Adaptive Wave Models for Option Pricing Evolution: Nonlinear and Quantum Schrödinger Approaches

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Abstract

Adaptive wave model for financial option pricing is proposed, as a high-complexity alternative to the standard Black–Scholes model. The new option-pricing model, representing a controlled Brownian motion, includes two wave-type approaches: nonlinear and quantum, both based on (adaptive form of) the Schrödinger equation. The nonlinear approach comes in two flavors: (i) for the case of constant volatility, it is defined by a single adaptive nonlinear Schrödinger (NLS) equation, while for the case of stochastic volatility, it is defined by an adaptive Manakov system of two coupled NLS equations. The linear quantum approach is defined in terms of de Broglie's plane waves and free-particle Schrödinger equation. In this approach, financial variables have quantum-mechanical interpretation and satisfy the Heisenberg-type uncertainty relations. Both models are capable of successful fitting of the Black–Scholes data, as well as defining Greeks.

Keywords: Black–Scholes option pricing, adaptive nonlinear Schrödinger equation, adaptive Manakov system, quantum-mechanical option pricing, market-heat potential

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1 Introduction

Recall that the celebrated Black–Scholes partial differential equation (PDE) describes the time–evolution of the market value of a *stock option* [1, 2]. Formally, for a function $u = u(t, s)$ defined on the domain $0 \leq s < \infty,\ 0 \leq t \leq T$ and describing the market value of a stock option with the stock (asset) price s, the *Black–Scholes PDE* can be written (using the physicist notation: $\partial_z u = \partial u/\partial z$) as a diffusion–type equation:

$$\partial_t u = -\frac{1}{2}(\sigma s)^2\, \partial_{ss} u - rs\, \partial_s u + ru, \tag{1}$$

where $\sigma > 0$ is the standard deviation, or *volatility* of s, r is the short–term prevailing continuously–compounded risk–free interest rate, and $T > 0$ is the time to maturity of the stock option. In this formulation it is assumed that the *underlying* (typically the stock) follows a *geometric Brownian motion* with 'drift' μ and volatility σ, given by the stochastic differential equation (SDE) [3]

$$ds(t) = \mu s(t)dt + \sigma s(t)dW(t), \tag{2}$$

where W is the standard Wiener process. The Black-Scholes PDE (1) is usually derived from SDEs describing the geometric Brownian motion (2), with the stock-price solution given by:

$$s(t) = s(0)\, \mathrm{e}^{(\mu - \frac{1}{2}\sigma^2)t + \sigma W(t)}.$$

In mathematical finance, derivation is usually performed using Itô lemma [4] (assuming that the underlying asset obeys the Itô SDE), while in physics it is performed using Stratonovich interpretation [5, 6] (assuming that the underlying asset obeys the Stratonovich SDE [8]).

The Black-Sholes PDE (1) can be applied to a number of one-dimensional models of interpretations of prices given to u, e.g., puts or calls, and to s, e.g., stocks or futures, dividends, etc. The most important examples are European call and put options, defined by:

$$u_{\mathrm{Call}}(s,t) = s\,\mathcal{N}(\mathrm{d}_1)\,\mathrm{e}^{-T\delta} - k\,\mathcal{N}(\mathrm{d}_2)\,\mathrm{e}^{-rT}, \tag{3}$$

$$u_{\mathrm{Put}}(s,t) = k\,\mathcal{N}(-\mathrm{d}_2)\,\mathrm{e}^{-rT} - s\,\mathcal{N}(-\mathrm{d}_1)\,\mathrm{e}^{-T\delta}, \tag{4}$$

$$\mathcal{N}(\lambda) = \frac{1}{2}\left(1 + \mathrm{erf}\left(\frac{\lambda}{\sqrt{2}}\right)\right),$$

$$\mathrm{d}_1 = \frac{\ln\left(\frac{s}{k}\right) + T\left(r - \delta + \frac{\sigma^2}{2}\right)}{\sigma\sqrt{T}}, \qquad \mathrm{d}_2 = \frac{\ln\left(\frac{s}{k}\right) + T\left(r - \delta - \frac{\sigma^2}{2}\right)}{\sigma\sqrt{T}},$$

where $\mathrm{erf}(\lambda)$ is the (real-valued) error function, k denotes the strike price and δ represents the dividend yield. In addition, for each of the call and put options, there are five Greeks (see, e.g. [9, 10]), or sensitivities, which are partial derivatives of the option-price with respect

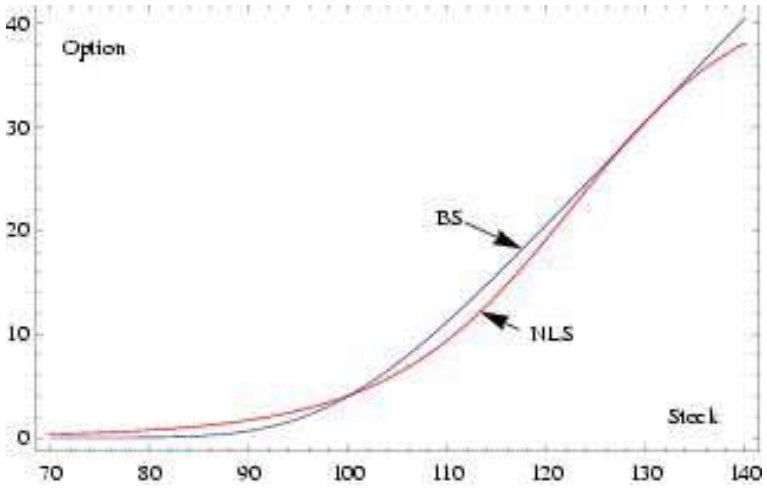

Figure 1: Fitting the Black–Scholes call option with $\beta(w)$-adaptive PDF of the shock-wave NLS-solution (10).

to stock price (Delta), interest rate (Rho), volatility (Vega), elapsed time since entering into the option (Theta), and the second partial derivative of the option-price with respect to the stock price (Gamma).

