Decoherence in Bose-Einstein Condensates:
Towards Bigger and Better Schrödinger Cats

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We consider a quantum superposition of Bose-Einstein condensates in two immiscible internal states. A decoherence rate for the resulting Schrödinger cat is calculated and shown to be a significant threat to this macroscopic quantum superposition of BEC's. An experimental scenario is outlined where the decoherence rate due to the thermal cloud is dramatically reduced thanks to trap engineering and "symmetrization" of the environment which allow the Schrödinger cat to be an approximate pointer state.

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I. MOTIVATION AND SUMMARY OF RESULTS

Microscopic quantum superpositions are an everyday physicist's experience. Macroscopic quantum superpositions, despite nearly a century of experimentation with quantum mechanics, are still encountered only very rarely. Fast decoherence of macroscopically distinct states is to be blamed [1]. In spite of that, recent years were a witness to an interesting quantum optics experiment [2] on decoherence of a few photon superpositions. Moreover, matter-wave interference in fullerene C_{60} has been observed [3]. Another recent experiment has succeeded in "engineering" the environment in the context of trapped ions [4]. This success plus rapid progress in Bose-Einstein condensation (BEC) of alkali metal atomic vapors [5] tempt one to push similar investigations even further into the macroscopic domain. The condensates can contain up to 10^7 atoms in the same quantum state. What is more, it is possible to prepare condensates in two different internal states of the atoms. Some of these pairs of internal levels are immiscible, and their condensates tend to phase separate into distinct domains with definite internal states [6]. The immiscibility seems to be a prerequisite to prepare a quantum superposition in which all atoms are in one or the other internal state, $|\psi\rangle = (|N,0\rangle + |0,N\rangle)/\sqrt{2}$, where N is the total number of condensed atoms. There are at least two theoretical proposals how to prepare a macroscopic quantum superposition in this framework [7,8]. Neither of them addresses the crucial question of decoherence.

We find the quantum superposition state involving significant number of BEC atoms to be practically impossible in the standard harmonic trap. Our master equation (derived in Section IV), when applied to the standard harmonic trap of frequency ω, gives for the decoherence rate due to the environment of noncondensed atoms

$$t_{\mathrm{dec}}^{-1} \approx 16\pi^3 \left(4\pi a^2 \frac{N_{\mathrm{E}}}{V} v_T\right) N^2 \ , \tag{1}$$

where N is a number of condensed atoms, $v_T = \sqrt{2k_{\mathrm{B}}T/m}$ is a thermal velocity in the noncondensed thermal cloud, a is a scattering length, V is a volume of the trap, and N_{E} is a number of atoms in the thermal cloud,

$$N_{\mathrm{E}} \approx e^{\frac{\mu}{k_{\mathrm{B}}T}} \left(\frac{k_{\mathrm{B}}T}{\hbar\omega}\right)^3 \ . \tag{2}$$

Here $\mu < 0$ is a chemical potential.

Equation (1) gives the rate of decoherence to the leading order in the fugacity $z = \exp(\beta\mu)$ and also to the leading order in the condensate size, N. Next-to-leading order terms are $O(z^2)$ and $O(N)$, in agreement with Refs. [9,10], so they were neglected here. We also assume that N is small enough for a condensate to live in a single mode as opposed

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to the large N Thomas-Fermi limit. Given all these assumptions, in the standard harmonic trap setting Eq.(1) is a lower estimate for the decoherence rate.

Even without going into details of our derivation, which are given in Section IV, it is easy to understand where a formula like Eq.(1) comes from. N^2 is the main factor which makes the decoherence rate large. It comes from the master equation of the Bloch-Lindblad form $\dot\rho \sim -[N_A - N_B, [N_A - N_B, \rho]]$, with A and B the two internal states of the atoms. N^2 is the distance squared between macroscopically different components of the superposition $(|N,0\rangle + |0,N\rangle)/\sqrt{2}$ - the common wisdom reason why macroscopic objects are classical [1]. The factor in brackets is a scattering rate of a condensate atom on noncondensate atoms - the very process by which the thermal cloud environment learns the quantum state of the system.

Let us estimate the decoherence rate for a set of typical parameters: $T = 1\mu K$, $\omega = 50 Hz$, and $a = 3 \ldots 5 nm$. The thermal velocity is $v_T \approx 10^{-2} m/s$. The volume of the trap can be approximated by $V = 4\pi a_{return}^3/3$, where $a_{return} = \sqrt{2k_B T/m\omega^2}$ is a return point in a harmonic trap at the energy of $k_B T$. We estimate the decoherence time as $t_{dec} \approx 10^5 sec/(N_E N^2)$. For $N_E = 10^0 \ldots 10^4$ and $N = 10 \ldots 10^7$ it can range from 1000s down to 10^{-13}s. For $N = 10$ our (over-)estimate for t_{dec} is large. However, already for $N = 1000$ and $N_E = 10$ (which are still below the Thomas-Fermi regime) t_{dec} shrinks down to milliseconds. Given that our t_{dec} is an upper estimate and that big condensates are more interesting as Schrödinger cats, it is clear that for the sake of cat's longevity, one must go beyond the standard harmonic trap setting.

From Eqs.(1,2) it is obvious that the decoherence rate depends a lot on temperature and on chemical potential. The two factors influence strongly both N_E and v_T. Both can be improved by the following scenario, which is a combination of present day experimental techniques. In the experiment of Ref. [11] a narrow optical dip was superposed at the bottom of a wide magnetic trap. The parameters of the dip were tuned so that it had just one bound state. The gap between this single condensate mode and the first excited state was $1.5\mu K$, which at $T = 1\mu K$ gives a fugacity of $z = \exp(-1.5)$. We need the gap so that we can use the single mode approximation. At low temperatures, the gap results in a small fugacity, which is convenient for calculations. We propose to prepare a condensate inside a similar combination of a wide magnetic and a narrow optical trap (or more generally: a wide well plus a narrow dip with a single bound mode) and then to open the magnetic trap and let the noncondensed atoms disperse. The aim is to get rid of the thermal cloud as much as possible. A similar technique was used in the experiment of Ref. [12].

Let us estimate the ultimate limit for the efficiency of this technique. At the typical initial temperature of $1\mu K$ the thermal velocity of atoms is $10^{-2} m/s$. An atom with this velocity can cross a $1\mu m$ dip in 10^{-4}s. If we wait for, say, 1s after opening the wide trap, then all atoms with velocities above $10^{-6} m/s$ will disperse away from the dip. A thermal velocity of $10^{-6} m/s$ corresponds to the temperature of $10^{-8}\mu K$. As the factor $N_E v_T \sim T^{7/2}$ in Eq.(1), then already 1s after opening the wide trap the decoherence rate due to non-condensed atoms is reduced by a factor of 10^{-28}!

