcal{A}_t$ for all $X, Y \in \mathcal{A}_t$ and $\alpha \mathcal{F}_t$ -measurable such that $0 \leq \alpha \leq 1$;*
2. *solid, i.e. $Y \in \mathcal{A}_t$ whenever $Y \geq X$ for some $X \in \mathcal{A}_t$;*
3. *such that $0 \in \mathcal{A}_t$ and $\text{ess inf} \{ X \in L_t^\infty \mid X \in \mathcal{A}_t \} = 0$.*

Moreover, ρ_t is uniquely determined through its acceptance set, since

$$\rho_t(X) = \text{ess inf} \{ Y \in L_t^\infty \mid X + Y \in \mathcal{A}_t \}. \quad (1)$$

Conversely, if some set $\mathcal{A}_t \subseteq L^\infty$ satisfies conditions 1)-3), then the functional $\rho_t : L^\infty \rightarrow L_t^\infty$ defined via (1) is a conditional convex risk measure.

Proof. Properties 1)-3) of the acceptance set follow easily from properties (i)-(iii) in Definition 1. To prove (1) note that by cash invariance $\rho_t(X) + X \in \mathcal{A}_t$ for all X , and this implies “ \geq ” in (1). On the other hand, for all $Z \in \{ Y \in L_t^\infty \mid X + Y \in \mathcal{A}_t \}$ we have

$$0 \geq \rho_t(Z + X) = \rho_t(X) - Z,$$

thus $\rho_t(X) \leq \text{ess inf} \{ Y \in L_t^\infty \mid X + Y \in \mathcal{A}_t \}$.

For the proof of the last part of the assertion we refer to [17, Proposition 3]. □

Due to (1), the value $\rho_t(X)$ can be viewed as the minimal conditional capital requirement needed to be added to the position X in order to make it acceptable at time t . The following example shows how risk measures can be defined via (1).

Example 3. Consider the set of all positions having non-negative conditional expected utility, i.e.

$$\mathcal{A}_t := \{X \in L^\infty \mid E[u_t(X)|\mathcal{F}_t] \geq 0\},$$

where u_t denotes some non-increasing and concave utility function. It is easy to check that the set \mathcal{A}_t has all properties 1)-3) from Proposition 2. A basic choice is the exponential utility function $u_t(x) = 1 - e^{-\gamma_t x}$, where $\gamma_t > 0$ P -a.s. denotes the risk aversion parameter such that $\gamma_t, \frac{1}{\gamma_t} \in L_t^\infty$. The corresponding conditional convex risk measure ρ_t associated to \mathcal{A}_t via (1) takes the form

$$\rho_t(X) = \frac{1}{\gamma_t} \log E[e^{-\gamma_t X} | \mathcal{F}_t], \quad X \in L^\infty,$$

and is called the conditional entropic risk measure. The entropic risk measure was introduced in [22] in the static setting, in the dynamical setting it appeared in [5, 28, 17, 12, 20, 13]. We characterize the dynamic entropic risk measure in Section 5.

3 Robust representation

As observed in [3, 22, 23] in the static setting, the axiomatic properties of a convex risk measure yield, under some regularity condition, a representation of the minimal capital requirement as a suitably modified worst expected loss over a whole class of probabilistic models. In the dynamical setting, such robust representations of conditional coherent risk measures were obtained on a finite probability space in [31] for random variables and in [30] for stochastic processes. On a general probability space, robust representations for conditional coherent and convex risk measures were proved in [17, 6, 9, 26, 20, 8] for random variables and in [12] for stochastic processes. In this section we mainly summarize the results from [17, 20, 8].

The alternative probability measures in a robust representation of a risk measure ρ_t contribute to the risk evaluation to a different degree. To formalize this aspect we use the notion of the minimal penalty function α_t^{\min} , defined for each $Q \in \mathcal{M}_1(P)$ as

$$\alpha_t^{\min}(Q) = Q\text{-ess sup}_{X \in \mathcal{A}_t} E_Q[-X | \mathcal{F}_t]. \quad (2)$$

The following property of the minimal penalty function is a standard result, that will be used in the proof of Theorem 5.

Lemma 4. For $Q \in \mathcal{M}_1(P)$ and $0 \leq s \leq t$,

$$E_Q[\alpha_t^{\min}(Q) | \mathcal{F}_s] = Q\text{-ess sup}_{Y \in \mathcal{A}_t} E_Q[-Y | \mathcal{F}_s] \quad Q\text{-a.s.}$$

and in particular

$$E_Q[\alpha_t^{\min}(Q)] = \sup_{Y \in \mathcal{A}_t} E_Q[-Y].$$

Proof. First we claim that the set

$$\{E_Q[-X | \mathcal{F}_t] \mid X \in \mathcal{A}_t\}$$

is directed upward for any $Q \in \mathcal{M}_1(P)$. Indeed, for $X, Y \in \mathcal{A}_t$ we can define $Z := XI_A + YI_{A^c}$, where $A := \{E_Q[-X|\mathcal{F}_t] \geq E_Q[-Y|\mathcal{F}_t]\} \in \mathcal{F}_t$. Conditional convexity of ρ_t implies that $Z \in \mathcal{A}_t$, and by definition of Z

$$E_Q[-Z|\mathcal{F}_t] = \max(E_Q[-X|\mathcal{F}_t], E_Q[-Y|\mathcal{F}_t]) \quad Q\text{-a.s.}$$

Hence there exists a sequence $(X_n^Q)_{n \in \mathbb{N}}$ in \mathcal{A}_t such that

$$\alpha_t^{\min}(Q) = \lim_n E_Q[-X_n^Q|\mathcal{F}_t] \quad Q\text{-a.s.}, \quad (3)$$

and by monotone convergence we get

$$\begin{aligned} E_Q[\alpha_t^{\min}(Q)|\mathcal{F}_s] &= \lim_n E_Q[E_Q[-X_n^Q|\mathcal{F}_t]|\mathcal{F}_s] \\ &\leq Q\text{-ess sup}_{Y \in \mathcal{A}_t} E_Q[-Y|\mathcal{F}_s] \quad Q\text{-a.s.} \end{aligned}$$

The converse inequality follows directly from the definition of $\alpha_t^{\min}(Q)$. \square

The following theorem relates robust representations to some continuity properties of conditional convex risk measures. It combines [17, Theorem 1] with [20, Corollary 2.4]; similar results can be found in [6, 26, 12].

Theorem 5. *For a conditional convex risk measure ρ_t the following are equivalent:*

1. ρ_t has a robust representation

$$\rho_t(X) = \operatorname{ess\,sup}_{Q \in \mathcal{Q}_t} (E_Q[-X|\mathcal{F}_t] - \alpha_t(Q)), \quad X \in L^\infty, \quad (4)$$

where

$$\mathcal{Q}_t := \{Q \in \mathcal{M}_1(P) \mid Q = P|_{\mathcal{F}_t}\}$$

and α_t is a map from \mathcal{Q}_t to the set of \mathcal{F}_t -measurable random variables with values in $\mathbb{R} \cup \{+\infty\}$, such that $\operatorname{ess\,sup}_{Q \in \mathcal{Q}_t} (-\alpha_t(Q)) = 0$.

2. ρ_t has the robust representation in terms of the minimal penalty function, i.e.

$$\rho_t(X) = \operatorname{ess\,sup}_{Q \in \mathcal{Q}_t} (E_Q[-X|\mathcal{F}_t] - \alpha_t^{\min}(Q)), \quad X \in L^\infty, \quad (5)$$

where α_t^{\min} is given in (2).

3. ρ_t has the robust representation

$$\rho_t(X) = \operatorname{ess\,sup}_{Q \in \mathcal{Q}_t^f} (E_Q[-X|\mathcal{F}_t] - \alpha_t^{\min}(Q)) \quad P\text{-a.s.}, \quad X \in L^\infty, \quad (6)$$

where

$$\mathcal{Q}_t^f := \{Q \in \mathcal{M}_1(P) \mid Q = P|_{\mathcal{F}_t} \text{ and } E_Q[\alpha_t^{\min}(Q)] < \infty\}.$$

4. ρ_t has the “Fatou-property”: for any bounded sequence $(X_n)_{n \in \mathbb{N}}$ which converges P -a.s. to some X ,

$$\rho_t(X) \leq \liminf_{n \rightarrow \infty} \rho_t(X_n) \quad P\text{-a.s.}$$

5. ρ_t is continuous from above, i.e.

$$X_n \searrow X \quad P\text{-a.s.} \implies \rho_t(X_n) \nearrow \rho_t(X) \quad P\text{-a.s.}$$

for any sequence $(X_n)_n \subseteq L^\infty$ and $X \in L^\infty$.

Proof. 3) \implies 1) and 2) \implies 1) are obvious. 1) \implies 4): Dominated convergence implies that $E_Q[X_n | \mathcal{F}_t] \rightarrow E_Q[X | \mathcal{F}_t]$ for each $Q \in \mathcal{Q}_t$, and $\liminf_{n \rightarrow \infty} \rho_t(X_n) \geq \rho_t(X)$ follows by using the robust representation of ρ_t as in the unconditional setting, see, e.g., [22, Lemma 4.20].

4) \implies 5): Monotonicity implies $\limsup_{n \rightarrow \infty} \rho_t(X_n) \leq \rho_t(X)$, and $\liminf_{n \rightarrow \infty} \rho_t(X_n) \geq \rho_t(X)$ follows by 4).

5) \implies 2): The inequality

$$\rho_t(X) \geq \operatorname{ess\,sup}_{Q \in \mathcal{Q}_t} (E_Q[-X | \mathcal{F}_t] - \alpha_t^{\min}(Q)) \quad (7)$$

follows from the definition of α_t^{\min} . In order to prove the equality we will show that

$$E_P[\rho_t(X)] \leq E_P \left[\operatorname{ess\,sup}_{Q \in \mathcal{Q}_t} (E_Q[-X | \mathcal{F}_t] - \alpha_t^{\min}(Q)) \right].$$

To this end, consider the map $\rho^P : L^\infty \rightarrow \mathbb{R}$ defined by $\rho^P(X) := E_P[\rho_t(X)]$. It is easy to check that ρ^P is a convex risk measure which is continuous from above. Hence [22, Theorem 4.31] implies that ρ^P has the robust representation

$$\rho^P(X) = \sup_{Q \in \mathcal{M}_1(P)} (E_Q[-X] - \alpha(Q)) \quad X \in L^\infty,$$

where the penalty function $\alpha(Q)$ is given by

$$\alpha(Q) = \sup_{X \in L^\infty : \rho^P(X) \leq 0} E_Q[-X].$$

Next we will prove that $Q \in \mathcal{Q}_t$ if $\alpha(Q) < \infty$. Indeed, let $A \in \mathcal{F}_t$ and $\lambda > 0$. Then

$$-\lambda P[A] = E_P[\rho_t(\lambda I_A)] = \rho^P(\lambda I_A) \geq E_Q[-\lambda I_A] - \alpha(Q),$$

so

$$P[A] \leq Q[A] + \frac{1}{\lambda} \alpha(Q) \quad \text{for all } \lambda > 0,$$

and hence $P[A] \leq Q[A]$ if $\alpha(Q) < \infty$. The same reasoning with $\lambda < 0$ implies $P[A] \geq Q[A]$, thus $P = Q$ on \mathcal{F}_t if $\alpha(Q) < \infty$. By Lemma 4, we have for every $Q \in \mathcal{Q}_t$

$$E_P[\alpha_t^{\min}(Q)] = \sup_{Y \in \mathcal{A}_t} E_P[-Y].$$

Since $\rho^P(Y) \leq 0$ for all $Y \in \mathcal{A}_t$, this implies

$$E_P[\alpha_t^{\min}(Q)] \leq \alpha(Q)$$

for all $Q \in \mathcal{Q}_t$, by definition of the penalty function $\alpha(Q)$.