Using the standard *Kolmogorov probability* approach, instead of the market value of an option given by the Black–Scholes equation (1), we could consider the corresponding probability density function (PDF) given by the backward Fokker–Planck equation (see [6, 7]). Alternatively, we can obtain the same PDF (for the market value of a stock option), using the *quantum–probability* formalism [11, 12], as a solution to a time–dependent linear or nonlinear *Schrödinger equation* for the evolution of the complex–valued wave $\psi-$function for which the absolute square, $|\psi|^2$, is the PDF. The adaptive nonlinear Schrödinger (NLS) equation was recently used in [10] as an approach to option price modelling, as briefly reviewed in this section. The new model, philosophically founded on adaptive markets hypothesis [13, 14] and Elliott wave market theory [15, 16], as well as my own recent work on quantum congition [17, 18], describes adaptively controlled Brownian market behavior. This nonlinear approach to option price modelling is reviewed in the next section. Its important limiting case with low interest-rate reduces to the linear Schrödinger equation. This linear approach to option price modelling is elaborated in the subsequent section.

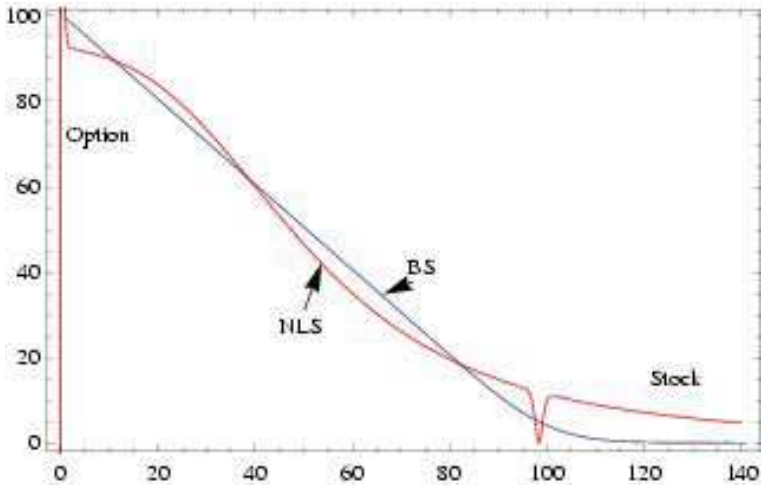

Figure 2: Fitting the Black–Scholes put option with $\beta(w)$–adaptive PDF of the shock-wave NLS $\psi_2(s,t)$ solution (10). Notice the kink near $s = 100$.

2 Nonlinear adaptive wave model for general option pricing

2.1 Adaptive NLS model

The adaptive, wave–form, nonlinear and stochastic option–pricing model with stock price s, volatility σ and interest rate r is formally defined as a complex-valued, focusing (1+1)–NLS equation, defining the time-dependent *option–price wave function* $\psi = \psi(s,t)$, whose absolute square $|\psi(s,t)|^2$ represents the probability density function (PDF) for the option price in terms of the stock price and time. In natural quantum units, this NLS equation reads:

$$\mathrm{i}\partial_t\psi = -\frac{1}{2}\sigma\partial_{ss}\psi - \beta|\psi|^2\psi, \qquad (\mathrm{i} = \sqrt{-1}), \tag{5}$$

where $\beta = \beta(r,w)$ denotes the adaptive market-heat potential (see [19]), so the term $V(\psi) = -\beta|\psi|^2$ represents the $\psi-$dependent potential field. In the simplest nonadaptive scenario β is equal to the interest rate r, while in the adaptive case it depends on the set of adjustable synaptic weights $\{w_j^i\}$ as:

$$\beta(r,w) = r\sum_{i=1}^{n} w_1^i \,\mathrm{erf}\left(\frac{w_2^i s}{w_3^i}\right). \tag{6}$$

Physically, the NLS equation (5) describes a nonlinear wave (e.g. in Bose-Einstein condensates) defined by the complex-valued wave function $\psi(s,t)$ of real space and time parameters. In the present context, the space-like variable s denotes the stock (asset) price.