It is not realistic to expect such a "cosmological" reduction factor. The "dip" which is left after the wide harmonic trap is gone could be, for example, a superposition of an ideal dip plus a wide shallow well (which was a negligible perturbation in presence of the wide harmonic trap). The well would have a band of width ΔE of bound states which would not disperse but preserve their occupation numbers from before the opening of the wide trap. They would stay in contact with the condensate and continue to "monitor" its quantum state. Even if such a truncated environment happens to be already relatively harmless, there are means to do better than that.

Further reduction of the decoherence rate can be achieved by "symmetrization" of the environmental state. Perfect symmetrization can be obtained provided that:

i) Atoms have two internal states $|A\rangle$ and $|B\rangle$.

ii) $|A\rangle$ and $|B\rangle$ feel the same trap potential.

iii) $A - A$ and $B - B$ scattering lengths are the same.

iv) A Hamiltonian has a term which drives coherent transitions $A \leftrightarrow B$ with a frequency of $\lambda/\hbar$.

v) $\Delta E \ll \lambda$.

The term in (iv) can be realized by driven coherent transitions like in the experiments of Refs. [13].

Given an ideal symmetry between A and B (assumptions i-iii), the eigenmodes of the dilute environment have annihilation operators $S_s \sim (a_s + b_s)$ and $O_s \sim (a_s - b_s)$, which are symmetric and antisymmetric respectively. a_s and b_s are annihilation operators for the two internal states of an atom in the trap eigenstate numbered as "s" with trap eigenenergy of ϵ_s. S_s has energy $(\epsilon_s - \lambda)$ and O_s has energy $(\epsilon_s + \lambda)$. The symmetric and antisymmetric ΔE-bands of states can be visualized as two ladders shifted with respect to each other by 2λ. In other words, the two sets of states feel the same, but shifted, trapping potentials. If (v) is satisfied, then the antisymmetric O_s's are nearly empty since they can evaporate into symmetric states and then leave the trap. The symmetric S_s's cannot distinguish between A and B so they do not destroy the quantum coherence between the Schrödinger cat's components. After symmetrization, N_E in Eq.(1) has to be replaced by the final number of atoms in the antisymmetric states only;

$$N_E^O \approx \begin{cases} n_\lambda \exp((\mu - \lambda)/k_B T), & \text{for } 2\lambda < \Delta E \\ 0, & \text{for } 2\lambda > \Delta E \end{cases} \tag{3}$$

Here T is the temperature before opening the wide trap. n_λ is the number of antisymmetric bound states which remain within the ΔE-band of symmetric states. Atoms in these antisymmetric bound states cannot disperse away. For $2\lambda > \Delta E$ this number n_λ is zero and there is no decoherence from the thermal cloud.

In this perfect symmetrization limit the states $|\pm\rangle \equiv (|N,0\rangle \pm |0,N\rangle)/\sqrt{2}$ exist within a decoherence-free pointer subspace of the Hilbert space, since they have degenerate eigenvalues of the interaction hamiltonian V [14,15,16]. Any state of that subspace can be written as $\alpha|N,0\rangle + \beta|0,N\rangle$, with α and β complex numbers. If $\mathcal{P}_{[\alpha|N,0\rangle+\beta|0,N\rangle]}$ denotes a projector onto that subspace, then

$$[V, \mathcal{P}_{[\alpha|N,0\rangle+\beta|0,N\rangle]}] = 0, \tag{4}$$

which means that any quantum superposition $\alpha|N,0\rangle+\beta|0,N\rangle$ in the subspace is an eigenstate of the system operators in the interaction hamiltonian (a perfect pointer state), and as such will retain its phase coherence and last forever. The interaction Hamiltonian between the condensate and the thermal cloud is a sum of products of condensate operators and environmental operators. Only terms with symmetric environmental operators are relevant because the antisymmetric states are empty. The total Hamiltonian is symmetric with respect to $A \leftrightarrow B$ so, to preserve this symmetry, the relevant terms with symmetric environmental operators also contain symmetric condensate operators. The argument simplifies a lot for small fugacity where there is only one leading term with the $N_A + N_B$ condensate operator. The states $|\pm\rangle$ are its eigenstates with the same eigenvalue N. They are also (almost) degenerate eigenstates of the condensate Hamiltonian build out of $N_{A,B}$. The coherent transitions $A \leftrightarrow B$ break this degeneracy of $|\pm\rangle$ but the difference of their eigenfrequencies is negligible as compared to the usual condensate lifetime of ~ 10s. In the next-to-leading order in fugacity there are symmetric interaction terms which change the number of condensed atoms. These terms drive the $|N,0\rangle$ and $|0,N\rangle$ states into slightly "squeezed-like" states $|S,0\rangle$ and $|0,S\rangle$ respectively [17]. There are also terms which exchange A with B. They give each state a small admixture of the opposite component. Superpositions of these are still decoherence-free pointer subspaces - there are no relevant antisymmetric operators to destroy their quantum coherence. When the antisymmetric environmental states begin to be occupied (see Eq.(3)), then the commutation relation Eq.(4) is only approximate and states within the subspace will decohere. To leading order in fugacity and condensate size, the decoherence rate is given by Eqs. (1) and (3).

This pointer subspace is not perfect – its existence is in apparent contradiction with the finite lifetimes of the condensates which can last for at most $10\ldots20$s. The reason is that the thermal cloud is not the only source of decoherence. The condensate loses atoms because of Rayleigh scattering, external heating, and three-body decay. The atoms which escape from the condensate carry information about its quantum state. They destroy its quantum coherence. Three body decay, the last process of the three above is the most important one [11]. In the experiment of Ref. [11] the measured loss rate per atom was 4/s for $N = 10^7$ or around 1 atom per 10^{-7}s. The last rate scales like N^3 so already for $N = 10^4$ just one atom is lost per second; decoherence time is 1s. Another possibility is to increase slightly the dip radius. The loss rate scales like density squared so an increase in the dip width by a factor of 2 reduces the loss rate by a factor of $2^6 = 64$.

Ambient magnetic fields are yet another source of coherence loss [18]. The condensed atoms have magnetic moments. If the magnetic moments of A and B were different the magnetic field would distinguish between them and would introduce an unknown phase into the quantum superposition, thus rendering its underlying coherent nature undetectable. Fortunately the much used $|F, m_F\rangle = |2,1\rangle, |1,-1\rangle$ states of ^{87}Rb have the same magnetic moments. For them the magnetic field is a "symmetric" environment [18].

One more source of decoherence is the typical $\approx 1\%$ difference between the $A-A$ and $B-B$ scattering lengths which violates the assumption (iii) above. The Hamiltonian is not perfectly symmetric under $A \leftrightarrow B$. Even for a perfectly symmetrized environment symmetric environmental operators couple to not fully symmetric condensate operators. This means that for the 1% difference of scattering lengths symmetrization can improve decoherence time by at most two orders of magnitude as compared to the unsymmetrized environment.