Finally we obtain

$$\begin{aligned} E_P[\rho_t(X)] &= \rho^P(X) = \sup_{Q \in \mathcal{M}_1(P), \alpha(Q) < \infty} (E_Q[-X] - \alpha(Q)) \\ &\leq \sup_{Q \in \mathcal{Q}_t, E_P[\alpha_t^{\min}(Q)] < \infty} (E_Q[-X] - \alpha(Q)) \\ &\leq \sup_{Q \in \mathcal{Q}_t, E_P[\alpha_t^{\min}(Q)] < \infty} E_P[E_Q[-X | \mathcal{F}_t] - \alpha_t^{\min}(Q)] \\ &\leq E_P \left[\operatorname{ess\,sup}_{Q \in \mathcal{Q}_t, E_P[\alpha_t^{\min}(Q)] < \infty} (E_Q[-X | \mathcal{F}_t] - \alpha_t^{\min}(Q)) \right] \\ &\leq E_P \left[\operatorname{ess\,sup}_{Q \in \mathcal{Q}_t} E_Q[-X | \mathcal{F}_t] - \alpha_t^{\min}(Q) \right], \end{aligned} \tag{8}$$

proving equality (5).

5) \Rightarrow 3) The inequality

$$\rho_t(X) \geq \operatorname{ess\,sup}_{Q \in \mathcal{Q}_t^f} (E_Q[-X | \mathcal{F}_t] - \alpha_t^{\min}(Q))$$

follows from (7) since $\mathcal{Q}_t^f \subseteq \mathcal{Q}_t$, and (8) proves the equality. \square

The penalty function $\alpha_t^{\min}(Q)$ is minimal in the sense that any other function α_t in a robust representation (4) of ρ_t satisfies

$$\alpha_t^{\min}(Q) \leq \alpha_t(Q) \text{ } P\text{-a.s.}$$

for all $Q \in \mathcal{Q}_t$. An alternative formula for the minimal penalty function is given by

$$\alpha_t^{\min}(Q) = \operatorname{ess\,sup}_{X \in L^\infty} (E_Q[-X | \mathcal{F}_t] - \rho_t(X)) \quad \text{for all } Q \in \mathcal{Q}_t.$$

This follows as in the unconditional case; see, e.g., [22, Theorem 4.15, Remark 4.16].

Remark 6. *Another characterization of a conditional convex risk measure ρ_t that is equivalent to the properties 1)-4) of Theorem 5 is the following: The acceptance set \mathcal{A}_t is weak*-closed, i.e., it is closed in L^∞ with respect to the topology $\sigma(L^\infty, L^1(\Omega, \mathcal{F}, P))$. This equivalence was shown in [12] in the context of risk measures for processes and in [26] for risk measures for random variables. Though in [26] a slightly different definition of a conditional risk measure is used, the reasoning given there works just the same in our case; cf. [26, Theorem 3.16].*

For the characterization of time consistency in Section 4 we will need a robust representation of a conditional convex risk measure ρ_t under any measure $Q \in \mathcal{M}_1(P)$, where possibly $Q \notin \mathcal{Q}_t$. Such representation can be obtained as in Theorem 5 by considering ρ_t as a risk measure under Q , as shown in the next corollary. This result is a version of [8, Proposition 1].

Corollary 7. *A conditional convex risk measure ρ_t is continuous from above if and only if it has the robust representations*

$$\rho_t(X) = \underset{R \in \mathcal{Q}_t(Q)}{\text{Q-ess sup}}(E_R[-X | \mathcal{F}_t] - \alpha_t^{\min}(R)) \quad (9)$$

$$= \underset{R \in \mathcal{Q}_t^f(Q)}{\text{Q-ess sup}}(E_R[-X | \mathcal{F}_t] - \alpha_t^{\min}(R)) \quad \text{Q-a.s.}, \quad \forall X \in L^\infty, \quad (10)$$

for all $Q \in \mathcal{M}_1(P)$, where

$$\mathcal{Q}_t(Q) = \{ R \in \mathcal{M}_1(P) \mid R = Q|_{\mathcal{F}_t} \}$$

and

$$\mathcal{Q}_t^f(Q) = \{ R \in \mathcal{M}_1(P) \mid R = Q|_{\mathcal{F}_t}, E_R[\alpha_t^{\min}(R)] < \infty \}.$$

Proof. To show that continuity from above implies representation (9), we can replace P by a probability measure $Q \in \mathcal{M}_1(P)$ and repeat all the reasoning of the proof of 5) \Rightarrow 2) in Theorem 5. In this case we consider the static convex risk measure

$$\rho^Q(X) = E_Q[\rho_t(X)] = \sup_{R \in \mathcal{M}_1(P)} (E_R[-X] - \alpha(R)), \quad X \in L^\infty,$$

instead of ρ^P . The proof of (10) follows in the same way from [20, Corollary 2.4]. Conversely, continuity from above follows from Theorem 5 since representation (9) holds under P . \square

Remark 8. *One can easily see that the set \mathcal{Q}_t in representations (4) and (5) can be replaced by $\mathcal{P}_t := \{ Q \in \mathcal{M}_1(P) \mid Q \approx P \text{ on } \mathcal{F}_t \}$. Moreover, representation (4) is also equivalent to*

$$\rho_t(X) = \text{ess sup}_{Q \in \mathcal{M}_1(P)} (E_Q[-X | \mathcal{F}_t] - \hat{\alpha}_t(Q)), \quad X \in L^\infty,$$

where the conditional expectation under $Q \in \mathcal{M}_1(P)$ is defined under P as

$$E_Q[X | \mathcal{F}_t] := \frac{E_P[Z_T X | \mathcal{F}_t]}{Z_t} I_{\{Z_t > 0\}},$$

and the extended penalty function $\hat{\alpha}_t$ is given by

$$\hat{\alpha}_t(Q) = \begin{cases} \alpha_t(Q) & \text{on } \{ \frac{dQ}{dP} |_{\mathcal{F}_t} > 0 \}; \\ +\infty & \text{otherwise.} \end{cases}$$

In the *coherent* case the penalty function $\alpha_t^{\min}(Q)$ can only take values 0 or ∞ due to positive homogeneity of ρ_t . Thus representation (9) takes the following form.

Corollary 9. *A conditional coherent risk measure ρ_t is continuous from above if and only if it is representable in the form*

$$\rho_t(X) = \text{ess sup}_{Q \in \mathcal{Q}_t^0} E_Q[-X | \mathcal{F}_t], \quad X \in L^\infty, \quad (11)$$

where

$$\mathcal{Q}_t^0 := \{ Q \in \mathcal{Q}_t \mid \alpha_t^{\min}(Q) = 0 \text{ Q-a.s.} \}.$$

Example 10. A notable example of a conditional coherent risk measure is conditional Average Value at Risk defined as

$$AV@R_{t,\lambda_t}(X) := \text{ess sup}\{E_Q[-X|\mathcal{F}_t] \mid Q \in \mathcal{Q}_t, \frac{dQ}{dP} \leq \lambda_t^{-1}\}$$

with $\lambda_t \in L_t^\infty$, $0 < \lambda_t \leq 1$. Static Average Value at Risk was introduced in [3] as a valid alternative to the widely used yet criticized Value at Risk. The conditional version of Average Value at Risk appeared in [4], and was also studied in [18, 34].

As observed, e.g., in [12, Remark 3.13], the minimal penalty function has the local property. In our context it means that for any $Q^1, Q^2 \in \mathcal{Q}_t(Q)$ with the corresponding density processes Z^1 and Z^2 with respect to P , and for any $A \in \mathcal{F}_t$, the probability measure R defined via $\frac{dR}{dP} := I_A Z_T^1 + I_{A^c} Z_T^2$ has the penalty function value

$$\alpha_t^{\min}(R) = I_A \alpha_t^{\min}(Q^1) + I_{A^c} \alpha_t^{\min}(Q^2) \quad Q\text{-a.s.}$$

In particular $R \in \mathcal{Q}_t^f(Q)$ if $Q^1, Q^2 \in \mathcal{Q}_t^f(Q)$. Standard arguments (cf., e.g., [17, Lemma 1]) imply then that the set

$$\left\{ E_R[-X | \mathcal{F}_t] - \alpha_t^{\min}(R) \mid R \in \mathcal{Q}_t^f(Q) \right\}$$

is directed upward, thus

$$E_Q[\rho_t(X)|\mathcal{F}_s] = \text{Q-ess sup}_{R \in \mathcal{Q}_t^f(Q)} (E_R[-X|\mathcal{F}_s] - E_R[\alpha_t^{\min}(R)|\mathcal{F}_s]) \quad (12)$$

for all $Q \in \mathcal{M}_1(P)$, $X \in L^\infty(\Omega, \mathcal{F}, P)$ and $0 \leq s \leq t$.

4 Time consistency properties

In the dynamical setting risk assessment of a financial position is updated when new information is released. This leads to the notion of a dynamic risk measure.

Definition 11. A sequence $(\rho_t)_{t \in \mathbb{T}}$ is called a dynamic convex risk measure if ρ_t is a conditional convex risk measure for each $t \in \mathbb{T}$.

A key question in the dynamical setting is how the conditional risk assessments at different times are interrelated. This question has led to several notions of time consistency discussed in the literature. A unifying view was suggested in [33].

Definition 12. Assume that $(\rho_t)_{t \in \mathbb{T}}$ is a dynamic convex risk measure and let \mathcal{Y}_t be a subset of L^∞ such that $0 \in \mathcal{Y}_t$ and $\mathcal{Y}_t + \mathbb{R} = \mathcal{Y}_t$ for each $t \in \mathbb{T}$. Then $(\rho_t)_{t \in \mathbb{T}}$ is called acceptance (resp. rejection) consistent with respect to $(\mathcal{Y}_t)_{t \in \mathbb{T}}$, if for all $t \in \mathbb{T}$ such that $t < T$ and for any $X \in L^\infty$ and $Y \in \mathcal{Y}_{t+1}$ the following condition holds:

$$\rho_{t+1}(X) \leq \rho_{t+1}(Y) \text{ (resp. } \geq) \implies \rho_t(X) \leq \rho_t(Y) \text{ (resp. } \geq). \quad (13)$$

The idea is that the degree of time consistency is determined by a sequence of benchmark sets $(\mathcal{Y}_t)_{t \in \mathbb{T}}$: if a financial position at some future time is always preferable to some element of the benchmark set, then it should also be preferable today. The bigger the benchmark set, the stronger is the resulting notion of time consistency. In the following we focus on three cases.

Definition 13. We call a dynamic convex risk measure $(\rho_t)_{t \in \mathbb{T}}$

1. strongly time consistent, if it is either acceptance consistent or rejection consistent with respect to $\mathcal{Y}_t = L^\infty$ for all t in the sense of Definition 12;
2. middle acceptance (resp. middle rejection) consistent, if for all t we have $\mathcal{Y}_t = L_t^\infty$ in Definition 12;
3. weakly acceptance (resp. weakly rejection) consistent, if for all t we have $\mathcal{Y}_t = \mathbb{R}$ in Definition 12.

Note that there is no difference between rejection consistency and acceptance consistency with respect to L^∞ , since the role of X and Y is symmetric in that case. Obviously strong time consistency implies both middle rejection and middle acceptance consistency, and middle rejection (resp. middle acceptance) consistency implies weak rejection (resp. weak acceptance) consistency. In the rest of the paper we drop the terms “middle” and “strong” in order to simplify the terminology.

4.1 Time consistency

Time consistency has been studied extensively in the recent work on dynamic risk measures, see [4, 15, 30, 17, 12, 26, 9, 8, 29, 20, 13, 16] and the references therein. In the next proposition we recall some equivalent characterizations of time consistency.

Proposition 14. A dynamic convex risk measure $(\rho_t)_{t \in \mathbb{T}}$ is time consistent if and only if any of the following conditions holds:

1. for all $t \in \mathbb{T}$ such that $t < T$ and for all $X, Y \in L^\infty$:

$$\rho_{t+1}(X) \leq \rho_{t+1}(Y) \text{ } P\text{-a.s.} \implies \rho_t(X) \leq \rho_t(Y) \text{ } P\text{-a.s.}; \quad (14)$$

2. for all $t \in \mathbb{T}$ such that $t < T$ and for all $X, Y \in L^\infty$:

$$\rho_{t+1}(X) = \rho_{t+1}(Y) \text{ } P\text{-a.s.} \implies \rho_t(X) = \rho_t(Y) \text{ } P\text{-a.s.}; \quad (15)$$

3. $(\rho_t)_{t \in \mathbb{T}}$ is recursive, i.e.

$$\rho_t = \rho_t(-\rho_{t+s}) \text{ } P\text{-a.s.}$$

for all $t, s \geq 0$ such that $t, t+s \in \mathbb{T}$.