The NLS equation (5) has been exactly solved using the power series expansion method [20, 21] of Jacobi elliptic functions [22]. Consider the $\psi-$function describing a single plane wave, with the wave number k and circular frequency ω:

$$\psi(s,t) = \phi(\xi)\, \mathrm{e}^{\mathrm{i}(ks-\omega t)}, \qquad \text{with} \;\; \xi = s - \sigma k t \;\; \text{and} \;\; \phi(\xi) \in \mathbb{R}. \tag{7}$$

Its substitution into the NLS equation (5) gives the nonlinear oscillator ODE:

$$\phi''(\xi) + [\omega - \frac{1}{2}\sigma k^2]\, \phi(\xi) + \beta \phi^3(\xi) = 0. \tag{8}$$

We can seek a solution $\phi(\xi)$ for (8) as a linear function [21]

$$\phi(\xi) = a_0 + a_1 \mathrm{sn}(\xi),$$

where $\mathrm{sn}(s) = \mathrm{sn}(s,m)$ are Jacobi elliptic sine functions with *elliptic modulus* $m \in [0,1]$, such that $\mathrm{sn}(s,0) = \sin(s)$ and $\mathrm{sn}(s,1) = \tanh(s)$. The solution of (8) was calculated in [10] to be

$$\begin{aligned} \phi(\xi) &= \pm m \sqrt{\frac{-\sigma}{\beta}}\, \mathrm{sn}(\xi), \qquad \text{for} \;\; m \in [0,1]; \;\; \text{and} \\ \phi(\xi) &= \pm \sqrt{\frac{-\sigma}{\beta}}\, \tanh(\xi), \qquad \text{for} \;\; m = 1. \end{aligned}$$

This gives the exact periodic solution of (5) as [10]

$$\psi_1(s,t) = \pm m \sqrt{\frac{-\sigma}{\beta(w)}}\, \mathrm{sn}(s - \sigma k t)\, \mathrm{e}^{\mathrm{i}[ks - \frac{1}{2}\sigma t(1+m^2+k^2)]}, \qquad \text{for} \;\; m \in [0,1); \tag{9}$$

$$\psi_2(s,t) = \pm \sqrt{\frac{-\sigma}{\beta(w)}}\, \tanh(s - \sigma k t)\, \mathrm{e}^{\mathrm{i}[ks - \frac{1}{2}\sigma t(2+k^2)]}, \qquad \text{for} \;\; m = 1, \tag{10}$$

where (9) defines the general solution, while (10) defines the *envelope shock-wave*[1] (or, 'dark soliton') solution of the NLS equation (5).

Alternatively, if we seek a solution $\phi(\xi)$ as a linear function of Jacobi elliptic cosine functions, such that $\mathrm{cn}(s,0) = \cos(s)$ and $\mathrm{cn}(s,1) = \mathrm{sech}(s)$,[2]

$$\phi(\xi) = a_0 + a_1 \mathrm{cn}(\xi),$$

[1] A shock wave is a type of fast-propagating nonlinear disturbance that carries energy and can propagate through a medium (or, field). It is characterized by an abrupt, nearly discontinuous change in the characteristics of the medium. The energy of a shock wave dissipates relatively quickly with distance and its entropy increases. On the other hand, a soliton is a self-reinforcing nonlinear solitary wave packet that maintains its shape while it travels at constant speed. It is caused by a cancelation of nonlinear and dispersive effects in the medium (or, field).

[2] A closely related solution of an anharmonic oscillator ODE:

$$\phi''(s) + \phi(s) + \phi^3(s) = 0$$

then we get [10]

$$\psi_3(s,t) = \pm m\sqrt{\frac{\sigma}{\beta(w)}}\,\mathrm{cn}(s-\sigma kt)\,\mathrm{e}^{\mathrm{i}[ks-\frac{1}{2}\sigma t(1-2m^2+k^2)]}, \qquad \text{for } m\in[0,1); \tag{11}$$

$$\psi_4(s,t) = \pm\sqrt{\frac{\sigma}{\beta(w)}}\,\mathrm{sech}(s-\sigma kt)\,\mathrm{e}^{\mathrm{i}[ks-\frac{1}{2}\sigma t(k^2-1)]}, \qquad \text{for } m=1, \tag{12}$$

where (11) defines the general solution, while (12) defines the *envelope solitary-wave* (or, 'bright soliton') solution of the NLS equation (5).

In all four solution expressions (9), (10), (11) and (12), the adaptive potential $\beta(w)$ is yet to be calculated using either unsupervised Hebbian learning, or supervised Levenberg–Marquardt algorithm (see, e.g. [23, 24]). In this way, the NLS equation (5) becomes the *quantum neural network* (see [18]). Any kind of numerical analysis can be easily performed using above closed-form solutions $\psi_i(s,t)$ $(i=1,...,4)$ as initial conditions.

The adaptive NLS–PDFs of the shock-wave type (10) has been used in [10] to fit the Black–Scholes call and put options (see Figures 1 and 2). Specifically, the adaptive heat potential (6) was combined with the spatial part of (10)

$$\phi(s) = \left|\sqrt{\frac{\sigma}{\beta}}\tanh(s-kt\sigma)\right|^2, \tag{13}$$

while parameter estimates where obtained using 100 iterations of the Levenberg–Marquardt algorithm.

As can be seen from Figure (2) there is a kink near $s=100$. This kink, which is a natural characteristic of the spatial shock-wave (13), can be smoothed out (Figure 3) by taking the sum of the spatial parts of the shock-wave solution (10) and the soliton solution (12) as:

$$\phi(s) = \left|\sqrt{\frac{\sigma}{\beta}}\left[d_1\tanh(s-kt\sigma)+d_2\,\mathrm{sech}(s-kt\sigma)\right]\right|^2. \tag{14}$$

The adaptive NLS–based Greeks (Delta, Rho, Vega, Theta and Gamma) have been defined in [10], as partial derivatives of the shock-wave solution (10).