In summary, we outlined a BEC scenario for an experimental realisation of a decoherence-free Schrödinger cat. This scenario has two ingredients:

1) opening of the wide trap followed by an evaporation of the thermal cloud,

2) and symmetrization of the environment.

The Schrödinger cat is expected to be a quantum superposition of number eigenstates, $|N,0\rangle$ and $|0,N\rangle$.

More details can be found in Sections II-IV, where we study the decoherence rate and the idea of symmetrized environment. Finally, Section V contains discussions.

II. SYMMETRIZED ENVIRONMENT

We introduce annihilation operators for $|A\rangle$ and $|B\rangle$,

$$\phi_A(\vec{x}) \;=\; a\,g(\vec{x}) \;+\; \sum_s a_s\,u_s(\vec{x})\;,$$

$$\phi_B(\vec{x}) \;=\; b\,g(\vec{x}) \;+\; \sum_s b_s\,u_s(\vec{x})\;. \tag{5}$$

where $g(\vec{x})$ is the ground state wave-function localized in the dip with energy $\epsilon_g < 0$, and s is an index running over excited (environmental) states with ortonormal wave-functions $u_s(\vec{x})$. The Hamiltonian of the system is

$$H = \int d^3x \left[v(\phi_A^\dagger \phi_A)(\phi_B^\dagger \phi_B) + \left\{ \frac{u}{2}\phi_A^\dagger\phi_A^\dagger\phi_A\phi_A + \nabla\phi_A^\dagger\nabla\phi_A + U(r)\phi_A^\dagger\phi_A - \lambda\phi_A^\dagger\phi_B \right\} + \{A \leftrightarrow B\} \right]. \tag{6}$$

Here $u = 4\pi\hbar^2 a_{AA}/m$ ($a_{AA} = a_{BB}$ are the inter-scattering lengths) and $v = 4\pi\hbar^2 a_{AB}/m$ (a_{AB} is the intra-scattering length). The immiscibility assumption implies $v > u$. $U(r)$ is the trap potential and λ is the strength of the coherent driving transitions $A \leftrightarrow B$. Substitution of (5) into (6) and subsequent linearization gives a Hamiltonian for the dilute environment

$$H_\mathrm{E} = \sum_s \left[\epsilon_s(a_s^\dagger a_s + b_s^\dagger b_s) - \lambda(a_s^\dagger b_s + b_s^\dagger a_s) \right] . \tag{7}$$

ϵ_s is the single particle energy of level s. We take the lowest enviromental energy $\min[\epsilon_s] = 0$. A transformation

$$S_s = \frac{a_s + b_s}{\sqrt{2}} \;\;,\;\; O_s = \frac{a_s - b_s}{\sqrt{2}} \tag{8}$$

brings H_E to a diagonal form

$$H_\mathrm{E} = \sum_s \left[(\epsilon_s - \lambda)S_s^\dagger S_s + (\epsilon_s + \lambda)O_s^\dagger O_s \right] . \tag{9}$$

Symmetric S's and antisymmetric O's form two identical ladders of states but shifted with respect to each other by 2λ. There is a gap $(-\lambda - \epsilon_g) > 0$ between the lowest S_s state and the ground state. In equilibrium the state S_s's occupation number is $n_s^\mathrm{S} = 1/[\exp(\beta(\epsilon_s - \lambda - \epsilon_g)) - 1] \approx z\exp(-\beta(\epsilon_s - \lambda))$, where the fugacity $z = \exp(-\beta|\epsilon_g|)$ is assumed to be small. The O_s's occupation number is $n_s^\mathrm{O} \approx n_s^\mathrm{S}\exp(-2\beta\lambda)$. If $2\lambda\beta \gg 1$, then O's would be nearly empty. The condensate two-mode Hamiltonian is

$$H_\mathrm{C} = \epsilon_g(a^\dagger a + b^\dagger b) - \lambda(a^\dagger b + b^\dagger a) + \frac{u_\mathrm{c}}{2}(a^\dagger a^\dagger aa + b^\dagger b^\dagger bb) + v_\mathrm{c}(a^\dagger b^\dagger ab) , \tag{10}$$

where e.g. $u_\mathrm{c} = u\int d^3x\, g^4(\vec{x})$ and $v_\mathrm{c} > u_\mathrm{c}$. This Hamiltonian was studied in detail in [7,19]. For a purity factor $\epsilon \equiv (\lambda/(v_\mathrm{c} - u_\mathrm{c})N)^N \ll 1$ the lowest energy subspace contains two macroscopic superpositions

$$|+\rangle = \frac{1}{\sqrt{2\,N!}}[(a^\dagger)^N + (b^\dagger)^N]\,|0,0\rangle \equiv \frac{1}{\sqrt{2}}(|N,0\rangle + |0,N\rangle) \;,$$

$$|-\rangle = \frac{1}{\sqrt{2\,N!}}[(a^\dagger)^N - (b^\dagger)^N]\,|0,0\rangle \equiv \frac{1}{\sqrt{2}}(|N,0\rangle - |0,N\rangle) \;. \tag{11}$$

The lower $|+\rangle$ and the higher $|-\rangle$ states are separated by a small energy gap of $N(u_\mathrm{c} - v_\mathrm{c})\epsilon\ln\epsilon$. If we just have $\epsilon < 1$, the $|\pm\rangle$ states contain an admixture of intermediate states $|N-1,1\rangle,\ldots,|1,N-1\rangle$ such that their overlap is $\langle +|-\rangle = \epsilon$. For $\epsilon \gg 1$ they shrink to

$$(a^\dagger \pm b^\dagger)^N\,|0,0\rangle \;. \tag{12}$$

¿From now on we assume the pure case, $\epsilon \ll 1$.

Finally, the interaction Hamiltonian V, which contains all condensate-noncondensate vertices, is a sum of a symmetric

$$V_\mathrm{S} = \left[(4u + 2v)(a^\dagger a + b^\dagger b) + (2v)(a^\dagger b + b^\dagger a) \right] \otimes \left[\sum_{s_1,s_2} S_{s_1}^\dagger S_{s_2}\alpha_{s_1^\star s_2} \right] +$$

$$\left[(u + v)(a^\dagger + b^\dagger) \right] \otimes \left[\sum_{s_1 s_2 s_3} S_{s_1}^\dagger S_{s_2} S_{s_3}\beta_{s_1^\star s_2 s_3} \right] + \mathrm{h.c.} +$$