Proof. It is obvious that time consistency implies condition (14), and that (14) implies (15). By cash invariance we have $\rho_{t+1}(-\rho_{t+1}(X)) = \rho_{t+1}(X)$ and hence one-step recursiveness follows from (15). We

prove that one-step recursiveness implies recursiveness by induction on s . For $s = 1$ the claim is true for all t . Assume that the induction hypothesis holds for each t and all $k \leq s$ for some $s \geq 1$. Then we obtain

$$\begin{aligned}\rho_t(-\rho_{t+s+1}(X)) &= \rho_t(-\rho_{t+s}(-\rho_{t+s+1}(X))) \\ &= \rho_t(-\rho_{t+s}(X)) \\ &= \rho_t(X),\end{aligned}$$

where we have applied the induction hypothesis to the random variable $-\rho_{t+s+1}(X)$. Hence the claim follows. Finally, due to monotonicity, recursiveness implies time consistency. \square

If we restrict a conditional convex risk measure ρ_t to the space L_{t+s}^∞ for some $s \geq 0$, the corresponding acceptance set is given by

$$\mathcal{A}_{t,t+s} := \{ X \in L_{t+s}^\infty \mid \rho_t(X) \leq 0 \text{ } P\text{-a.s.} \},$$

and the minimal penalty function by

$$\alpha_{t,t+s}^{\min}(Q) := \underset{X \in \mathcal{A}_{t,t+s}}{\text{Q-ess sup}} E_Q[-X \mid \mathcal{F}_t], \quad Q \in \mathcal{M}_1(P). \quad (16)$$

The following lemma recalls equivalent characterizations of recursive inequalities in terms of acceptance sets from [20, Lemma 4.6]; property (17) was shown in [15].

Lemma 15. *Let $(\rho_t)_{t \in \mathbb{T}}$ be a dynamic convex risk measure. Then the following equivalences hold for all s, t such that $t, t+s \in \mathbb{T}$ and all $X \in L^\infty$:*

$$X \in \mathcal{A}_{t,t+s} + \mathcal{A}_{t+s} \iff -\rho_{t+s}(X) \in \mathcal{A}_{t,t+s} \quad (17)$$

$$\mathcal{A}_t \subseteq \mathcal{A}_{t,t+s} + \mathcal{A}_{t+s} \iff \rho_t(-\rho_{t+s}) \leq \rho_t \text{ } P\text{-a.s.} \quad (18)$$

$$\mathcal{A}_t \supseteq \mathcal{A}_{t,t+s} + \mathcal{A}_{t+s} \iff \rho_t(-\rho_{t+s}) \geq \rho_t \text{ } P\text{-a.s.} \quad (19)$$

Proof. To prove “ \Rightarrow ” in (17) let $X = X_{t,t+s} + X_{t+s}$ with $X_{t,t+s} \in \mathcal{A}_{t,t+s}$ and $X_{t+s} \in \mathcal{A}_{t+s}$. Then

$$\rho_{t+s}(X) = \rho_{t+s}(X_{t+s}) - X_{t,t+s} \leq -X_{t,t+s}$$

by cash invariance, and monotonicity implies

$$\rho_t(-\rho_{t+s}(X)) \leq \rho_t(X_{t,t+s}) \leq 0.$$

The converse direction follows immediately from $X = X + \rho_{t+s}(X) - \rho_{t+s}(X)$ and $X + \rho_{t+s}(X) \in \mathcal{A}_{t+s}$ for all $X \in L^\infty$.

In order to show “ \Rightarrow ” in (18), fix $X \in L^\infty$. Since $X + \rho_t(X) \in \mathcal{A}_t \subseteq \mathcal{A}_{t,t+s} + \mathcal{A}_{t+s}$, we obtain

$$\rho_{t+s}(X) - \rho_t(X) = \rho_{t+s}(X + \rho_t(X)) \in -\mathcal{A}_{t,t+s},$$

by (17) and cash invariance. Hence

$$\rho_t(-\rho_{t+s}(X)) - \rho_t(X) = \rho_t(-(\rho_{t+s}(X) - \rho_t(X))) \leq 0 \text{ } P\text{-a.s.}$$

To prove “ \Leftarrow ” let $X \in \mathcal{A}_t$. Then $-\rho_{t+s}(X) \in \mathcal{A}_{t,t+s}$ by the right hand side of (18), and hence $X \in \mathcal{A}_{t,t+s} + \mathcal{A}_{t+s}$ by (17).

Now let $X \in L^\infty$ and assume $\mathcal{A}_t \supseteq \mathcal{A}_{t,t+s} + \mathcal{A}_{t+s}$. Then

$$\begin{aligned} \rho_t(-\rho_{t+s}(X)) + X &= \rho_t(-\rho_{t+s}(X)) - \rho_{t+s}(X) + \rho_{t+s}(X) + X \\ &\in \mathcal{A}_{t,t+s} + \mathcal{A}_{t+s} \subseteq \mathcal{A}_t. \end{aligned}$$

Hence

$$\rho_t(X) - \rho_t(-\rho_{t+s}(X)) = \rho_t(X + \rho_t(-\rho_{t+s}(X))) \leq 0$$

by cash invariance, and this proves “ \Rightarrow ” in (19). For the converse direction let $X \in \mathcal{A}_{t,t+s} + \mathcal{A}_{t+s}$. Since $-\rho_{t+s}(X) \in \mathcal{A}_{t,t+s}$ by (17), we obtain

$$\rho_t(X) \leq \rho_t(-\rho_{t+s}(X)) \leq 0,$$

hence $X \in \mathcal{A}_t$. □

We also have the following relation between acceptance sets and penalty functions; cf. [29, Lemma 2.2.5].

Lemma 16. *Let $(\rho_t)_{t \in \mathbb{T}}$ be a dynamic convex risk measures. Then the following implications hold for all t, s such that $t, t+s \in \mathbb{T}$ and for all $Q \in \mathcal{M}_1(P)$:*

$$\begin{aligned} \mathcal{A}_t \subseteq \mathcal{A}_{t,t+s} + \mathcal{A}_{t+s} &\Rightarrow \alpha_t^{\min}(Q) \leq \alpha_{t,t+s}^{\min}(Q) + E_Q[\alpha_{t+s}^{\min}(Q) | \mathcal{F}_t] \quad Q\text{-a.s.} \\ \mathcal{A}_t \supseteq \mathcal{A}_{t,t+s} + \mathcal{A}_{t+s} &\Rightarrow \alpha_t^{\min}(Q) \geq \alpha_{t,t+s}^{\min}(Q) + E_Q[\alpha_{t+s}^{\min}(Q) | \mathcal{F}_t] \quad Q\text{-a.s.} \end{aligned}$$

Proof. Straightforward from the definition of the minimal penalty function and Lemma 4. □

The following theorem gives equivalent characterizations of time consistency in terms of acceptance sets, penalty functions, and a supermartingale property of the risk process.

Theorem 17. *Let $(\rho_t)_{t \in \mathbb{T}}$ be a dynamic convex risk measure such that each ρ_t is continuous from above. Then the following conditions are equivalent:*

1. $(\rho_t)_{t \in \mathbb{T}}$ is time consistent.
2. $\mathcal{A}_t = \mathcal{A}_{t,t+s} + \mathcal{A}_{t+s}$ for all t, s such that $t, t+s \in \mathbb{T}$.
3. $\alpha_t^{\min}(Q) = \alpha_{t,t+s}^{\min}(Q) + E_Q[\alpha_{t+s}^{\min}(Q) | \mathcal{F}_t]$ Q -a.s. for all t, s such that $t, t+s \in \mathbb{T}$ and all $Q \in \mathcal{M}_1(P)$.
4. For all $X \in L^\infty(\Omega, \mathcal{F}, P)$ and all t, s such that $t, t+s \in \mathbb{T}$ and all $Q \in \mathcal{M}_1(P)$ we have

$$E_Q[\rho_{t+s}(X) + \alpha_{t+s}^{\min}(Q) | \mathcal{F}_t] \leq \rho_t(X) + \alpha_t^{\min}(Q) \quad Q\text{-a.s.}$$

Equivalence of properties 1) and 2) of Theorem 17 was proved in [15]. Characterizations of time consistency in terms of penalty functions as in 3) of Theorem 17 appeared in [20, 7, 13, 8]; similar results for risk measures for processes were given in [12, 13]. The supermartingale property as in 4) of Theorem 17 was obtained in [20]; cf. also [8] for the absolutely continuous case.

Proof. The proof of 1) \Rightarrow 2) \Rightarrow 3) follows from Lemma 15 and Lemma 16. To prove 3) \Rightarrow 4) fix $Q \in \mathcal{M}_1(P)$. By (12) we have

$$E_Q[\rho_{t+s}(X)|\mathcal{F}_t] = \text{Q-ess sup}_{R \in \mathcal{Q}_{t+s}^f(Q)} (E_R[-X|\mathcal{F}_t] - E_R[\alpha_{t+s}^{\min}(R)|\mathcal{F}_t]).$$

On the set $\{\alpha_t^{\min}(Q) = \infty\}$ property 4) holds trivially. On the set $\{\alpha_t^{\min}(Q) < \infty\}$ property 3) implies $E_Q[\alpha_{t+s}^{\min}(Q)|\mathcal{F}_t] < \infty$ and $\alpha_{t,t+s}^{\min}(Q) < \infty$, then for $R \in \mathcal{Q}_{t+s}^f(Q)$

$$\alpha_t^{\min}(R) = \alpha_{t,t+s}^{\min}(Q) + E_R[\alpha_{t+s}^{\min}(R)|\mathcal{F}_t] < \infty \quad \text{Q-a.s.}$$

Thus

$$E_Q[\rho_{t+s}(X) + \alpha_{t+s}^{\min}(Q)|\mathcal{F}_t] = \text{Q-ess sup}_{R \in \mathcal{Q}_{t+s}^f(Q)} (E_R[-X|\mathcal{F}_t] - \alpha_t^{\min}(R)) + \alpha_t^{\min}(Q)$$

on $\{\alpha_t^{\min}(Q) < \infty\}$. Moreover, since $\mathcal{Q}_{t+s}^f(Q) \subseteq \mathcal{Q}_t(Q)$, (9) implies

$$E_Q[\rho_{t+s}(X) + \alpha_{t+s}^{\min}(Q)|\mathcal{F}_t] \leq \text{Q-ess sup}_{R \in \mathcal{Q}_t(Q)} (E_R[-X|\mathcal{F}_t] - \alpha_t^{\min}(R)) + \alpha_t^{\min}(Q) = \rho_t(X) + \alpha_t^{\min}(Q) \quad \text{Q-a.s.}$$

It remains to prove 4) \Rightarrow 1). To this end fix $Q \in \mathcal{Q}_t^f$ and $X, Y \in L^\infty$ such that $\rho_{t+1}(X) \leq \rho_{t+1}(Y)$. Note that $E_Q[\alpha_{t+s}(Q)] < \infty$ due to 4), hence $Q \in \mathcal{Q}_{t+s}^f(Q)$. Using 4) and representation (10) for ρ_{t+s} under Q , we obtain

$$\begin{aligned} \rho_t(Y) + \alpha_t^{\min}(Q) &\geq E_Q[\rho_{t+1}(Y) + \alpha_{t+1}^{\min}(Q)|\mathcal{F}_t] \\ &\geq E_Q[\rho_{t+1}(X) + \alpha_{t+1}^{\min}(Q)|\mathcal{F}_t] \\ &\geq E_Q[E_Q[-X|\mathcal{F}_{t+1}] - \alpha_{t+1}^{\min}(Q) + \alpha_{t+1}^{\min}(Q)|\mathcal{F}_t] \\ &= E_Q[-X|\mathcal{F}_t]. \end{aligned}$$

Hence representation (6) yields $\rho_t(y) \geq \rho_t(X)$, and time consistency follows from Proposition 14. \square

Properties 3) and 4) of Theorem 17 imply in particular supermartingale properties of penalty function processes and risk processes. This allows to apply martingale theory for characterization the the dynamics of these processes, as we do in Proposition 18 and Proposition 21; cf. also [15, 20, 29, 8, 16].