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is given by

$$\phi(s) = \sqrt{\frac{2m}{1-2m}}\,\mathrm{cn}\left(\sqrt{1+\frac{2m}{1-2m}}\;s,\,m\right).$$

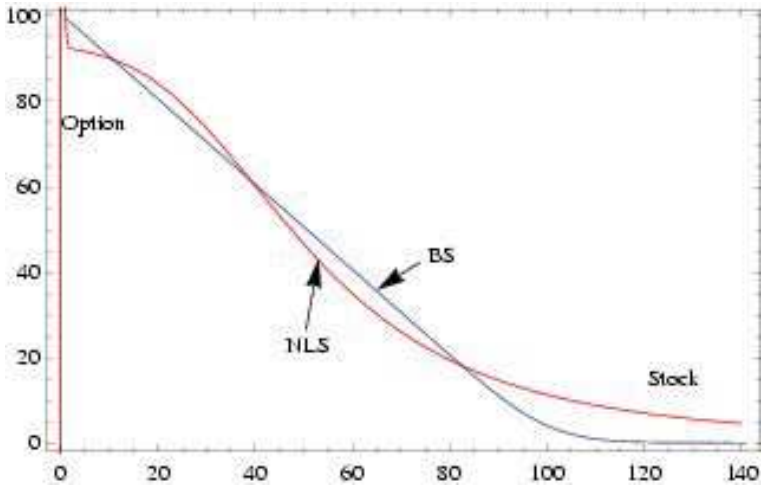

Figure 3: Smoothing out the kink in the put option fit, by combining the shock-wave solution with the soliton solution, as defined by (14).

2.2 Adaptive Manakov system

Next, for the purpose of including a *controlled stochastic volatility*[3] into the adaptive–NLS model (5), the full bidirectional quantum neural computation model [18] for option-price forecasting has been formulated in [10] as a self-organized system of two coupled self-focusing NLS equations: one defining the *option–price wave function* $\psi = \psi(s,t)$ and the other defining the *volatility wave function* $\sigma = \sigma(s,t)$:

$$\text{Volatility NLS}: \quad \mathrm{i}\partial_t \sigma = -\frac{1}{2}\partial_{ss}\sigma - \beta(r,w)\left(|\sigma|^2 + |\psi|^2\right)\sigma, \tag{15}$$

$$\text{Option price NLS}: \quad \mathrm{i}\partial_t \psi = -\frac{1}{2}\partial_{ss}\psi - \beta(r,w)\left(|\sigma|^2 + |\psi|^2\right)\psi. \tag{16}$$

In this coupled model, the σ–NLS (15) governs the $(s,t)-$evolution of stochastic volatility, which plays the role of a nonlinear coefficient in (16); the ψ–NLS (16) defines the $(s,t)-$evolution of option price, which plays the role of a nonlinear coefficient in (15). The purpose of this coupling is to generate a *leverage effect*, i.e. stock volatility is (negatively) correlated to stock returns[4] (see, e.g. [27]). This bidirectional associative memory effectively performs quantum neural computation [18], by giving a spatio-temporal and quantum generalization of Kosko's BAM family of neural networks [28, 29]. In addition, the shock-wave and solitary-wave nature of the coupled NLS equations may describe brain-like effects frequently

[3]Controlled stochastic volatility here represents volatility evolving in a stochastic manner but within the controlled boundaries.

[4]The hypothesis that financial leverage can explain the leverage effect was first discussed by F. Black [26].

occurring in financial markets: volatility/price propagation, reflection and collision of shock and solitary waves (see [30]).

The coupled NLS-system (15)–(16), without an embedded $w-$learning (i.e., for constant $\beta = r$ – the interest rate), actually defines the well-known *Manakov system*,[5] proven by S. Manakov in 1973 [31] to be completely integrable, by the existence of infinite number of involutive integrals of motion. It admits 'bright' and 'dark' soliton solutions. The simplest solution of (15)–(16), the so-called *Manakov bright 2–soliton*, has the form resembling that of the sech-solution (12) (see [34, 35, 36, 37, 38, 39, 40]), and is formally defined by:

$$\psi_{\rm sol}(s,t) = 2b\,\mathbf{c}\,\mathrm{sech}(2b(s+4at))\,\mathrm{e}^{-2\mathrm{i}(2a^2t+as-2b^2t)}, \tag{17}$$

where $\psi_{\rm sol}(s,t) = \begin{pmatrix} \sigma(s,t) \\ \psi(s,t) \end{pmatrix}$, $\mathbf{c} = (c_1, c_2)^T$ is a unit vector such that $|c_1|^2 + |c_2|^2 = 1$. Real-valued parameters a and b are some simple functions of (σ, β, k), which can be determined by the Levenberg–Marquardt algorithm. I have argued in [10] that in some short-time financial situations, the adaptation effect on β can be neglected, so our option-pricing model (15)–(16) can be reduced to the Manakov 2–soliton model (17), as depicted and explained in Figure 4.