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$$\left[(4u + 2v)(a^\dagger a + b^\dagger b) - (2v)(a^\dagger b + b^\dagger a)\right] \otimes \left[\sum_{s_1,s_2} O_{s_1}^\dagger O_{s_2} \alpha_{s_1^\star s_2}\right] +$$

$$\left[a^\dagger + b^\dagger\right] \otimes \left[(u - v) \sum_{s_1 s_2 s_3} S_{s_1}^\dagger O_{s_2} O_{s_3} \beta_{s_1^\star s_2 s_3} + (4u) \sum_{s_1 s_2 s_3} O_{s_1}^\dagger O_{s_2} S_{s_3} \beta_{s_1^\star s_2 s_3}\right] + \text{h.c.} \tag{13}$$

and an antisymmetric part

$$V_{\rm O} = \left[(4u - 2v)(a^\dagger a - b^\dagger b) + (2v)(a^\dagger b - b^\dagger a)\right] \otimes \left[\sum_{s_1,s_2} S_{s_1}^\dagger O_{s_2} \alpha_{s_1^\star s_2} + \text{h.c.}\right] +$$

$$\left[a^\dagger - b^\dagger\right] \otimes \left[\sum_{s_1 s_2 s_3} \beta_{s_1^\star s_2 s_3}((u + v)O_{s_1}^\dagger O_{s_2} O_{s_3} + (u - v)O_{s_1}^\dagger S_{s_2} S_{s_3} + (2u)S_{s_1}^\dagger S_{s_2} O_{s_3})\right] + \text{h.c.} \tag{14}$$

The coefficients are given by integrals

$$\alpha_{s_1^\star s_2} = \frac{1}{4} \int d^3x \; g^2 u_{s_1}^\star u_{s_2} \;, \quad \beta_{s_1^\star s_2^\star s_3} = \frac{1}{2\sqrt{2}} \int d^3x \; g u_{s_1}^\star u_{s_2}^\star u_{s_3} \;. \tag{15}$$

In $V_{\rm S}$ and $V_{\rm O}$ we neglected vertices C + C → C + NC and C + C → NC + NC, where C is a condensate and NC is a noncondensate particle, and their hermitian conjugates. They are forbiden by energy conservation due to the gap between the condensate mode and the lowest environmental state.

All the terms in $V_{\rm S,O}$ were arranged in the form [condensate operator]⊗[environment operator]. $V_{\rm S}$ contains only [c.o.]'s which are symmetric under $a \leftrightarrow b$. They act in precisely the same way on both components of the superposition and as such they do not destroy coherence between the macroscopic components. To illustrate this let us calculate a commutator of the leading order term in V_S with the projector onto the subspace of states $\alpha|N,0\rangle + \beta|0,N\rangle$,

$$\left[(4u + 2v)(a^\dagger a + b^\dagger b) \otimes \sum_{s_1,s_2} S_{s_1}^\dagger S_{s_2} \alpha_{s_1^\star s_2} \;,\; \mathcal{P}_{[\alpha|N,0\rangle + \beta|0,N\rangle]}\right] = 0 \tag{16}$$

This commutator vanishes because states of the form $\alpha|N,0\rangle + \beta|0,N\rangle$ are eigenstates of $N_{\rm C} = a^\dagger a + b^\dagger b$. Therefore, this subspace would be a pointer subspace [14] if this leading term were the only term in the interaction Hamiltonian. However, $V_{\rm S}$ has other terms that are not simply functions of the total condensate number operator. Coherent states of an annihilation operator $a + b$ are exact eigenstates of the [c.o.] $a + b$ and approximate eigenstates of $a^\dagger + b^\dagger$. These coherent states, however, are combinations of (12) and as such they are not in the lowest energy subspace of $H_{\rm C}$. What is more, the matrix elements of $a + b$ are of the order $O(\sqrt{N})$ which is negligible as compared to the matrix elements of the number operator. If we project $a + b$ and its h.c. on the subspace (11), then their approximate eigenstates for large $|z|$ are coherent states $\alpha|z,0\rangle + \beta|0,z\rangle$. The decoherence effects of $a + b$ and $N_{\rm C}$ put together lead to a superposition of macroscopic "squeezed-like" states $\alpha|S,0\rangle + \beta|0,S\rangle$. This result is by now well established for a single component condensate [17]. Finally, the [c.o.] $a^\dagger b + b^\dagger a$ drives the state out of the pointer subspace (11). Its effect is supressed by the purity condition $\epsilon \ll 1$ and is also negligible as compared to the direct effect of the λ-term in H_C. Similar comment applies to the "out of the subspace" action of $a^\dagger + b^\dagger$. Therefore, the effect of all these terms in $V_{\rm S}$ imply that the subspace spanned by $|N,0\rangle$ and $|0,N\rangle$ is no longer an exact decoherence-free pointer subspace, i.e. $[V_{\rm S}, \mathcal{P}_{[\alpha|N,0\rangle + \beta|0,N\rangle]}] \neq 0$. The correct pointer subspace would be one spanned by those "squeezed-like" states. However, to leading order in fugacity and in the condensate size, N, the dominant terms are those depending on $N_{\rm C}$; the subspace of $\alpha|N,0\rangle + \beta|0,N\rangle$ is (approximately) the exact decoherence-free pointer subspace.

The antisymmetric [c.o.]'s in $V_{\rm O}$ act in opposite way on both components; they destroy their quantum coherence. To illustrate this let us calculate a commutator of the leading term in $V_{\rm O}$ with the projector operator onto the subspace $\alpha|N,0\rangle + \beta|0,N\rangle$,

$$\left[(4u - 2v)(a^\dagger a - b^\dagger b) \otimes \left(\sum_{s_1,s_2} S_{s_1}^\dagger O_{s_2} \alpha_{s_1^\star s_2} + \text{h.c.}\right) \;,\; \mathcal{P}_{[\alpha|N,0\rangle + \beta|0,N\rangle]}\right] =$$

$$2N(4u - 2v)\left(\alpha\beta^\star|N,0\rangle\langle 0,N| - \alpha^\star\beta|0,N\rangle\langle N,0|\right) \otimes \left(\sum_{s_1,s_2} S_{s_1}^\dagger O_{s_2} \alpha_{s_1^\star s_2} + \text{h.c.}\right) \tag{17}$$

The commutator is $O(N)$. The occupation numbers of O's are supressed by a Boltzmann factor $n^{\rm S}\exp(-2\lambda\beta)$; for $2\lambda\beta \gg 1$ the O's are unoccupied. As such they cannot scatter into condensate particles. It is also impossible to

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scatter S's into O's thanks to energy conservation. The unoccupied O's are irrelevant for decoherence and as such can be neglected in V and in the above commutator. This effectively sets the dangerous V_O to zero and leaves us only with $V_S \neq 0$. In the absence of O's the states $\alpha|N,0\rangle + \beta|0,N\rangle$ are a pointer subspace (or a decoherence-free subspace). The symmetrized environment of just S's defines the quantum states of the components but it does not destroy their mutual coherence.