Proposition 18. *Let $(\rho_t)_{t \in \mathbb{T}}$ be a time consistent dynamic convex risk measure such that each ρ_t is continuous from above. Then the process*

$$V_t^Q(X) := \rho_t(X) + \alpha_t^{\min}(Q), \quad t \in \mathbb{T}$$

is a Q-supermartingale for all $X \in L^\infty$ and all $Q \in \mathcal{Q}_0$, where

$$\mathcal{Q}_0 := \{Q \in \mathcal{M}_1(P) \mid \alpha_0^{\min}(Q) < \infty\}.$$

Moreover, $(V_t^Q(X))_{t \in \mathbb{T}}$ is a Q-martingale if $Q \in \mathcal{Q}_0$ is a ‘‘worst case’’ measure for X at time 0, i.e. if the supremum in the robust representation of $\rho_0(X)$ is attained at Q :

$$\rho_0(X) = E_Q[-X] - \alpha_0^{\min}(Q) \quad \text{Q-a.s.}$$

In this case Q is a “worst case” measure for X at any time t , i.e.

$$\rho_t(X) = E_Q[-X|\mathcal{F}_t] - \alpha_t^{\min}(Q) \quad Q\text{-a.s. for all } t \in \mathbb{T}.$$

The converse holds if $T < \infty$ or $\lim_{t \rightarrow \infty} \rho_t(X) = -X$ P -a.s. (what is called asymptotic precision in [20]): If $(V_t^Q(X))_{t \in \mathbb{T}}$ is a Q -martingale then $Q \in \mathcal{Q}_0$ is a “worst case” measure for X at any time $t \in \mathbb{T}$.

Proof. The supermartingale property of $(V_t^Q(X))_{t \in \mathbb{T}}$ under each $Q \in \mathcal{Q}_0$ follows directly from properties 3) and 4) of Theorem 17. To prove the remaining part of the claim, fix $Q \in \mathcal{Q}_0$ and $X \in L^\infty$. If Q is a “worst case” measure for X at time 0, the process

$$U_t(X) := V_t^Q(X) - E_Q[-X|\mathcal{F}_t], \quad t \in \mathbb{T}$$

is a non-negative Q -supermartingale beginning at 0. Indeed, the supermartingale property follows from that of $(V_t^Q(X))_{t \in \mathbb{T}}$, and non-negativity follows from the representation (10), since $Q \in \mathcal{Q}_t^f(Q)$. Thus $U_t = 0$ Q -a.s. for all t , and this proves the “if” part of the claim. To prove the converse direction, note that if $(V_t^Q(X))_{t \in \mathbb{T}}$ is a Q -martingale and $\rho_T(X) = -X$ (resp. $\lim_{t \rightarrow \infty} \rho_t(X) = -X$ P -a.s.), the process $U(X)$ is a Q -martingale ending at 0 (resp. converging to 0 in $L^1(Q)$), and thus $U_t(X) = 0$ Q -a.s. for all $t \in \mathbb{T}$. \square

Remark 19. The fact that a worst case measure for X at time 0, if it exists, remains a worst case measure for X at any time $t \in \mathbb{T}$ was also shown in [13, Theorem 3.9] for a time consistent dynamic risk measure without using the supermartingale property from Proposition 18.

Remark 20. In difference to [20, Theorem 4.5], without the additional assumption that the set

$$\mathcal{Q}^* := \{ Q \in \mathcal{M}^e(P) \mid \alpha_0^{\min}(Q) < \infty \} \quad (20)$$

is nonempty, the supermartingale property of $(V_t^Q(X))_{t \in \mathbb{T}}$ for all $X \in L^\infty$ and all $Q \in \mathcal{Q}^*$ is not sufficient to prove time consistency. In this case we also do not have the robust representation of ρ_t in terms of the set \mathcal{Q}^* .

The process $(\alpha_t^{\min}(Q))_{t \in \mathbb{T}}$ is a Q -supermartingale for all $Q \in \mathcal{Q}_0$ due to Property 3) of Theorem 17. The next proposition provides the explicit form of its Doob- and its Riesz-decomposition; cf. also [29, Proposition 2.3.2].

Proposition 21. Let $(\rho_t)_{t \in \mathbb{T}}$ be a time consistent dynamic convex risk measure such that each ρ_t is continuous from above. Then for each $Q \in \mathcal{Q}_0$ the process $(\alpha_t^{\min}(Q))_{t \in \mathbb{T}}$ is a non-negative Q -supermartingale with the Riesz decomposition

$$\alpha_t^{\min}(Q) = Z_t^Q + M_t^Q \quad Q\text{-a.s.}, \quad t \in \mathbb{T},$$

where

$$Z_t^Q := E_Q \left[\sum_{k=t}^{T-1} \alpha_{k,k+1}^{\min}(Q) \mid \mathcal{F}_t \right] \quad Q\text{-a.s.}, \quad t \in \mathbb{T}$$

is a Q -potential and

$$M_t^Q := \begin{cases} 0 & \text{if } T < \infty, \\ \lim_{s \rightarrow \infty} E_Q [\alpha_s(Q) | \mathcal{F}_t] & \text{if } T = \infty \end{cases} \quad Q\text{-a.s.}, \quad t \in \mathbb{T}$$

is a non-negative Q -martingale.

Moreover, the Doob decomposition of $(\alpha_t^{\min}(Q))_{t \in \mathbb{T}}$ is given by

$$\alpha_t^{\min}(Q) = E_Q \left[\sum_{k=0}^{T-1} \alpha_{k,k+1}^{\min}(Q) | \mathcal{F}_t \right] + M_t^Q - \sum_{k=0}^{t-1} \alpha_{k,k+1}^{\min}(Q), \quad t \in \mathbb{T}$$

with the Q -martingale

$$E_Q \left[\sum_{k=0}^{T-1} \alpha_{k,k+1}^{\min}(Q) | \mathcal{F}_t \right] + M_t^Q, \quad t \in \mathbb{T}$$

and the non-decreasing predictable process $(\sum_{k=0}^{t-1} \alpha_{k,k+1}^{\min}(Q))_{t \in \mathbb{T}}$.

Proof. We fix $Q \in \mathcal{M}_1(P)$ and applying property 3) of Theorem 17 step by step we obtain

$$\alpha_t^{\min}(Q) = E_Q \left[\sum_{k=t}^{t+s-1} \alpha_{k,k+1}^{\min}(Q) | \mathcal{F}_t \right] + E_Q [\alpha_{t+s}^{\min}(Q) | \mathcal{F}_t] \quad Q\text{-a.s.} \quad (21)$$

for all t, s such that $t, t+s \in \mathbb{T}$. If $T < \infty$, the Doob- and Riesz-decompositions follow immediately from (21), since $\alpha_T(Q) = 0$ Q -a.s.. If $T = \infty$, by monotonicity there exists the limit

$$Z_t^Q = \lim_{s \rightarrow \infty} E_Q \left[\sum_{k=t}^s \alpha_{k,k+1}^{\min}(Q) | \mathcal{F}_t \right] = E_Q \left[\sum_{k=t}^{\infty} \alpha_{k,k+1}^{\min}(Q) | \mathcal{F}_t \right] \quad Q\text{-a.s.}$$

for all $t \in \mathbb{T}$, where we have used the monotone convergence theorem for the second equality. Equality (21) implies then that there exists

$$M_t^Q = \lim_{s \rightarrow \infty} E_Q [\alpha_{t+s}^{\min}(Q) | \mathcal{F}_t] \quad Q\text{-a.s.}, \quad t \in \mathbb{T}$$

and

$$\alpha_t^{\min}(Q) = Z_t^Q + M_t^Q \quad Q\text{-a.s.}$$

for all $t \in \mathbb{T}$.

The process $(Z_t^Q)_{t \in \mathbb{T}}$ is a non-negative Q -supermartingale. Indeed,

$$E_Q [Z_t^Q] \leq E_Q \left[\sum_{k=0}^{\infty} \alpha_{k,k+1}^{\min}(Q) \right] \leq \alpha_0^{\min}(Q) < \infty \quad (22)$$

and $E_Q [Z_{t+1}^Q | \mathcal{F}_t] \leq Z_t^Q$ Q -a.s. for all $t \in \mathbb{T}$ by definition. Moreover, monotone convergence implies

$$\lim_{t \rightarrow \infty} E_Q [Z_t^Q] = E_Q \left[\lim_{t \rightarrow \infty} \sum_{k=t}^{\infty} \alpha_{k,k+1}^{\min}(Q) \right] = 0 \quad Q\text{-a.s.},$$

since $\sum_{k=0}^{\infty} \alpha_{k,k+1}^{\min}(Q) < \infty$ Q -a.s. by (22). Hence the process $(Z_t^Q)_{t \in \mathbb{T}}$ is a Q -potential.

The process $(M_t^Q)_{t \in \mathbb{T}}$ is a non-negative Q -martingale, since

$$E_Q[M_t^Q] \leq E_Q[\alpha_t^{\min}(Q)] \leq \alpha_0^{\min}(Q) < \infty$$

and

$$\begin{aligned} E_Q[M_{t+1}^Q - M_t^Q | \mathcal{F}_t] &= E_Q[\alpha_{t+1}^{\min}(Q) | \mathcal{F}_t] - \alpha_t^{\min}(Q) - E_Q[Z_{t+1}^Q - Z_t^Q | \mathcal{F}_t] \\ &= \alpha_{t,t+1}^{\min}(Q) - \alpha_{t,t+1}^{\min}(Q) = 0 \quad Q\text{-a.s.} \end{aligned}$$

for all $t \in \mathbb{T}$ by property 3) of Theorem 17 and the definition of $(Z_t^Q)_{t \in \mathbb{T}}$.

The Doob-decomposition follows straightforward from the Riesz-decomposition. \square

Remark 22. *It was shown in [20, Theorem 5.4] that the martingale M^Q in the Riesz decomposition of $(\alpha_t^{\min}(Q))_{t \in \mathbb{T}}$ vanishes if and only if $\lim_{t \rightarrow \infty} \rho_t(X) \geq -X$ P -a.s., i.e. the dynamic risk measure $(\rho_t)_{t \in \mathbb{T}}$ is asymptotically safe. This is not always the case; see [20, Example 5.5].*

For a *coherent* risk measure we have

$$\mathcal{Q}_t^f(Q) = \mathcal{Q}_t^0(Q) := \{ R \in \mathcal{M}^1(P) \mid R = Q|_{\mathcal{F}_t}, \alpha_t^{\min}(R) = 0 \text{ } Q\text{-a.s.} \}.$$

In order to give an equivalent characterization of property 3) of Theorem 17 in the coherent case, we introduce the sets

$$\mathcal{Q}_{t,t+s}^0(Q) = \{ R \ll P|_{\mathcal{F}_{t+s}} \mid R = Q|_{\mathcal{F}_t}, \alpha_{t,t+s}^{\min}(R) = 0 \text{ } Q\text{-a.s.} \} \quad \forall t, s \geq 0 \text{ such that } t, t+s \in \mathbb{T}.$$

For $Q^1 \in \mathcal{Q}_{t,t+s}^0(Q)$ and $Q^2 \in \mathcal{Q}_{t+s}^0(Q)$ we denote by $Q^1 \oplus^{t+s} Q^2$ the pasting of Q^1 and Q^2 in $t+s$ via Ω , i.e. the measure \tilde{Q} defined via

$$\tilde{Q}(A) = E_{Q^1} [E_{Q^2}[I_A | \mathcal{F}_{t+s}]], \quad A \in \mathcal{F}. \quad (23)$$

The relation between stability under pasting and time consistency of coherent risk measures that can be represented in terms of equivalent probability measures was studied in [4, 15, 26, 20]. In our present setting, Theorem 17 applied to a coherent risk measure takes the following form.