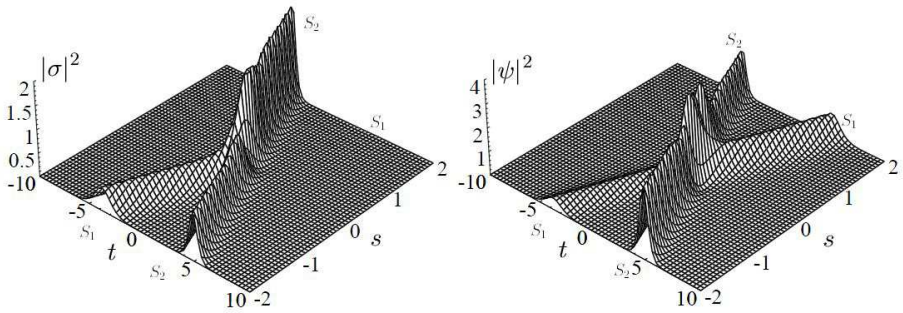

Figure 4: Hypothetical market scenario including sample PDFs for volatility $|\sigma|^2$ and $|\psi|^2$ of the Manakov 2–soliton (17). On the left, we observe the $(s,t)-$evolution of stochastic volatility: we have a collision of two volatility component-solitons, $S_1(s,t)$ and $S_2(s,t)$, which join together into the resulting soliton $S_2(s,t)$, annihilating the $S_1(s,t)$ component in the process. On the right, we observe the $(s,t)-$evolution of option price: we have a collision of two option component-solitons, $S_1(s,t)$ and $S_2(s,t)$, which pass through each other without much change, except at the collision point. Due to symmetry of the Manakov system, volatility and option price can exchange their roles.

[5] Manakov system has been used to describe the interaction between wave packets in dispersive conservative media, and also the interaction between orthogonally polarized components in nonlinear optical fibres (see, e.g. [32, 33] and references therein).

3 Quantum wave model for low interest-rate option pricing

In the case of a low interest-rate $r \ll 1$, we have $\beta(r) \ll 1$, so $V(\psi) \to 0$, and therefore equation (5) can be approximated by a quantum-like *option wave packet.* It is defined by a continuous superposition of *de Broglie's plane waves*, 'physically' associated with a free quantum particle of unit mass. This linear wave packet, given by the time-dependent complex-valued wave function $\psi = \psi(s,t)$, is a solution of the *linear Schrödinger equation* with zero potential energy, Hamiltonian operator $\hat{H}$ and volatility σ playing the role similar to the Planck constant. This equation can be written as:

$$\mathrm{i}\sigma\partial_t\psi = \hat{H}\psi, \qquad \text{where} \qquad \hat{H} = -\frac{\sigma^2}{2}\partial_{ss}. \tag{18}$$

Thus, we consider the $\psi-$function describing a single de Broglie's plane wave, with the wave number k, linear momentum $p = \sigma k$, wavelength $\lambda_k = 2\pi/k$, angular frequency $\omega_k = \sigma k^2/2$, and oscillation period $T_k = 2\pi/\omega_k = 4\pi/\sigma k^2$. It is defined by (compare with [41, 42, 12])

$$\psi_k(s,t) = A\mathrm{e}^{\mathrm{i}(ks-\omega_k t)} = A\mathrm{e}^{\mathrm{i}(ks-\frac{\sigma k^2}{2}t)} = A\cos(ks - \frac{\sigma k^2}{2}t) + A\mathrm{i}\sin(ks - \frac{\sigma k^2}{2}t), \tag{19}$$

where A is the amplitude of the wave, the angle $(ks-\omega_k t) = (ks - \frac{\sigma k^2}{2}t)$ represents the phase of the wave ψ_k with the *phase velocity:* $v_k = \omega_k/k = \sigma k/2$.

The space-time wave function $\psi(s,t)$ that satisfies the linear Schrödinger equation (18) can be decomposed (using Fourier's separation of variables) into the spatial part $\phi(s)$ and the temporal part $\mathrm{e}^{-\mathrm{i}\omega t}$ as:

$$\psi(s,t) = \phi(s)\,\mathrm{e}^{-\mathrm{i}\omega t} = \phi(s)\,\mathrm{e}^{-\frac{\mathrm{i}}{\sigma}Et}.$$

The spatial part, representing *stationary* (or, *amplitude*) *wave function*, $\phi(s) = A\mathrm{e}^{\mathrm{i}ks}$, satisfies the *linear harmonic oscillator,* which can be formulated in several equivalent forms:

$$\phi'' + k^2\phi = 0, \qquad \phi'' + \left(\frac{p}{\sigma}\right)^2\phi = 0, \qquad \phi'' + \left(\frac{\omega_k}{v_k}\right)^2\phi = 0, \qquad \phi'' + \frac{2E_k}{\sigma^2}\phi = 0. \tag{20}$$

Planck's *energy quantum* of the option wave ψ_k is given by: $E_k = \sigma\omega_k = \frac{1}{2}(\sigma k)^2$.

From the plane-wave expressions (19) we have: $\psi_k(s,t) = A\mathrm{e}^{\frac{\mathrm{i}}{\sigma}(ps-E_k t)}-$ for the wave going to the 'right' and $\psi_k(s,t) = A\mathrm{e}^{-\frac{\mathrm{i}}{\sigma}(ps+E_k t)}-$ for the wave going to the 'left'.

The general solution to (18) is formulated as a linear combination of de Broglie's option waves (19), comprising the option wave-packet:

$$\psi(s,t) = \sum_{i=0}^{n} c_i\psi_{k_i}(s,t), \qquad (\text{with } n \in \mathbb{N}). \tag{21}$$

Its absolute square, $|\psi(s,t)|^2$, represents the probability density function at a time t.