III. MASTER EQUATION

Symmetrized environment is a robust idea whose validity does not depend on detailed calculations. Nevertheless, for the sake of illustration we derived (by a perturbative expansion in V) an approximate Bloch-Lindblad form master equation for the reduced density matrix $\rho(t)$ of the condensate modes. The calculations are long but rather straightforward, their details can be found in Section IV. Here we just give the final result.

$$\dot{\rho} = \frac{i}{\hbar}[\rho, H_C^{\text{ren}}] + \dot{\rho}_S + \dot{\rho}_O \ . \tag{18}$$

H_C^{ren} is a renormalized condensate Hamiltonian,

$$H_C^{\text{ren}} = H_C + \langle H_C \rangle + c_1 \Delta_1^\dagger \Delta_1 + c_2 \Delta_2^\dagger \Delta_2 + c_3 \Delta_3^\dagger \Delta_3 + c_4 \Delta_4^\dagger \Delta_4 \ . \tag{19}$$

$\langle \ldots \rangle$ means an average over an initial environment thermal density matrix and $\dot{\rho}_S$ and $\dot{\rho}_O$ are contributions from V_S and V_O respectively. They read

$$\dot{\rho}_S = d_1 D[\Delta_1]\rho + d_2 D[\Delta_2]\rho \ , \tag{20}$$

$$\dot{\rho}_O = d_3 D[\Delta_3]\rho + \tilde{d}_3 D[\Delta_3^\dagger]\rho + d_4 D[\Delta_4]\rho \tag{21}$$

with $D[\Delta]\rho \equiv \{\Delta^\dagger \Delta, \rho\} - 2\Delta\rho\Delta^\dagger$ the Lindblad operator. The operators introduced in this expression are defined as

$$\Delta_1 = (4u + 2v)(a^\dagger a + b^\dagger b) + (2v)(a^\dagger b + b^\dagger a) \ , \tag{22}$$

$$\Delta_2 = (u + v)(a^\dagger + b^\dagger) \ , \tag{23}$$

$$\Delta_3 = (4u - 2v)(a^\dagger a - b^\dagger b) + (2v)(a^\dagger b - b^\dagger a) \ , \tag{24}$$

$$\Delta_4 = a^\dagger - b^\dagger \ . \tag{25}$$

The coefficients c_i and d_i are proportional to fugacity z. What is more $c_{3,4}/c_{1,2} \sim e^{-2\lambda\beta}$ and $d_{3,4}/d_{1,2} \sim e^{-2\lambda\beta}$. As anticipated, the antisymmetric contributions are supressed by a Boltzmann factor due to O's. This approximate master equation confirms our heuristic arguments.

Any asymmetry between A and B is a source of decoherence. If, for example, the $A - A$ and $B - B$ scattering lengths were different, $u_{AA} = u + \delta u$ and $u_{BB} - \delta u$, then V_O would acquire extra terms $\sim \delta u$. They would show up in $\dot{\rho}_O$ with coefficients $d \sim z\, u\delta u$. The ratio of these to the "symmetric" coefficients $d_{1,2}$ is $\delta u/u$ (which is approximately 0.03 for $|F = 1; m_F = -1\rangle$ and $|F = 2; m_F = 1\rangle$ states of ^{87}Rb) as compared to $d_{3,4}/d_{1,2} \sim \exp(-2\lambda\beta)$. If $\delta u/u$ is less than the Boltzmann factor, then this asymmetry is not a leading source of decoherence.

IV. DERIVATION OF THE MASTER EQUATION

We derive a master equation by perturbative expansion in V or in the coupling constants u, v which we regard to be of the same order. We assume the initial density matrix at $t = 0$ to be a product $\tilde{\rho}(0) = \rho(0) \otimes \rho_E$ of the system (condensate) and environment (noncondensate) density matrices. To take a more accurate starting point we make a rearrangement $V \to V - \langle V \rangle$ and $H_C \to H_C + \langle V \rangle$ where $\langle \ldots \rangle = \text{Tr}_E[\ldots \rho_E]$ is a trace over the environment thermal density matrix at the initial time. The new H_C differs from the old one by a renormalization

$$\epsilon_g \to \epsilon_g^{\text{eff}} = \epsilon_g + (4u + 2v)\sum_s n_s^S \alpha_{s^\star s} \ ,$$

$$\lambda \to \lambda^{\text{eff}} = \lambda - (2v)\sum_s n_s^S \alpha_{s^\star s} \ . \tag{26}$$

We compute $\dot{\tilde{\rho}}(t)$ up to second order in the perturbation Hamiltonian $V = \sum_i \Delta_i \otimes E_i$, with Δ_i an operator for the condensate and E_i one for the environment. The master equation for $\rho(t)$ alone is obtained by tracing $\dot{\tilde{\rho}}(t)$ over the non-condensed modes. The result is [20]

$$\dot{\rho}(t) = \frac{i}{\hbar}[\rho, H_{\mathrm{C}}] - \frac{1}{2\hbar^2}\sum_{ij}\int_0^t d\tau\, g_{ij}^{\mathrm{sym}}(\tau)[\Delta_i(0),[\Delta_j(-\tau),\rho]] - \frac{1}{2\hbar^2}\sum_{ij}\int_0^t d\tau\, g_{ij}^{\mathrm{asym}}(\tau)[\Delta_i(0),\{\Delta_j(-\tau),\rho\}] \,, \quad (27)$$

where

$$g_{ij}^{\mathrm{sym}}(\tau) = \langle\{E_i(\tau), E_j(0)\}\rangle \,, \tag{28}$$
$$g_{ij}^{\mathrm{asym}}(\tau) = \langle[E_i(\tau), E_j(0)]\rangle \,. \tag{29}$$

We have used that after the rearrangement $V \to V - \langle V \rangle$ and $H_{\mathrm{C}} \to H_{\mathrm{C}} + \langle V \rangle$, there are no linear terms in the perturbative expansion because $\langle V \rangle = 0$. We first start to study the contribution of the symmetric terms V_{S} in the interaction Hamiltonian, and later we shall deal with the antisymmetric ones V_{O}. It is easy to show that the are no cross terms S-O in the calculation of the master equation.