Corollary 23. *Let $(\rho_t)_{t \in \mathbb{T}}$ be a dynamic coherent risk measure such that each ρ_t is continuous from above. Then the following conditions are equivalent:*

1. $(\rho_t)_{t \in \mathbb{T}}$ is time consistent.
2. For all $Q \in \mathcal{M}_1(P)$ and all t, s such that $t, t+s \in \mathbb{T}$

$$\mathcal{Q}_t^0(Q) = \{ Q^1 \oplus^{t+s} Q^2 \mid Q^1 \in \mathcal{Q}_{t,t+s}^0(Q), Q^2 \in \mathcal{Q}_{t+s}^0(Q^1) \}.$$

3. For all $Q \in \mathcal{M}_1(P)$ such that $\alpha_t^{\min}(Q) = 0$ Q -a.s.,

$$E_Q[\rho_{t+s}(X) | \mathcal{F}_t] \leq \rho_t(X) \quad \text{and} \quad \alpha_{t+s}^{\min}(Q) = 0 \text{ } Q\text{-a.s.}$$

for all $X \in L^\infty(\Omega, \mathcal{F}, P)$ and for all t, s such that $t, t+s \in \mathbb{T}$.

Proof. 1) \Rightarrow 2): Time consistency implies property 3) of Theorem 17, and we will show that this implies property 2) of Corollary 23. Fix $Q \in \mathcal{M}_1(P)$. To prove “ \supseteq ” let $Q^1 \in \mathcal{Q}_t^0(Q)$, $Q^2 \in \mathcal{Q}_{t+s}^0(Q^1)$, and consider \tilde{Q} defined as in (23). Note that $\tilde{Q} = Q^1$ on \mathcal{F}_{t+s} and

$$E_{\tilde{Q}}[X|\mathcal{F}_{t+s}] = E_{Q^2}[X|\mathcal{F}_{t+s}] \quad Q^1\text{-a.s. for all } X \in L^\infty(\Omega, \mathcal{F}, P).$$

Hence, using 3) of Theorem 17 we obtain

$$\begin{aligned} \alpha_t^{\min}(\tilde{Q}) &= \alpha_{t,t+s}^{\min}(\tilde{Q}) + E_{\tilde{Q}}[\alpha_{t,t+s}^{\min}(\tilde{Q})|\mathcal{F}_t] \\ &= \alpha_{t,t+s}^{\min}(Q^1) + E_{Q^1}[\alpha_{t,t+s}^{\min}(Q^2)|\mathcal{F}_t] = 0 \quad Q\text{-a.s.}, \end{aligned}$$

and thus $\tilde{Q} \in \mathcal{Q}_t^0(Q)$. Conversely, for every $\tilde{Q} \in \mathcal{Q}_t^0(Q)$ we have $\alpha_{t,t+s}^{\min}(\tilde{Q}) = \alpha_{t,t+s}^{\min}(\tilde{Q}) = 0$ \tilde{Q} -a.s. by 3) of Theorem 17, and $\tilde{Q} = \tilde{Q} \oplus \tilde{Q}$. This proves “ \subseteq ”.

2) \Rightarrow 3): Let $R \in \mathcal{M}_1(P)$ with $\alpha_t^{\min}(R) = 0$ R -a.s.. Then $R \in \mathcal{Q}_t^0(R)$, and thus $R = Q^1 \otimes^{t+s} Q^2$ for some $Q^1 \in \mathcal{Q}_{t,t+s}^0(R)$ and $Q^2 \in \mathcal{Q}_{t+s}^0(Q^1)$. This implies $R = Q^1$ on \mathcal{F}_{t+s} and

$$E_R[X|\mathcal{F}_{t+s}] = E_{Q^2}[X|\mathcal{F}_{t+s}] \quad R\text{-a.s.}$$

Hence $\alpha_{t,t+s}^{\min}(R) = \alpha_{t,t+s}^{\min}(Q^1) = 0$ R -a.s., and $\alpha_{t+s}^{\min}(R) = \alpha_{t+s}^{\min}(Q^2) = 0$ R -a.s.. To prove the inequality 3) note that due to (12)

$$\begin{aligned} E_R[\rho_{t+s}(X) | \mathcal{F}_t] &= \text{R-ess sup}_{Q \in \mathcal{Q}_{t+s}^0(R)} E_Q[-X | \mathcal{F}_t] \\ &\leq \text{R-ess sup}_{Q \in \mathcal{Q}_t^0(R)} E_Q[-X | \mathcal{F}_t] = \rho_t(X) \quad R\text{-a.s.}, \end{aligned}$$

where we have used that the pasting of $R|_{\mathcal{F}_{t+s}}$ and Q belongs to $\mathcal{Q}_t^0(R)$.

3) \Rightarrow 1): Obviously property 3) of Corollary 23 implies property 4) of Theorem 17 and thus time consistency. \square

4.2 Rejection and acceptance consistency

Rejection and acceptance consistency were introduced and studied in [33, 18, 29]. These properties can be characterized via recursive inequalities as stated in the next proposition; see [33, Theorem 3.1.5] and [18, Proposition 3.5].

Proposition 24. *A dynamic convex risk measure $(\rho_t)_{t \in \mathbb{T}}$ is rejection (resp. acceptance) consistent if and only if for all $t \in \mathbb{T}$ such that $t < T$*

$$\rho_t(-\rho_{t+1}) \leq \rho_t \quad (\text{resp. } \geq) \quad P\text{-a.s.} \quad (24)$$

Proof. We argue for the case of rejection consistency; the case of acceptance consistency follows in the same manner. Assume first that $(\rho_t)_{t \in \mathbb{T}}$ satisfies (24) and let $X \in L^\infty$ and $Y \in L^\infty(\mathcal{F}_{t+1})$ such that $\rho_{t+1}(X) \geq \rho_{t+1}(Y)$. Using cash invariance, (24), and monotonicity, we obtain

$$\rho_t(X) \geq \rho_t(-\rho_{t+1}(X)) \geq \rho_t(-\rho_{t+1}(Y)) = \rho_t(Y).$$

The converse implication follows due to cash invariance by applying (13) to $Y = -\rho_{t+1}(X)$. \square

Remark 25. For a dynamic coherent risk measure, weak acceptance consistency and acceptance consistency are equivalent. This was shown in [18, Proposition 3.9].

Another way to characterize rejection consistency was suggested in [29].

Proposition 26. A dynamic convex risk measure $(\rho_t)_{t \in \mathbb{T}}$ is rejection consistent if and only if any of the following conditions holds:

1. For all $t \in \mathbb{T}$ such that $t < T$ and all $X \in L^\infty$

$$\rho_t(X) - \rho_{t+1}(X) \in \mathcal{A}_{t,t+1}; \quad (25)$$

2. For all $t \in \mathbb{T}$ such that $t < T$ and all $X \in \mathcal{A}_t$, we have $-\rho_{t+1}(X) \in \mathcal{A}_t$.

Proof. Since

$$\rho_t(-\rho_{t+1}(X)) = \rho_t(\rho_t(X) - \rho_{t+1}(X)) + \rho_t(X)$$

by cash invariance, (25) implies rejection consistency, and obviously rejection consistency implies condition 2). If 2) holds, then for any $X \in L^\infty$

$$\rho_t(\rho_t(X) - \rho_{t+1}(X)) = \rho_t(-\rho_{t+1}(X + \rho_t(X))) \leq 0,$$

due to cash invariance and the fact that $X + \rho_t(X) \in \mathcal{A}_t$. □

Property (25) was introduced in [29] under the name *prudence*. It means that the adjustment $\rho_{t+1}(X) - \rho_t(X)$ of the minimal capital requirement for X at time $t + 1$ is acceptable at time t . In other words, one stays on the safe side at each period of time by making capital reserves according to a rejection consistent dynamic risk measure.

Similar to time consistency, rejection and acceptance consistency can be characterized in terms of acceptance sets and penalty functions.

Theorem 27. Let $(\rho_t)_{t \in \mathbb{T}}$ be a dynamic convex risk measure such that each ρ_t is continuous from above. Then the following properties are equivalent:

1. $(\rho_t)_{t \in \mathbb{T}}$ is rejection consistent (resp. acceptance consistent).
2. The inclusion

$$\mathcal{A}_t \subseteq \mathcal{A}_{t,t+1} + \mathcal{A}_{t+1} \quad \text{resp.} \quad \mathcal{A}_t \supseteq \mathcal{A}_{t,t+1} + \mathcal{A}_{t+1}$$

holds for all $t \in \mathbb{T}$ such that $t < T$.

3. The inequality

$$\alpha_t^{\min}(Q) \leq (\text{resp. } \geq) \alpha_{t,t+1}^{\min}(Q) + E_Q[\alpha_{t+1}^{\min}(Q) | \mathcal{F}_t] \quad Q\text{-a.s.}$$

holds for all $t \in \mathbb{T}$ such that $t < T$ and all $Q \in \mathcal{M}_1(P)$.

Proof. Equivalence of 1) and 2) was proved in Proposition 24 and Lemma 15, and the proof of 2) \Rightarrow 3) is given in Lemma 16.

Let us show that property 3) implies property 1). We argue for the case of rejection consistency; the case of acceptance consistency follows in the same manner. We fix $t \in \mathbb{T}$ such that $t < T$, and consider the risk measure

$$\tilde{\rho}_t(X) := \rho_t(-\rho_{t+1}(X)), \quad X \in L^\infty.$$

It is easily seen that $\tilde{\rho}_t$ is a conditional convex risk measure that is continuous from above. Moreover, the dynamic risk measure $(\tilde{\rho}_t, \rho_{t+1})$ is time consistent by definition, and thus it fulfills properties 2) and 3) of Theorem 17. We denote by $\tilde{\mathcal{A}}_t$ and $\tilde{\mathcal{A}}_{t,t+1}$ the acceptance sets of the risk measure $\tilde{\rho}_t$, and by $\tilde{\alpha}_t^{\min}$ its penalty function. Since

$$\tilde{\rho}_t(X) = \rho_t(-\rho_{t+1}(X)) = \rho_t(X)$$

for all $X \in L_{t+1}$, we have $\tilde{\mathcal{A}}_{t,t+1} = \mathcal{A}_{t,t+1}$, and thus

$$\tilde{\mathcal{A}}_t = \mathcal{A}_{t,t+1} + \mathcal{A}_{t+1}$$

by 2) of Theorem 17. Lemma 16 and property 3) then imply

$$\tilde{\alpha}_t^{\min}(Q) = \alpha_{t,t+1}^{\min}(Q) + E_Q[\alpha_{t+1}^{\min}(Q) | \mathcal{F}_t] \geq \alpha_t^{\min}(Q)$$

for all $Q \in \mathcal{Q}_t$. Thus

$$\rho_t(X) \geq \tilde{\rho}_t(X) = \rho_t(-\rho_{t+1}(X))$$

for all $X \in L^\infty$, due to representation (6). \square

Remark 28. *Similar to Corollary 23, condition 3) of Theorem 27 can be restated for a dynamic coherent risk measure $(\rho_t)_{t \in \mathbb{T}}$ as follows:*

$$\mathcal{Q}_t^0(Q) \supseteq \{ Q^1 \oplus^{t+1} Q^2 \mid Q^1 \in \mathcal{Q}_{t,t+1}^0(Q), Q^2 \in \mathcal{Q}_{t+1}^0(Q^1) \} \quad (\text{resp. } \subseteq)$$

for all $t \in \mathbb{T}$ such that $t < T$ and all $Q \in \mathcal{M}_1(P)$.

The following proposition provides an additional equivalent characterization of rejection consistency, that can be viewed as an analogon of the supermartingale property 4) of Theorem 17.