The *group velocity* of an option wave-packet is given by: $v_g = d\omega_k/dk$. It is related to the phase velocity v_k of a plane wave as: $v_g = v_k - \lambda_k dv_k/d\lambda_k$. Closely related is the *center* of the option wave-packet (the point of maximum amplitude), given by: $s = td\omega_k/dk$.

The following quantum-motivated assertions can be stated:

1. Volatility σ has dimension of *financial action*, or *energy* $\times$ *time*.

2. The total energy E of an option wave-packet is (in the case of similar plane waves) given by Planck's superposition of the energies E_k of n individual waves: $E = n\sigma\omega_k = \frac{n}{2}(\sigma k)^2$, where $L = n\sigma$ denotes the *angular momentum* of the option wave-packet, representing the shift between its growth and decay, and *vice versa.*

3. The average energy $\langle E\rangle$ of an option wave-packet is given by Boltzmann's partition function:
$$\langle E\rangle = \frac{\sum_{n=0}^{\infty} nE_k \mathrm{e}^{-\frac{nE_k}{bT}}}{\sum_{n=0}^{\infty} \mathrm{e}^{-\frac{nE_k}{bT}}} = \frac{E_k}{\mathrm{e}^{\frac{E_k}{bT}} - 1},$$
where b is the Boltzmann-like kinetic constant and T is the market temperature.

4. The energy form of the Schrödinger equation (18) reads: $E\psi = \mathrm{i}\sigma\partial_t\psi$.

5. The eigenvalue equation for the Hamiltonian operator $\hat{H}$ is the *stationary Schrödinger equation:*
$$\hat{H}\phi(s) = E\phi(s), \qquad \text{or} \qquad E\phi(s) = -\frac{\sigma^2}{2}\partial_{ss}\phi(s),$$
which is just another form of the harmonic oscillator (20). It has oscillatory solutions of the form:
$$\phi_E(s) = c_1 \mathrm{e}^{\frac{\mathrm{i}}{\sigma}\sqrt{2E_k}\,s} + c_2 \mathrm{e}^{-\frac{\mathrm{i}}{\sigma}\sqrt{2E_k}\,s},$$
called *energy eigen-states* with energies E_k and denoted by: $\hat{H}\phi_E(s) = E_k\phi_E(s)$.

The Black–Scholes put and call options have been fitted with the quantum PDFs (see Figures 5 and 6) given by the absolute square of (21) with $n = 7$ and $n = 3$, respectively. Using supervised Levenberg–Marquardt algorithm and *Mathematica* 7, the following coefficients were obtained for the Black–Scholes put option:

$\sigma^* = -0.0031891,\ t^* = -0.0031891,\ k_1 = 2.62771,\ k_2 = 2.62777,\ k_3 = 2.65402,$
$k_4 = 2.61118,\ k_5 = 2.64104,\ k_6 = 2.54737,\ k_7 = 2.62778,\ c_1 = 1.26632,\ c_2 = 1.26517,$
$c_3 = 2.74379,\ c_4 = 1.35495,\ c_5 = 1.59586,\ c_6 = 0.263832,\ c_7 = 1.26779,$

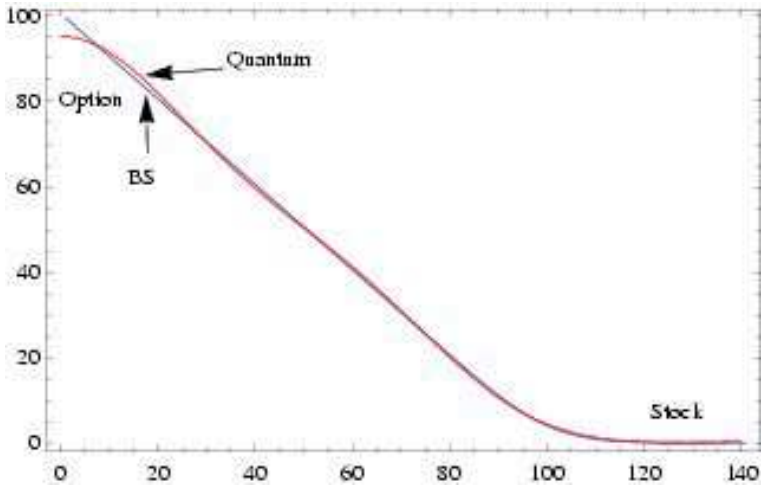

Figure 5: Fitting the Black–Scholes put option with the quantum PDF given by the absolute square of (21) with $n = 7$.

with $\sigma_{BS} = -94.0705\sigma^*,\ t_{BS} = -31.3568t^*$.

Using the same algorithm, the following coefficients were obtained for the Black–Scholes call option:

$\sigma^* = -11.9245,\ t^* = -11.9245,\ k_1 = 0.851858,\ k_2 = 0.832409,$
$k_3 = 0.872061,\ c_1 = 2.9004,\ c_2 = 2.72592,\ c_3 = 2.93291,$
with $\sigma_{BS} - 0.0251583\sigma^*,\ t = -0.00838609t^*$.