A. Free Hamiltonian

The next step in the calculation is to solve the dynamics of the free condensate Hamiltonian

$$H_{\mathrm{C}} = (u_{\mathrm{c}} - v_{\mathrm{c}})J_z^2 + \lambda^{\mathrm{eff}}J_x \,, \tag{30}$$

where we introduced angular momentum operators

$$J_x = \frac{1}{2}(a^\dagger b + b^\dagger a) \,, \tag{31}$$

$$J_y = \frac{i}{2}(b^\dagger a - a^\dagger b) \,, \tag{32}$$

$$J_z = \frac{1}{2}(a^\dagger a - b^\dagger b), \tag{33}$$

and discarded constant terms proportional to the total number of particles $N = a^\dagger a + b^\dagger b$ [19]. The Heisenberg equations of motion for the condensate operators $a(t)$ and $b(t)$ are

$$i\hbar\dot{a} = \epsilon_g^{\mathrm{eff}}a - \lambda^{\mathrm{eff}}b + (u_{\mathrm{c}}a^\dagger a + v_{\mathrm{c}}b^\dagger b)a \,,$$
$$i\hbar\dot{b} = \epsilon_g^{\mathrm{eff}}b - \lambda^{\mathrm{eff}}a + (u_{\mathrm{c}}b^\dagger b + v_{\mathrm{c}}a^\dagger a)b \,. \tag{34}$$

These equations cannot be solved exactly for $\lambda^{\mathrm{eff}} \neq 0$, when numbers of a's and b's are not conserved independently due to the coherent transfer of particles from one state to the other. In their mean-field version, these equations correspond to the well-known macroscopic self-trapping equation, which have the feature that below a critical "purity" the oscillations between states A and B are not complete; for $\epsilon \ll 1$ the systems is self-locked in either of these states [21]. In this limit we can set λ^{eff} to zero, and for the immiscible case $v_{\mathrm{c}} > u_{\mathrm{c}}$ the ground state of H_C corresponds to maximum eigenstates of J_z^2, i.e. pure cat states (11). Also in this case the mean-field solutions are

$$a(t) = a(0)e^{-\frac{it}{\hbar}[\epsilon_g^{\mathrm{eff}} + N(u_{\mathrm{c}}+v_{\mathrm{c}})/2]} \,, b(t) = b(0)e^{-\frac{it}{\hbar}[\epsilon_g^{\mathrm{eff}} + N(u_{\mathrm{c}}+v_{\mathrm{c}})/2]} \,. \tag{35}$$

We believe that these solutions are qualitatively correct as long as the purity factor $\epsilon \ll 1$.

B. Symmetric Interaction Hamiltonian

The symmetric Hamiltonian can be written as

$$V_{\mathrm{S}} = \Delta_1 E_1 + \tilde{\Delta}_1 \tilde{E}_1 + \left[\Delta_2 E_2 + \tilde{\Delta}_2 \tilde{E}_2 + \mathrm{h.c.}\right] \,, \tag{36}$$

where we have defined condensate operators

$$\Delta_1 = (4u + 2v)(a^\dagger a + b^\dagger b) + (2v)(a^\dagger b + b^\dagger a) \ , \tag{37}$$

$$\tilde{\Delta}_1 = (4u + 2v)(a^\dagger a + b^\dagger b) - (2v)(a^\dagger b + b^\dagger a) \ , \tag{38}$$

$$\Delta_2 = (u + v)(a^\dagger + b^\dagger) \ , \tag{39}$$

$$\tilde{\Delta}_2 = (u + v)(a^\dagger + b^\dagger) \ , \tag{40}$$

and the corresponding non-condensate operators

$$E_1 = \sum_{s_1,s_2} \alpha_{s_1^*,s_2} S_{s_1}^\dagger S_{s_1} - \sum_s |\alpha_{s^*,s}|^2 S_s^\dagger S_s \ , \tag{41}$$

$$\tilde{E}_1 = \sum_{s_1,s_2} \alpha_{s_1^*,s_2} O_{s_1}^\dagger O_{s_1} - \sum_s |\alpha_{s^*,s}|^2 O_s^\dagger O_s \ , \tag{42}$$

$$E_2 = \sum_{s_1,s_2,s_3} \beta_{s_1^*,s_2,s_3} (S_{s_1}^\dagger S_{s_2} S_{s_3} + O_{s_1}^\dagger S_{s_2} O_{s_3}) \ , \tag{43}$$

$$\tilde{E}_2 = \sum_{s_1,s_2,s_3} \beta_{s_1^*,s_2,s_3} (O_{s_1}^\dagger O_{s_2} S_{s_3} + S_{s_1}^\dagger O_{s_2} O_{s_3}) \ . \tag{44}$$

It is clear that the terms Δ_i and $\tilde{\Delta}_i$ and their corresponding environment operators have the same structure, and therefore will give the same qualitatively contribution to the master equation. In the following we shall keep only the Δ_i terms.

The expectation values of multiple-points noncondensate operators are written in terms of the two-point functions

$$\langle S_{s_1}^\dagger(t_1) S_{s_2}(t_2) \rangle = \delta_{s_1 s_2} n_{s_1}^{\mathrm{S}} e^{\frac{i}{\hbar}(\epsilon_{s_1} - \lambda)(t_1 - t_2)} e^{-\gamma_{s_1}^{\mathrm{S}}|t_1 - t_2|} \ ,$$

$$\langle O_{s_1}^\dagger(t_1) O_{s_2}(t_2) \rangle = \delta_{s_1 s_2} n_{s_1}^{\mathrm{O}} e^{\frac{i}{\hbar}(\epsilon_{s_1} + \lambda)(t_1 - t_2)} e^{-\gamma_{s_1}^{\mathrm{O}}|t_1 - t_2|} \ ,$$

$$\langle S_{s_1}^\dagger(t_1) O_{s_2}(t_2) \rangle = 0 \tag{45}$$

via Wick's theorem. Here $n_s^{\mathrm{S,O}} = [z e^{\beta(\epsilon_s \pm \lambda)} - 1]^{-1}$ are Bose occupation numbers, and the γ_s's are inverse finite lifetimes of the environmental states. We also expand these expectation values to leading order in fugacity z. The kernels are

$$\langle E_1(\tau) E_1(0) \rangle = \langle E_1(0) E_1(\tau) \rangle^* = \sum_{s_1,s_2} |\alpha_{s_1^*,s_2}|^2 n_{s_1}^{\mathrm{S}} e^{\frac{i\tau}{\hbar}(\epsilon_{s_1} - \epsilon_{s_2})} e^{-\tau(\gamma_{s_1}^{\mathrm{S}} + \gamma_{s_2}^{\mathrm{S}})} + O(z^2) \ , \tag{46}$$

$$\langle E_2(\tau) E_2^\dagger(0) \rangle = \langle E_2(0) E_2^\dagger(\tau) \rangle^* = \sum_{s_1,s_2,s_3} |\beta_{s_1^*,s_2,s_3}|^2 e^{\frac{i\tau}{\hbar}(\epsilon_{s_1} - \epsilon_{s_2} - \epsilon_{s_3} + \lambda)} \left[2 n_{s_1}^{\mathrm{S}} e^{-\tau(\gamma_{s_1}^{\mathrm{S}} + \gamma_{s_2}^{\mathrm{S}} + \gamma_{s_3}^{\mathrm{S}})} + \right.$$

$$\left. n_{s_1}^{\mathrm{O}} e^{-\tau(\gamma_{s_1}^{\mathrm{O}} + \gamma_{s_2}^{\mathrm{S}} + \gamma_{s_3}^{\mathrm{O}})} \right] + O(z^2) \ , \tag{47}$$

$$\langle E_2^\dagger(\tau) E_2(0) \rangle = \langle E_2^\dagger(0) E_2(\tau) \rangle = O(z^2) \ . \tag{48}$$