Proposition 29. *Let $(\rho_t)_{t \in \mathbb{T}}$ be a dynamic convex risk measure such that each ρ_t is continuous from above. Then $(\rho_t)_{t \in \mathbb{T}}$ is rejection consistent if and only if the inequality*

$$E_Q[\rho_{t+1}(X) | \mathcal{F}_t] \leq \rho_t(X) + \alpha_{t,t+1}^{\min}(Q) \quad Q\text{-a.s.} \quad (26)$$

holds for all $Q \in \mathcal{M}_1(P)$ and all $t \in \mathbb{T}$ such that $t < T$. In this case the process

$$U_t^Q(X) := \rho_t(X) - \sum_{k=0}^{t-1} \alpha_{k,k+1}^{\min}(Q), \quad t \in \mathbb{T}$$

is a Q -supermartingale for all $X \in L^\infty$ and all $Q \in \mathcal{Q}^f$, where

$$\mathcal{Q}^f := \left\{ Q \in \mathcal{M}_1(P) \mid E_Q \left[\sum_{k=0}^t \alpha_{k,k+1}^{\min}(Q) \right] < \infty \quad \forall t \in \mathbb{T} \right\}.$$

The proof of Proposition 29 is a special case of Theorem 31, which involves the notion of sustainability; cf. [29].

Definition 30. Let $(\rho_t)_{t \in \mathbb{T}}$ be a dynamic convex risk measure. We call a bounded adapted process $X = (X_t)_{t \in \mathbb{T}}$ sustainable with respect to the risk measure $(\rho_t)_{t \in \mathbb{T}}$ if

$$\rho_t(X_t - X_{t+1}) \leq 0 \quad \text{for all } t \in \mathbb{T} \text{ such that } t < T.$$

Consider X to be a cumulative investment process. If it is sustainable, then for all $t \in \mathbb{T}$ the adjustment $X_{t+1} - X_t$ is acceptable with respect to ρ_t .

The next theorem characterizes sustainable processes in terms of a supermartingale inequality; it is a generalization of [29, Corollary 2.4.10].

Theorem 31. Let $(\rho_t)_{t \in \mathbb{T}}$ be a dynamic convex risk measure such that each ρ_t is continuous from above and let $(X_t)_{t \in \mathbb{T}}$ be a bounded adapted process. Then the following properties are equivalent:

1. The process $(X_t)_{t \in \mathbb{T}}$ is sustainable with respect to the risk measure $(\rho_t)_{t \in \mathbb{T}}$.
2. For all $Q \in \mathcal{M}_1(P)$ and all $t \in \mathbb{T}, t \geq 1$, we have

$$E_Q[X_t | \mathcal{F}_{t-1}] \leq X_{t-1} + \alpha_{t-1,t}^{\min}(Q) \quad Q\text{-a.s.} \quad (27)$$

Proof. The proof of 1) \Rightarrow 2) follows directly from the definition of sustainability and the definition of the minimal penalty function.

To prove 2) \Rightarrow 1), let $(X_t)_{t \in \mathbb{T}}$ be a bounded adapted process such that (27) holds. In order to prove

$$X_t - X_{t-1} =: A_t \in -\mathcal{A}_{t-1,t} \quad \text{for all } t \in \mathbb{T}, t \geq 1,$$

suppose by way of contradiction that $A_t \notin -\mathcal{A}_{t-1,t}$. Since the set $\mathcal{A}_{t-1,t}$ is convex and weak*-closed due to Remark 6, the Hahn-Banach separation theorem (see, e.g., [22, Theorem A.56]) ensures the existence of $Z \in L^1(\mathcal{F}_t, P)$ such that

$$a := \sup_{X \in \mathcal{A}_{t-1,t}} E[Z(-X)] < E[Z A_t] =: b < \infty. \quad (28)$$

Since $\lambda I_{\{Z < 0\}} \in \mathcal{A}_{t-1,t}$ for every $\lambda \geq 0$, (28) implies $Z \geq 0$ P -a.s., and in particular $E[Z] > 0$. Define a probability measure $Q \in \mathcal{M}_1(P)$ via $\frac{dQ}{dP} := \frac{Z}{E[Z]}$ and note that, due to Lemma 4 and (28), we have

$$E_Q[\alpha_{t-1,t}^{\min}(Q)] = \sup_{X \in \mathcal{A}_{t-1,t}} E_Q[(-X)] = \sup_{X \in \mathcal{A}_{t-1,t}} E[Z(-X)] \frac{1}{E[Z]} = \frac{a}{E[Z]} < \infty. \quad (29)$$

Moreover, (28) and (29) imply

$$E_Q[(X_t - X_{t-1} - \alpha_{t-1,t}^{\min}(Q))] = E[Z](E[Z A_t] - E_Q[\alpha_{t-1,t}^{\min}(Q)]) = E[Z](b - a) > 0,$$

which cannot be true if (27) holds under Q . □

Remark 32. In particular, property 2) of Theorem 31 implies that the process

$$X_t - \sum_{k=0}^{t-1} \alpha_{k,k+1}^{\min}(Q), \quad t \in \mathbb{T}$$

is a Q -supermartingale for all $Q \in \mathcal{Q}^f$, if X is sustainable with respect to (ρ_t) . As shown in [29, Theorem 2.4.6, Corollary 2.4.8], this supermartingale property is equivalent to sustainability of X under some additional assumptions.

4.3 Weak time consistency

In this section we characterize the weak notions of time consistency from Definition 13. Due to cash invariance, they can be restated as follows: A dynamic convex risk measure $(\rho_t)_{t \in \mathbb{T}}$ is weakly acceptance (resp. weakly rejection) consistent, if and only if

$$\rho_{t+1}(X) \leq 0 \quad (\text{resp. } \geq) \quad \implies \quad \rho_t(X) \leq 0 \quad (\text{resp. } \geq)$$

for any $X \in L^\infty$ and for all $t \in \mathbb{T}$ such that $t < T$. This means that if some position is accepted (or rejected) for any scenario tomorrow, it should be already accepted (or rejected) today. In this form, weak acceptance consistency was introduced in [4]. Both weak acceptance and weak rejection consistency appeared in [35, 32].

Weak acceptance consistency was characterized in terms of acceptance sets in [33, Corollary 3.6], and in terms of a supermartingale property of penalty functions in [9, Lemma 3.17]. We summarize these characterizations in our present setting in the next proposition.

Proposition 33. Let $(\rho_t)_{t \in \mathbb{T}}$ be a dynamic convex risk measure such that each ρ_t is continuous from above. Then the following properties are equivalent:

1. $(\rho_t)_{t \in \mathbb{T}}$ is weakly acceptance consistent.
2. $\mathcal{A}_{t+1} \subseteq \mathcal{A}_t$ for all $t \in \mathbb{T}$ such that $t < T$.
3. The inequality

$$E_Q[\alpha_{t+1}^{\min}(Q) | F_t] \leq \alpha_t^{\min}(Q) \quad Q\text{-a.s.} \quad (30)$$

holds for all $Q \in \mathcal{M}_1(P)$ and all $t \in \mathbb{T}$ such that $t < T$. In particular $(\alpha_t^{\min}(Q))_{t \in \mathbb{T}}$ is a Q -supermartingale for all $Q \in \mathcal{Q}_0$.

Proof. The equivalence of 1) and 2) follows directly from the definition of weak acceptance consistency. Property 2) implies 3), since by Lemma 4

$$\begin{aligned} E_Q[\alpha_{t+1}^{\min}(Q) | F_t] &= \underset{X_{t+1} \in \mathcal{A}_{t+1}}{Q\text{-ess sup}} E_Q[-X_{t+1} | \mathcal{F}_t] \\ &\leq \underset{X \in \mathcal{A}_t}{Q\text{-ess sup}} E_Q[-X | \mathcal{F}_t] = \alpha_t^{\min}(Q) \quad Q\text{-a.s.} \end{aligned}$$

for all $Q \in \mathcal{M}_1(P)$.

To prove that 3) implies 2), we fix $X \in \mathcal{A}_{t+1}$ and note that

$$E_Q[-X|\mathcal{F}_{t+1}] \leq \alpha_{t+1}^{\min}(Q) \quad Q\text{-a.s. for all } Q \in \mathcal{M}_1(P)$$

by the definition of the minimal penalty function. Using (30) we obtain

$$E_Q[-X|\mathcal{F}_t] \leq E_Q[\alpha_{t+1}^{\min}(Q)|F_t] \leq \alpha_t^{\min}(Q) \quad Q\text{-a.s.}$$

for all $Q \in \mathcal{M}_1(P)$, in particular for $Q \in \mathcal{Q}_t^f(P)$. Thus $\rho_t(X) \leq 0$ by (6). \square

4.4 A recursive construction

In this section we assume that the time horizon T is finite. Then one can define a time consistent dynamic convex risk measure $(\tilde{\rho}_t)_{t=0,\dots,T}$ in a recursive way, starting with an arbitrary dynamic convex risk measure $(\rho_t)_{t=0,\dots,T}$, via

$$\begin{aligned} \tilde{\rho}_T(X) &:= \rho_T(X) = -X \\ \tilde{\rho}_t(X) &:= \rho_t(-\tilde{\rho}_{t+1}(X)), \quad t = 0, \dots, T-1, \quad X \in L^\infty. \end{aligned} \tag{31}$$

The recursive construction (31) was introduced in [12, Section 4.2], and also studied in [18, 13]. It is easy to see that $(\tilde{\rho}_t)_{t=0,\dots,T}$ is indeed a time consistent dynamic convex risk measure, and each $\tilde{\rho}_t$ is continuous from above if each ρ_t has this property.

Remark 34. *If the original dynamic convex risk measure $(\rho_t)_{t=0,\dots,T}$ is rejection (resp. acceptance) consistent, then the time consistent dynamic convex risk measure $(\tilde{\rho}_t)_{t=0,\dots,T}$ defined via (31) lies below (resp. above) $(\rho_t)_{t=0,\dots,T}$, i.e.*

$$\tilde{\rho}_t(X) \leq (\text{resp. } \geq) \rho_t(X) \quad \text{for all } t = 0, \dots, T \text{ and all } X \in L^\infty.$$

This can be easily proved by backward induction using Proposition 24, monotonicity, and (31). Moreover, as shown in [18, Theorem 3.10] in the case of rejection consistency, $(\tilde{\rho}_t)_{t=0,\dots,T}$ is the biggest time consistent dynamic convex risk measure that lies below $(\rho_t)_{t=0,\dots,T}$.

For all $X \in L^\infty$, the process $(\tilde{\rho}_t(X))_{t=0,\dots,T}$ has the following properties: $\tilde{\rho}_T(X) \geq -X$, and

$$\rho_t(\tilde{\rho}_t(X) - \tilde{\rho}_{t+1}(X)) = -\tilde{\rho}_t(X) + \rho_t(-\tilde{\rho}_{t+1}(X)) = 0 \quad \forall t = 0, \dots, T-1, \tag{32}$$

by definition and cash invariance. In other words, the process $(\tilde{\rho}_t(X))_{t=0,\dots,T}$ covers the final loss $-X$ and is sustainable with respect to the original risk measure $(\rho_t)_{t=0,\dots,T}$. The next proposition shows that $(\tilde{\rho}_t(X))_{t=0,\dots,T}$ is in fact the smallest process with both these properties. This result is a generalization of [29, Proposition 2.5.2], and, in the coherent case, related to [15, Theorem 6.4].

Proposition 35. *Let $(\rho_t)_{t=0,\dots,T}$ be a dynamic convex risk measure such that each ρ_t is continuous from above. Then, for each $X \in L^\infty$, the risk process $(\tilde{\rho}_t(X))_{t=0,\dots,T}$ defined via (31) is the smallest bounded adapted process $(U_t)_{t=0,\dots,T}$ such that $(U_t)_{t=0,\dots,T}$ is sustainable with respect to $(\rho_t)_{t=0,\dots,T}$ and $U_T \geq -X$.*

Proof. We have already seen that $\tilde{\rho}_T(X) \geq -X$ and $(\tilde{\rho}_t(X))_{t=0,\dots,T}$ is sustainable with respect to $(\rho_t)_{t=0,\dots,T}$ due to (32). Now let $(U_t)_{t=0,\dots,T}$ be another bounded adapted process with both these properties. We will show by backward induction that

$$U_t \geq \tilde{\rho}_t(X) \quad P\text{-a.s.} \quad \forall t = 0, \dots, T. \quad (33)$$

Indeed, we have

$$U_T \geq -X = \tilde{\rho}_T(X) \quad P\text{-a.s.}$$

If (33) holds for $t + 1$, Theorem 31 yields for all $Q \in \mathcal{Q}_t^f$:

$$\begin{aligned} U_t &\geq E_Q [U_{t+1} - \alpha_{t,t+1}^{\min}(Q) | \mathcal{F}_t] \\ &\geq E_Q [\tilde{\rho}_{t+1}(X) - \alpha_{t,t+1}^{\min}(Q) | \mathcal{F}_t] \quad P\text{-a.s.} \end{aligned}$$

Thus

$$\begin{aligned} U_t &\geq \operatorname{ess\,sup}_{Q \in \mathcal{Q}_t^f} (E_Q [\tilde{\rho}_{t+1}(X) | \mathcal{F}_t] - \alpha_{t,t+1}^{\min}(Q)) \\ &= \rho_t(-\tilde{\rho}_{t+1}(X)) = \tilde{\rho}_t(X) \quad P\text{-a.s.}, \end{aligned}$$

where we have used representation (6). This proves (33). \square

The recursive construction (31) can be used to construct a time consistent dynamic Average Value at Risk, as shown in the next example.