Now, given some initial option wave function, $\psi(s,0) = \psi_0(s)$, a solution to the initial-value problem for the linear Schrödinger equation (18) is, in terms of the pair of Fourier transforms $(\mathcal{F}, \mathcal{F}^{-1})$, given by (see [42])

$$\psi(s,t) = \mathcal{F}^{-1}\left[\mathrm{e}^{-\mathrm{i}\omega t}\mathcal{F}(\psi_0)\right] = \mathcal{F}^{-1}\left[\mathrm{e}^{-\mathrm{i}\frac{\sigma k^2}{2}t}\mathcal{F}(\psi_0)\right]. \tag{22}$$

For example (see [42]), suppose we have an initial option wave-function at time $t = 0$ given by the complex-valued Gaussian function:

$$\psi(s,0) = \mathrm{e}^{-as^2/2}\mathrm{e}^{\mathrm{i}\sigma ks},$$

where a is the width of the Gaussian, while p is the average momentum of the wave. Its

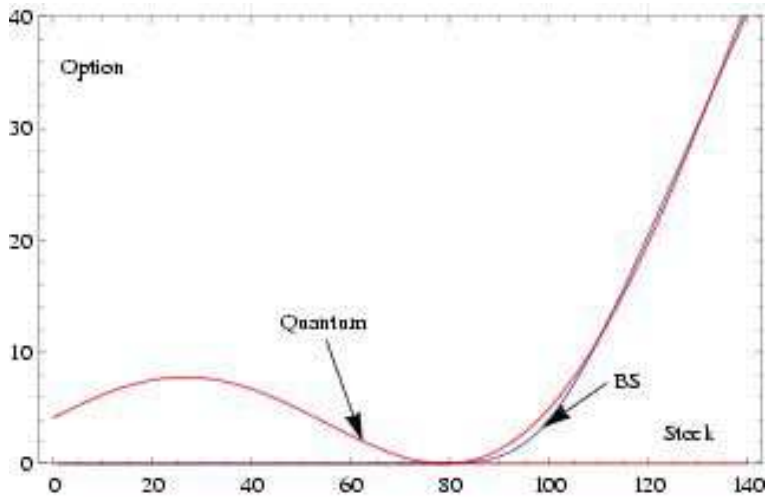

Figure 6: Fitting the Black–Scholes call option with the quantum PDF given by the absolute square of (21) with $n = 3$. Note that fit is good in the realistic stock region: $s \in [75, 140]$.

Fourier transform, $\hat{\psi}_0(k) = \mathcal{F}[\psi(s,0)]$, is given by

$$\hat{\psi}_0(k) = \frac{\mathrm{e}^{-\frac{(k-p)^2}{2a}}}{\sqrt{a}}.$$

The solution at time t of the initial value problem is given by

$$\psi(s,t) = \frac{1}{\sqrt{2\pi a}} \int_{-\infty}^{+\infty} \mathrm{e}^{\mathrm{i}(ks - \frac{\sigma k^2}{2}t)}\, \mathrm{e}^{-\frac{a(k-p)^2}{2a}}\, dk,$$

which, after some algebra becomes

$$\psi(s,t) = \frac{\exp(-\frac{as^2 - 2\mathrm{i}sp + \mathrm{i}p^2 t}{2(1+\mathrm{i}at)})}{\sqrt{1+\mathrm{i}at}}, \qquad (\text{with } \; p = \sigma k).$$

As a simpler example,[6] if we have an initial option wave-function given by the real-valued

[6] An example of a more general Gaussian wave-packet solution of (18) is given by:

$$\psi(s,t) = \sqrt{\frac{\sqrt{a/\pi}}{1+\mathrm{i}at}}\, \exp\left(\frac{-\frac{1}{2}a(s-s_0)^2 - \frac{\mathrm{i}}{2}p_0^2 t + \mathrm{i}p_0(s-s_0)}{1+\mathrm{i}at}\right),$$

where s_0, p_0 are initial stock-price and average momentum, while a is the width of the Gaussian. At time $t = 0$ the 'particle' is at rest around $s = 0$, its average momentum $p_0 = 0$. The wave function spreads with time while its maximum decreases and stays put at the origin. At time $-t$ the wave packet is the complex-conjugate of the wave-packet at time t.

Gaussian function,

$$\psi(s,0) = \frac{\mathrm{e}^{-s^2/2}}{\sqrt[4]{\pi}},$$

the solution of (18) is given by the complex-valued $\psi-$function,

$$\psi(s,t) = \frac{\exp(-\frac{s^2}{2(1+\mathrm{i}t)})}{\sqrt[4]{\pi}\sqrt{1+\mathrm{i}t}}.$$

From (22) it follows that a stationary option wave-packet is given by:

$$\phi(s) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} \mathrm{e}^{\frac{\mathrm{i}}{\sigma}ks}\, \hat{\psi}(k)\, dk, \qquad \text{where} \qquad \hat{\psi}(k) = \mathcal{F}[\phi(s)].$$

As $|\phi(s)|^2$ is the stationary stock PDF, we can calculate the *expectation values* of the stock and the wave number of the whole option wave-packet, consisting of n measured plane waves, as:

$$\langle s\rangle = \int_{-\infty}^{+\infty} s|\phi(s)|^2 ds \qquad \text{and} \qquad \langle k\rangle = \int_{-\infty}^{+\infty} k|\hat{\psi}(k)|^2 dk. \tag{23}$$