Given the kernels and the free evolution of condensate operators, we have to perform the time integrals in (27). We calculate the master equation for late times $t \gg \gamma_s^{-1}$. Hence the exponential decay of the propagators dominates their behavior, and we can apply the Markovian approximation replacing the upper limit in the time integration by infinity. This calculation results in two different contributions, one being a renormalization of the free Hamiltonian

$$\delta H_{\mathrm{C}} = c_1 \Delta_1^\dagger \Delta_1 + c_2 \Delta_2^\dagger \Delta_2 \ , \tag{49}$$

where

$$c_1 = \frac{1}{\hbar^2} \sum_{s_1,s_2} |\alpha_{s_1^*,s_2}|^2 n_{s_1}^{\mathrm{S}} \frac{\hbar^{-1}(\epsilon_{s_1} - \epsilon_{s_2})}{(\gamma_{s_1}^{\mathrm{S}} + \gamma_{s_2}^{\mathrm{S}})^2 + (\hbar^{-1}(\epsilon_{s_1} - \epsilon_{s_2}))^2} \ , \tag{50}$$

$$c_2 = \frac{1}{\hbar^2} \sum_{s_1,s_2,s_3} |\beta_{s_1^*,s_2,s_3}|^2 \left[2 n_{s_1}^{\mathrm{S}} \frac{\hbar^{-1}(\epsilon_{s_1} - \epsilon_{s_2} - \epsilon_{s_3} + \lambda + \epsilon_{\mathrm{g}}^{\mathrm{eff}} + N(u_{\mathrm{c}} + v_{\mathrm{c}})/2)}{(\gamma_{s_1}^{\mathrm{S}} + \gamma_{s_2}^{\mathrm{S}} + \gamma_{s_3}^{\mathrm{S}})^2 + (\hbar^{-1}(\epsilon_{s_1} - \epsilon_{s_2} - \epsilon_{s_3} + \lambda + \epsilon_{\mathrm{g}}^{\mathrm{eff}} + N(u_{\mathrm{c}} + v_{\mathrm{c}})/2))^2} + \right.$$

$$\left. n_{s_1}^{\mathrm{O}} \frac{\hbar^{-1}(\epsilon_{s_1} - \epsilon_{s_2} - \epsilon_{s_3} + \lambda - \epsilon_{\mathrm{g}}^{\mathrm{eff}} - N(u_{\mathrm{c}} + v_{\mathrm{c}})/2)}{(\gamma_{s_1}^{\mathrm{O}} + \gamma_{s_2}^{\mathrm{S}} + \gamma_{s_3}^{\mathrm{O}})^2 + (\hbar^{-1}(\epsilon_{s_1} - \epsilon_{s_2} - \epsilon_{s_3} + \lambda - \epsilon_{\mathrm{g}}^{\mathrm{eff}} - N(u_{\mathrm{c}} + v_{\mathrm{c}})/2))^2} \right] \ . \tag{51}$$

The other contribution has the Lindblad form $D[\Delta]\rho \equiv \{\Delta^\dagger\Delta, \rho\} - 2\Delta\rho\Delta^\dagger$ and reads

$$\dot\rho_{\rm S} = d_1 D[\Delta_1]\rho + d_2 D[\Delta_2]\rho \ , \tag{52}$$

where

$$d_1 = -\frac{1}{\hbar^2} \sum_{s_1,s_2} |\alpha_{s_1^*,s_2}|^2 n_{s_1}^{\rm S} \frac{\gamma_{s_1}^{\rm S} + \gamma_{s_2}^{\rm S}}{(\gamma_{s_1}^{\rm S} + \gamma_{s_2}^{\rm S})^2 + (\hbar^{-1}(\epsilon_{s_1} - \epsilon_{s_2}))^2} \ , \tag{53}$$

$$d_2 = -\frac{1}{\hbar^2} \sum_{s_1,s_2,s_3} |\beta_{s_1^*,s_2,s_3}|^2 \left[2n_{s_1}^{\rm S} \frac{\gamma_{s_1}^{\rm S} + \gamma_{s_2}^{\rm S} + \gamma_{s_3}^{\rm S}}{(\gamma_{s_1}^{\rm S} + \gamma_{s_2}^{\rm S} + \gamma_{s_3}^{\rm S})^2 + (\hbar^{-1}(\epsilon_{s_1} - \epsilon_{s_2} - \epsilon_{s_3} + \lambda + \epsilon_{\rm g}^{\rm eff} + N(u_{\rm c} + v_{\rm c})/2))^2} + \right.$$
$$\left. n_{s_1}^{\rm O} \frac{\gamma_{s_1}^{\rm S} + \gamma_{s_2}^{\rm S} + \gamma_{s_3}^{\rm S}}{(\gamma_{s_1}^{\rm O} + \gamma_{s_2}^{\rm S} + \gamma_{s_3}^{\rm O})^2 + (\hbar^{-1}(\epsilon_{s_1} - \epsilon_{s_2} - \epsilon_{s_3} + \lambda + \epsilon_{\rm g}^{\rm eff} + N(u_{\rm c} + v_{\rm c})/2))^2} \right] \ . \tag{54}$$

C. Antisymmetric Interaction Hamiltonian

The antisymmetric Hamiltonian is

$$V_{\rm O} = \Delta_3 E_3 + \Delta_4 E_4 + {\rm h.c.} \ , \tag{55}$$

where now

$$\Delta_3 = (4u - 2v)(a^\dagger a - b^\dagger b) + (2v)(a^\dagger b - b^\dagger a) \ , \tag{56}$$
$$\Delta_4 = a^\dagger - b^\dagger \ , \tag{57}$$

and the corresponding environment operators

$$E_3 = \sum_{s_1,s_2} \alpha_{s_1^*,s_2} O_{s_1}^\dagger S_{s_2} \ , \tag{58}$$

$$E_4 = \sum_{s_1,s_2,s_3} \beta_{s_1^*,s_2,s_3} \left[(u+v)O_{s_1}^\dagger O_{s_2} O_{s_3} - (u-v)O_{s_1}^\dagger S_{s_2} S_{s_3} + 2u S_{s_1}^\dagger S_{s_2} O_{s_3} \right] \ . \tag{59}$$