Example 36. *It is well known that dynamic Average Value at Risk $(AV@R_{t,\lambda_t})_{t=0,\dots,T}$ (cf. Example 10) is not time consistent, and does not even satisfy weaker notions of time consistency from Definition 13; see, e.g., [4, 32]. Moreover, since $\alpha_0^{\min}(P) = 0$ in this case, the set \mathcal{Q}^* in (20) is not empty, and [20, Corollary 4.12] implies that there exists no time consistent dynamic convex risk measure $(\rho_t)_{t \in \mathbb{T}}$ such that each ρ_t is continuous from above and $\rho_0 = AV@R_{0,\lambda_0}$. However, for $T < \infty$, the recursive construction (31) can be applied to $(AV@R_{t,\lambda_t})_{t=0,\dots,T}$ in order to modify it to a time consistent dynamic coherent risk measure $(\tilde{\rho}_t)_{t=0,\dots,T}$. This modified risk measure takes the form*

$$\begin{aligned} \tilde{\rho}_t(X) &= \operatorname{ess\,sup} \left\{ E_Q[-X | \mathcal{F}_t] \mid Q \in \mathcal{Q}_t, \frac{Z_{s+1}^Q}{Z_s^Q} \leq \lambda_s^{-1}, s = t, \dots, T-1 \right\} \\ &= \operatorname{ess\,sup} \left\{ E_P \left[-X \prod_{s=t+1}^T L_s \mid \mathcal{F}_t \right] \mid L_s \in L_s^\infty, 0 \leq L_s \leq \lambda_s^{-1}, E[L_s | \mathcal{F}_{s-1}] = 1, s = t+1, \dots, T \right\} \end{aligned}$$

for all $t = 0, \dots, T-1$, where $Z_t^Q = \frac{dQ}{dP} |_{\mathcal{F}_t}$. This was shown, e.g., in [13, Example 3.3.1].

5 The dynamic entropic risk measure

In this section we study time consistency properties of the dynamic entropic risk measure

$$\rho_t(X) = \frac{1}{\gamma_t} \log E[e^{-\gamma_t X} | \mathcal{F}_t], \quad t \in \mathbb{T}, \quad X \in L^\infty,$$

where the risk aversion parameter γ_t satisfies $\gamma_t > 0$ P -a.s. and $\gamma_t, \frac{1}{\gamma_t} \in L_t^\infty$ for all $t \in \mathbb{T}$ (cf. Example 3).

It is well known (see, e.g., [17, 20]) that the conditional entropic risk measure ρ_t has the robust representation (5) with the minimal penalty function α_t given by

$$\alpha_t(Q) = \frac{1}{\gamma_t} H_t(Q|P), \quad Q \in \mathcal{Q}_t,$$

where $H_t(Q|P)$ denotes the conditional relative entropy of Q with respect to P at time t :

$$H_t(Q|P) = E_Q \left[\log \frac{dQ}{dP} \mid \mathcal{F}_t \right], \quad Q \in \mathcal{Q}_t.$$

The dynamic entropic risk measure with constant risk aversion parameter $\gamma_t = \gamma_0 \in \mathbb{R}$ for all t was studied in [17, 12, 20, 13]. It plays a particular role since, as proved in [27], it is the only law invariant time consistent relevant dynamic convex risk measure.

In this section we consider an *adapted* risk aversion process $(\gamma_t)_{t \in \mathbb{T}}$, that depends both on time and on the available information. As shown in the next proposition, the process $(\gamma_t)_{t \in \mathbb{T}}$ determines time consistency properties of the corresponding dynamic entropic risk measure. This result corresponds to [29, Proposition 4.1.4], and generalizes [18, Proposition 3.13].

Proposition 37. *Let $(\rho_t)_{t \in \mathbb{T}}$ be the dynamic entropic risk measure with risk aversion given by an adapted process $(\gamma_t)_{t \in \mathbb{T}}$ such that $\gamma_t > 0$ P -a.s. and $\gamma_t, 1/\gamma_t \in L_t^\infty$. Then the following assertions hold:*

1. $(\rho_t)_{t \in \mathbb{T}}$ is rejection consistent if $\gamma_t \geq \gamma_{t+1}$ P -a.s. for all $t \in \mathbb{T}$ such that $t < T$;
2. $(\rho_t)_{t \in \mathbb{T}}$ is acceptance consistent if $\gamma_t \leq \gamma_{t+1}$ P -a.s. for all $t \in \mathbb{T}$ such that $t < T$;
3. $(\rho_t)_{t \in \mathbb{T}}$ is time consistent if $\gamma_t = \gamma_0 \in \mathbb{R}$ P -a.s. for all $t \in \mathbb{T}$ such that $t < T$.

Moreover, assertions 1), 2) and 3) hold with “if and only if”, if $\gamma_t \in \mathbb{R}$ for all t , or if the filtration $(\mathcal{F}_t)_{t \in \mathbb{T}}$ is rich enough in the sense that for all t and for all $B \in \mathcal{F}_t$ such that $P[B] > 0$ there exists $A \subset B$ such that $A \notin \mathcal{F}_t$ and $P[A] > 0$.

Proof. Fix $t \in \mathbb{T}$ and $X \in L^\infty$. Then

$$\begin{aligned} \rho_t(-\rho_{t+1}(X)) &= \frac{1}{\gamma_t} \log \left(E \left[\exp \left\{ \frac{\gamma_t}{\gamma_{t+1}} \log (E [e^{-\gamma_{t+1}X} | \mathcal{F}_{t+1}]) \right\} \mid \mathcal{F}_t \right] \right) \\ &= \frac{1}{\gamma_t} \log \left(E \left[E [e^{-\gamma_{t+1}X} | \mathcal{F}_{t+1}]^{\frac{\gamma_t}{\gamma_{t+1}}} \mid \mathcal{F}_t \right] \right). \end{aligned}$$

Thus $\rho_t(-\rho_{t+1}) = \rho_t$ if $\gamma_t = \gamma_{t+1}$ and this proves time consistency. Rejection (resp. acceptance) consistency follow by the generalized Jensen inequality that will be proved in Lemma 38. We apply this inequality at time $t + 1$ to the bounded random variable $Y := e^{-\gamma_{t+1}X}$ and the $\mathcal{B}((0, \infty)) \otimes \mathcal{F}_{t+1}$ -measurable function

$$u : (0, \infty) \times \Omega \rightarrow \mathbb{R}, \quad u(x, \omega) := x^{\frac{\gamma_t(\omega)}{\gamma_{t+1}(\omega)}}.$$

Note that $u(\cdot, \omega)$ is convex if $\gamma_t(\omega) \geq \gamma_{t+1}(\omega)$ and concave if $\gamma_t(\omega) \leq \gamma_{t+1}(\omega)$. Moreover, $u(X, \cdot) \in L^\infty$ for all $X \in L^\infty$ and $u(\cdot, \omega)$ is differentiable on $(0, \infty)$ with

$$|u'(x, \cdot)| = \frac{\gamma_t}{\gamma_{t+1}} x^{\frac{\gamma_t}{\gamma_{t+1}} - 1} \leq ax^b \quad P\text{-a.s.}$$

for some $a, b \in \mathbb{R}$ if $\gamma_t \geq \gamma_{t+1}$, due to our assumption $\frac{\gamma_t}{\gamma_{t+1}} \in L^\infty$. On the other hand, for $\gamma_t \leq \gamma_{t+1}$ we obtain

$$|u'(x, \cdot)| = \frac{\gamma_t}{\gamma_{t+1}} x^{\frac{\gamma_t}{\gamma_{t+1}} - 1} \leq a \frac{1}{x^c} \quad P\text{-a.s.}$$

for some $a, c \in \mathbb{R}$. Thus the assumptions of Lemma 38 are satisfied and we obtain

$$\rho_t(-\rho_{t+1}) \leq \rho_t \quad \text{if } \gamma_t \geq \gamma_{t+1} \quad P\text{-a.s. for all } t \in \mathbb{T} \text{ such that } t < T$$

and

$$\rho_t(-\rho_{t+1}) \geq \rho_t \quad \text{if } \gamma_t \leq \gamma_{t+1} \quad P\text{-a.s. for all } t \in \mathbb{T} \text{ such that } t < T.$$

The ‘‘only if’’ direction for constant γ_t follows by the classical Jensen inequality.

Now we assume that the sequence $(\rho_t)_{t \in \mathbb{T}}$ is rejection consistent and our assumption on the filtration $(\mathcal{F}_t)_{t \in \mathbb{T}}$ holds. We will show that the sequence $(\gamma_t)_{t \in \mathbb{T}}$ is decreasing in this case. Indeed, for $t \in \mathbb{T}$ such that $t < T$, consider $B := \{\gamma_t < \gamma_{t+1}\}$ and suppose that $P[B] > 0$. Our assumption on the filtration allows us to choose $A \subset B$ with $P[B] > P[A] > 0$ and $A \notin \mathcal{F}_{t+1}$. We define a random variable $X := -xI_A$ for some $x > 0$. Then

$$\begin{aligned} \rho_t(-\rho_{t+1}(X)) &= \frac{1}{\gamma_t} \log \left(E \left[\exp \left(\frac{\gamma_t}{\gamma_{t+1}} \log (E [e^{\gamma_{t+1}xI_A} | \mathcal{F}_{t+1}]) \right) \middle| \mathcal{F}_t \right] \right) \\ &= \frac{1}{\gamma_t} \log \left(E \left[\exp \left(\frac{\gamma_t}{\gamma_{t+1}} I_B \log (E [e^{\gamma_{t+1}xI_A} | \mathcal{F}_{t+1}]) \right) \middle| \mathcal{F}_t \right] \right), \end{aligned}$$

where we have used that $A \subset B$. Setting

$$Y := E [e^{\gamma_{t+1}xI_A} | \mathcal{F}_{t+1}] = e^{\gamma_{t+1}x} P[A | \mathcal{F}_{t+1}] + P[A^c | \mathcal{F}_{t+1}]$$

and bringing $\frac{\gamma_t}{\gamma_{t+1}}$ inside of the logarithm we obtain

$$\rho_t(-\rho_{t+1}(X)) = \frac{1}{\gamma_t} \log \left(E \left[\exp \left(I_B \log \left(Y^{\frac{\gamma_t}{\gamma_{t+1}}} I_B \right) \right) \middle| \mathcal{F}_t \right] \right). \quad (34)$$

The function $x \mapsto x^{\gamma_t(\omega)/\gamma_{t+1}(\omega)}$ is strictly concave for almost each $\omega \in B$, and thus

$$\begin{aligned} Y^{\frac{\gamma_t}{\gamma_{t+1}}} &= (e^{\gamma_{t+1}x} P[A | \mathcal{F}_{t+1}] + (1 - P[A | \mathcal{F}_{t+1}]))^{\frac{\gamma_t}{\gamma_{t+1}}} \\ &\geq e^{\gamma_t x} P[A | \mathcal{F}_{t+1}] + (1 - P[A | \mathcal{F}_{t+1}]) \quad P\text{-a.s. on } B, \end{aligned} \quad (35)$$

with strict inequality on the set

$$C := \{P[A | \mathcal{F}_{t+1}] > 0\} \cap \{P[A | \mathcal{F}_{t+1}] < 1\} \cap B.$$