The recordings of n individual option plane waves (19) will be scattered around the mean values (23). The width of the distribution of the recorded $s-$ and $k-$values are uncertainties Δs and Δk, respectively. They satisfy the Heisenberg-type uncertainty relation:

$$\Delta s\, \Delta k \geq \frac{n}{2},$$

which imply the similar relation for the total option energy and time:

$$\Delta E\, \Delta t \geq \frac{n}{2}.$$

Finally, Greeks for both put and call options are defined as the following partial derivatives of the option $\psi-$function PDF:

$$\text{Delta} = \partial_s|\psi(s,t)|^2 =$$
$$2\mathrm{i}\sum_{j=1}^{n} c_j k_j\, \mathrm{e}^{k_j\left(\mathrm{i}s-\frac{1}{2}\mathrm{i}\sigma k_j t\right)} \mathrm{Abs}\left[\sum_{j=1}^{n} c_j\, \mathrm{e}^{k_j\left(\mathrm{i}s-\frac{1}{2}\mathrm{i}\sigma k_j t\right)}\right] \mathrm{Abs}'\left[\sum_{j=1}^{n} c_j\, \mathrm{e}^{k_j\left(\mathrm{i}s-\frac{1}{2}\mathrm{i}\sigma k_j t\right)}\right]$$

$$\text{Vega} = \partial_\sigma|\psi(s,t)|^2 =$$
$$-\mathrm{i}t\sum_{j=1}^{n} c_j {k_j}^2\, \mathrm{e}^{k_j\left(\mathrm{i}s-\frac{1}{2}\mathrm{i}\sigma k_j t\right)} \mathrm{Abs}\left[\sum_{j=1}^{n} c_j\, \mathrm{e}^{k_j\left(\mathrm{i}s-\frac{1}{2}\mathrm{i}\sigma k_j t\right)}\right] \mathrm{Abs}'\left[\sum_{j=1}^{n} c_j\, \mathrm{e}^{k_j\left(\mathrm{i}s-\frac{1}{2}\mathrm{i}\sigma k_j t\right)}\right]$$

$$\text{Theta} = \partial_t|\psi(s,t)|^2 =$$
$$-\mathrm{i}\sigma\sum_{j=1}^{n} c_j {k_j}^2\, \mathrm{e}^{k_j\left(\mathrm{i}s-\frac{1}{2}\mathrm{i}\sigma k_j t\right)} \mathrm{Abs}\left[\sum_{j=1}^{n} c_j\, \mathrm{e}^{k_j\left(\mathrm{i}s-\frac{1}{2}\mathrm{i}\sigma k_j t\right)}\right] \mathrm{Abs}'\left[\sum_{j=1}^{n} c_j\, \mathrm{e}^{k_j\left(\mathrm{i}s-\frac{1}{2}\mathrm{i}\sigma k_j t\right)}\right]$$

$$\text{Gamma} = \partial_{ss}|\psi(s,t)|^2 =$$
$$-2\sum_{j=1}^{n} c_j {k_j}^2 \,\mathrm{e}^{k_j\left(\mathrm{i}s-\frac{1}{2}\mathrm{i}\sigma k_j t\right)} \mathrm{Abs}\left[\sum_{j=1}^{n} c_j \,\mathrm{e}^{k_j\left(\mathrm{i}s-\frac{1}{2}\mathrm{i}\sigma k_j t\right)}\right] \mathrm{Abs}'\left[\sum_{j=1}^{n} c_j \,\mathrm{e}^{k_j\left(\mathrm{i}s-\frac{1}{2}\mathrm{i}\sigma k_j t\right)}\right] -$$
$$\left(\sum_{j=1}^{n} c_j k_j \,\mathrm{e}^{k_j\left(\mathrm{i}s-\frac{1}{2}\mathrm{i}\sigma k_j t\right)}\right)^2 \mathrm{Abs}'\left[\sum_{j=1}^{n} c_j \,\mathrm{e}^{k_j\left(\mathrm{i}s-\frac{1}{2}\mathrm{i}\sigma k_j t\right)}\right]^2 -$$
$$\left(\sum_{j=1}^{n} c_j k_j \,\mathrm{e}^{k_j\left(\mathrm{i}s-\frac{1}{2}\mathrm{i}\sigma k_j t\right)}\right)^2 \mathrm{Abs}\left[\sum_{j=1}^{n} c_j \,\mathrm{e}^{k_j\left(\mathrm{i}s-\frac{1}{2}\mathrm{i}\sigma k_j t\right)}\right] \mathrm{Abs}''\left[\sum_{j=1}^{n} c_j \,\mathrm{e}^{k_j\left(\mathrm{i}s-\frac{1}{2}\mathrm{i}\sigma k_j t\right)}\right],$$

where Abs denotes the absolute value, while Abs' and Abs'' denote its first and second derivatives.

4 Conclusion

I have proposed an adaptive–wave alternative to the standard Black-Scholes option pricing model. The new model, philosophically founded on adaptive markets hypothesis [13, 14] and Elliott wave market theory [15, 16], describes adaptively controlled Brownian market behavior. Two approaches have been proposed: (i) a nonlinear one based on the adaptive NLS (solved by means of Jacobi elliptic functions) and the adaptive Manakov system (of two coupled NLS equations); (ii) a linear quantum-mechanical one based on the free-particle Schrödinger equation and de Broglie's plane waves. For the purpose of fitting the Black-Scholes data, the Levenberg-Marquardt algorithm was used.

The presented adaptive and quantum wave models are spatio-temporal dynamical systems of much higher complexity [25] then the Black-Scholes model. This makes the new wave models harder to analyze, but at the same time, their immense variety is potentially much closer to the real financial market complexity, especially at the time of financial crisis.

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