Our assumptions $P[A] > 0$, $A \subset B$ and $A \notin \mathcal{F}_{t+1}$ imply $P[C] > 0$ and using

$$e^{\gamma_t x} P[A | \mathcal{F}_{t+1}] + (1 - P[A | \mathcal{F}_{t+1}]) = E [e^{\gamma_t x I_A} | \mathcal{F}_{t+1}] \quad (36)$$

we obtain from (34), (35) and (36)

$$\rho_t(-\rho_{t+1}(X)) \geq \frac{1}{\gamma_t} \log (E [\exp (I_B \log (E [e^{\gamma_t x I_A} | \mathcal{F}_{t+1}]) \middle| \mathcal{F}_t)]), \quad (37)$$

with the strict inequality on some set of positive probability due to strict monotonicity of the exponential and the logarithmic functions. For the right hand side of (37) we have

$$\begin{aligned}
& \frac{1}{\gamma_t} \log (E [\exp (I_B \log (E [e^{\gamma_t x I_A} | \mathcal{F}_{t+1}])) | \mathcal{F}_t]) = \\
& = \frac{1}{\gamma_t} \log (E [I_B E [e^{\gamma_t x I_A} | \mathcal{F}_{t+1}] + I_{B^c} | \mathcal{F}_t]) \\
& = \frac{1}{\gamma_t} \log (E [\exp (\gamma_t x I_A) | \mathcal{F}_t]) \\
& = \rho_t (X),
\end{aligned}$$

where we have used $A \subset B$ and $B \in \mathcal{F}_{t+1}$. This is a contradiction to rejection consistency of $(\rho_t)_{t \in \mathbb{T}}$, and we conclude that $\gamma_{t+1} \leq \gamma_t$ for all t . The proof in the case of acceptance consistency follows in the same manner. And since time consistent dynamic risk measure is both acceptance and rejection consistent, we obtain $\gamma_{t+1} = \gamma_t$ for all t . \square

The following lemma concludes the proof of Proposition 37.

Lemma 38. *Let (Ω, \mathcal{F}, P) be a probability space and $\mathcal{F}_t \subseteq \mathcal{F}$ a σ -field. Let $I \subseteq \mathbb{R}$ be an open interval and*

$$u : I \times \Omega \rightarrow \mathbb{R}$$

be a $\mathcal{B}(I) \otimes \mathcal{F}_t$ -measurable function such that $u(\cdot, \omega)$ is convex (resp. concave) and finite on I for P -a.e. ω . Assume further that

$$|u'_+(x, \cdot)| \leq c(x) \quad P\text{-a.s. with some } c(x) \in \mathbb{R} \text{ for all } x \in I,$$

where $u'_+(\cdot, \omega)$ denotes the right-hand derivative of $u(\cdot, \omega)$. Let $X : \Omega \rightarrow [a, b] \subseteq I$ be an \mathcal{F} -measurable bounded random variable such that $E[|u(X, \cdot)|] < \infty$. Then

$$E[u(X, \cdot) | \mathcal{F}_t] \geq u(E[X | \mathcal{F}_t], \cdot) \quad (\text{resp } \leq) \quad P\text{-a.s.}$$

Proof. We will prove the assertion for the convex case; the concave one follows in the same manner. Fix $\omega \in \Omega$ such that $u(\cdot, \omega)$ is convex. Due to convexity we obtain for all $x_0 \in I$

$$u(x, \omega) \geq u(x_0, \omega) + u'_+(x_0, \omega)(x - x_0) \quad \text{for all } x \in I.$$

Take $x_0 = E[X | \mathcal{F}_t](\omega)$ and $x = X(\omega)$. Then

$$u(X(\omega), \omega) \geq u(E[X | \mathcal{F}_t](\omega), \omega) + u'_+(E[X | \mathcal{F}_t](\omega), \omega)(X(\omega) - E[X | \mathcal{F}_t](\omega)) \quad (38)$$

for P -almost all $\omega \in \Omega$. Note further that $\mathcal{B}(I) \otimes \mathcal{F}_t$ -measurability of u implies $\mathcal{B}(I) \otimes \mathcal{F}_t$ -measurability of u_+ . Thus

$$\omega \rightarrow u(E[X | \mathcal{F}_t](\omega), \omega) \quad \text{and} \quad \omega \rightarrow u'_+(E[X | \mathcal{F}_t](\omega), \omega)$$

are \mathcal{F}_t -measurable random variables, and $\omega \rightarrow u(X(\omega), \omega)$ is \mathcal{F} -measurable. Moreover, due to our assumption on X , there are constants $a, b \in I$ such that $a \leq E[X|\mathcal{F}_t] \leq b$ P -a.s.. Since $u'_+(\cdot, \omega)$ is increasing by convexity, by using our assumption on the boundedness of u'_+ we obtain

$$-c(a) \leq u'_+(a, \omega) \leq u'_+(E[X|\mathcal{F}_t], \omega) \leq u'_+(b, \omega) \leq c(b),$$

i.e. $u'_+(E[X|\mathcal{F}_t], \cdot)$ is bounded. Since $E[|u(X, \cdot)|] < \infty$, we can build conditional expectation on the both sides of (38) and we obtain

$$\begin{aligned} E[u(X, \cdot) | \mathcal{F}_t] &\geq E[u(E[X|\mathcal{F}_t], \cdot) + u'_+(E[X|\mathcal{F}_t], \cdot)(X - E[X|\mathcal{F}_t]) | \mathcal{F}_t] \\ &= E[u(E[X|\mathcal{F}_t], \cdot) | \mathcal{F}_t] \quad P\text{-a.s.}, \end{aligned}$$

where we have used \mathcal{F}_t -measurability of $u(E[X|\mathcal{F}_t], \cdot)$ and of $u'_+(E[X|\mathcal{F}_t], \cdot)$ and the boundedness of $u'_+(E[X|\mathcal{F}_t], \cdot)$. This proves our claim. \square

References

- [1] Acciaio, Beatrice and Föllmer, Hans and Penner, Irina (2010). Risk assessment for uncertain cash flows: Model ambiguity, and discounting ambiguity, and the role of bubbles. Submitted
- [2] Artzner, Philippe and Delbaen, Freddy and Eber, Jean-Marc and Heath, David (1997). Thinking coherently, *RISK* **10**, 68–71
- [3] Artzner, Philippe and Delbaen, Freddy and Eber, Jean-Marc and Heath, David (1999). Coherent measures of risk, *Math. Finance* **9/3**, 203–228
- [4] Artzner, Philippe and Delbaen, Freddy and Eber, Jean-Marc and Heath, David and Ku, Hyejin (2007). Coherent multiperiod risk adjusted values and Bellman's principle, *Ann. Oper. Res.* **152**, 5–22
- [5] Barrieu, Pauline and El Karoui, Nicole (2004). Optimal derivatives design under dynamic risk measures, *Mathematics of Finance, Contemporary Mathematics (A.M.S. Proceedings)*, 13–26
- [6] Bion-Nadal, Jocelyne (2004). Conditional risk measure and robust representation of convex conditional risk measures. CMAP preprint 557, Ecole Polytechnique Palaiseau
- [7] Bion-Nadal, Jocelyne (2008). Dynamic risk measures: time consistency and risk measures from BMO martingales, *Finance Stoch.* **12/2**, 219–244
- [8] Bion-Nadal, Jocelyne (2008). Time consistent dynamic risk processes, *Stochastic Processes and their Applications* **119**, 633–654
- [9] Burgert, Christian (2005). Darstellungssätze fuer statische und dynamische Risikomaße mit Anwendungen. Universität Freiburg

- [10] Cheridito, Patrick and Delbaen, Freddy and Kupper, Michael (2004). Coherent and convex monetary risk measures for bounded càdlàg processes, *Stochastic Process. Appl.* **112/1**, 1–22
- [11] Cheridito, Patrick and Delbaen, Freddy and Kupper, Michael (2005). Coherent and convex monetary risk measures for unbounded càdlàg processes, *Finance Stoch.* **9/3**, 369–387
- [12] Cheridito, Patrick and Delbaen, Freddy and Kupper, Michael (2006). Dynamic monetary risk measures for bounded discrete-time processes, *Electron. J. Probab.* **11/3**, 57–106
- [13] Cheridito, Patrick and Kupper, Michael (2006). Composition of time-consistent dynamic monetary risk measures in discrete time. Preprint
- [14] Delbaen, Freddy (2000). Coherent risk measures. Cattedra Galileiana, Scuola Normale Superiore, Classe di Scienze, Pisa
- [15] Delbaen, Freddy (2006). The structure of m-stable sets and in particular of the set of risk neutral measures. In: In memoriam Paul-André Meyer: Séminaire de Probabilités XXXIX, Lecture Notes in Math. **1874**, p. 215–258, Springer, Berlin
- [16] Delbaen, Freddy and Peng, Shige and Rosazza Gianin, Emanuela. Representation of the penalty term of dynamic concave utilities, *Finance Stoch.*, to appear
- [17] Detlefsen, Kai and Scandolo, Giacomo (2005). Conditional and dynamic convex risk measures, *Finance Stoch.* **9/4**, 539–561
- [18] Drapeau, Samuel (2006). Dynamics of Optimized Certainty Equivalents and φ -Divergence. Humboldt-Universität zu Berlin
- [19] El Karoui, Nicole and Ravanelli, Claudia (2009). Cash subadditive risk measures and interest rate ambiguity, *Math. Finance* **19/4**, 561–590
- [20] Föllmer, Hans and Penner, Irina (2006). Convex risk measures and the dynamics of their penalty functions, *Statist. Decisions* **24/1**, 61–96
- [21] Föllmer, Hans and Schied, Alexander (2002). Convex measures of risk and trading constraints, *Finance Stoch.* **6/4**, 429–447
- [22] Föllmer, Hans and Schied, Alexander (2004). *Stochastic finance. An introduction in discrete time.* De Gruyter Studies in Mathematics **27**, Berlin
- [23] Frittelli, Marco and Rosazza Gianin, Emanuela (2002). Putting order in risk measures, *Journal of Banking Finance* **26/7**, 1473–1486
- [24] Frittelli, Marco and Scandolo, Giacomo (2006). Risk measures and capital requirements for processes, *Math. Finance* **16/4**, 589–612
- [25] Jobert, Arnaud and Rogers, L. C. G. (2008). Valuations and dynamic convex risk measures, *Math. Finance* **18/1**, 1–22

- [26] Klöppel, Susanne and Schweizer, Martin (2005). Dynamic Utility Indifference Valuation Via Convex Risk Measure. Working Paper N 209: National Centre of Competence in Research Financial Valuation and Risk Management
- [27] Kupper, Michael and Schachermayer, Walter (2009). Representation Results for Law Invariant Time Consistent Functions, *Mathematics and Financial Economics* **2/3**, 189–210
- [28] Mania, Michael and Schweizer, Martin (2005). Dynamic exponential utility indifference valuation, *Ann. Appl. Probab.* **15/3**, 2113–2143
- [29] Penner, Irina (2007). Dynamic convex risk measures: time consistency, prudence, and sustainability. Humboldt-Universität zu Berlin.
- [30] Riedel, Frank (2004). Dynamic coherent risk measures, *Stochastic Process. Appl.* **112/2**, 185–200
- [31] Roorda, Berend and Schumacher, J. M. and Engwerda, Jacob (2005). Coherent acceptability measures in multiperiod models, *Math. Finance* **15/4**, 589–612
- [32] Roorda, Berend and Schumacher, J. M. (2007). Time consistency conditions for acceptability measures, with an application to Tail Value at Risk, *Insurance Math. Econom.*, **40/2**, 209–230
- [33] Tutsch, Sina (2008). Update rules for convex risk measures, *Quant. Finance* **8**, 833–843
- [34] Vogelpoth, Nicolas (2006). Some results on dynamic risk measures. Ludwig-Maximilians-Universität München
- [35] Weber, Stefan (2006). Distribution-Invariant Risk Measures, Information, and Dynamic Consistency, *Mathematical Finance* **16/2**, 419–441