The kernels are

$$\langle E_3(\tau) E_3^\dagger(0)\rangle = \langle E_3(0) E_3^\dagger(\tau)\rangle^* = \sum_{s_1,s_2} |\alpha_{s_1^*,s_2}|^2 n_{s_1}^{\rm O} e^{\frac{i\tau}{\hbar}(\epsilon_{s_1} - \epsilon_{s_2} + 2\lambda)} e^{-\tau(\gamma_{s_1}^{\rm O} + \gamma_{s_2}^{\rm S})} + O(z^2) \ , \tag{60}$$

$$\langle E_3^\dagger(\tau) E_3(0)\rangle = \langle E_3^\dagger(0) E_3(\tau)\rangle^* = \sum_{s_1,s_2} |\alpha_{s_1^*,s_2}|^2 n_{s_1}^{\rm O} e^{\frac{i\tau}{\hbar}(\epsilon_{s_1} - \epsilon_{s_2} - 2\lambda)} e^{-\tau(\gamma_{s_1}^{\rm S} + \gamma_{s_2}^{\rm O})} + O(z^2) \ , \tag{61}$$

$$\langle E_3^\dagger(\tau) E_3(0)\rangle = \langle E_3^\dagger(0) E_3(\tau)\rangle = O(z^2) \ , \tag{62}$$

$$\langle E_4(\tau) E_4^\dagger(0)\rangle = \langle E_4(0) E_4^\dagger(\tau)\rangle^* = \sum_{s_1,s_2,s_3} |\beta_{s_1^*,s_2,s_3}|^2 \left[2(u+v)^2 n_{s_1}^{\rm O} e^{\frac{i\tau}{\hbar}(\epsilon_{s_1} - \epsilon_{s_2} - \epsilon_{s_3} - \lambda) - \tau(\gamma_{s_1}^{\rm O} + \gamma_{s_2}^{\rm O} + \gamma_{s_3}^{\rm O})} + \right.$$
$$\left. 2(u-v)^2 n_{s_1}^{\rm O} e^{\frac{i\tau}{\hbar}(\epsilon_{s_1} - \epsilon_{s_2} - \epsilon_{s_3} + 3\lambda) - \tau(\gamma_{s_1}^{\rm O} + \gamma_{s_2}^{\rm O} + \gamma_{s_3}^{\rm O})} \right] \ , \tag{63}$$

$$\langle E_4^\dagger(0) E_4(\tau)\rangle = \langle E_4^\dagger(\tau) E_4(0)\rangle = O(z^2) \ . \tag{64}$$

The renormalization terms are

$$\delta H_{\rm C} = c_3 \Delta_3^\dagger \Delta_3 + c_4 \Delta_4^\dagger \Delta_4 \tag{65}$$

with

$$c_3 = \frac{1}{\hbar^2} \sum_{s_1,s_2} |\alpha_{s_1^*,s_2}|^2 \left[n_{s_1}^{\rm O} \frac{\hbar^{-1}(\epsilon_{s_1} - \epsilon_{s_2} - 2\lambda)}{(\gamma_{s_1}^{\rm S} + \gamma_{s_2}^{\rm O})^2 + (\hbar^{-1}(\epsilon_{s_1} - \epsilon_{s_2} - 2\lambda))^2} - n_{s_1}^{\rm O} \frac{\hbar^{-1}(\epsilon_{s_1} - \epsilon_{s_2} + 2\lambda)}{(\gamma_{s_1}^{\rm O} + \gamma_{s_2}^{\rm S})^2 + (\hbar^{-1}(\epsilon_{s_1} - \epsilon_{s_2} + 2\lambda))^2} \right] \ , \tag{66}$$

$$c_4 = \frac{1}{\hbar^2} \sum_{s_1,s_2,s_3} |\beta_{s_1^*,s_2,s_3}|^2 \left[2(u+v)^2 n_{s_1}^{\rm O} \frac{\hbar^{-1}(\epsilon_{s_1} - \epsilon_{s_2} - \epsilon_{s_3} - \lambda + \epsilon_{\rm g}^{\rm eff} + N(u_{\rm c} + v_{\rm c})/2)}{(\gamma_{s_1}^{\rm O} + \gamma_{s_2}^{\rm O} + \gamma_{s_3}^{\rm O})^2 + (\hbar^{-1}(\epsilon_{s_1} - \epsilon_{s_2} - \epsilon_{s_3} - \lambda + \epsilon_{\rm g}^{\rm eff} + N(u_{\rm c} + v_{\rm c})/2))^2} + \right.$$
$$\left. 2(u-v)^2 n_{s_1}^{\rm O} \frac{\hbar^{-1}(\epsilon_{s_1} - \epsilon_{s_2} - \epsilon_{s_3} + \lambda - \epsilon_{\rm g}^{\rm eff} - N(u_{\rm c} + v_{\rm c})/2)}{(\gamma_{s_1}^{\rm O} + \gamma_{s_2}^{\rm S} + \gamma_{s_3}^{\rm S})^2 + (\hbar^{-1}(\epsilon_{s_1} - \epsilon_{s_2} - \epsilon_{s_3} + 3\lambda - \epsilon_{\rm g}^{\rm eff} - N(u_{\rm c} + v_{\rm c})/2))^2} \right] \ . \tag{67}$$

Finally, the Lindblad part is

$$\dot{\rho}_{\mathrm{O}} = d_3 D[\Delta_3]\rho + \tilde{d}_3 D[\Delta_3^\dagger]\rho + d_4 D[\Delta_4]\rho \ , \tag{68}$$

where

$$d_3 = -\frac{1}{\hbar^2} \sum_{s_1,s_2} |\alpha_{s_1^*,s_2}|^2 n_{s_1}^{\mathrm{O}} \frac{\gamma_{s_1}^{\mathrm{S}} + \gamma_{s_2}^{\mathrm{O}}}{(\gamma_{s_1}^{\mathrm{S}} + \gamma_{s_2}^{\mathrm{O}})^2 + (\hbar^{-1}(\epsilon_{s_1} - \epsilon_{s_2} - 2\lambda))^2} \ , \tag{69}$$

$$\tilde{d}_3 = -\frac{1}{\hbar^2} \sum_{s_1,s_2} |\alpha_{s_1^*,s_2}|^2 n_{s_1}^{\mathrm{O}} \frac{\gamma_{s_1}^{\mathrm{O}} + \gamma_{s_2}^{\mathrm{S}}}{(\gamma_{s_1}^{\mathrm{S}} + \gamma_{s_2}^{\mathrm{O}})^2 + (\hbar^{-1}(\epsilon_{s_1} - \epsilon_{s_2} + 2\lambda))^2} \ , \tag{70}$$

$$d_4 = -\frac{1}{\hbar^2} \sum_{s_1,s_2,s_3} |\beta_{s_1^*,s_2,s_3}|^2 \left[2(u+v)^2 n_{s_1}^{\mathrm{O}} \frac{\gamma_{s_1}^{\mathrm{O}} + \gamma_{s_2}^{\mathrm{O}} + \gamma_{s_3}^{\mathrm{O}}}{(\gamma_{s_1}^{\mathrm{O}} + \gamma_{s_2}^{\mathrm{O}} + \gamma_{s_3}^{\mathrm{O}})^2 + (\hbar^{-1}(\epsilon_{s_1} - \epsilon_{s_2} - \epsilon_{s_3} - \lambda + \epsilon_{\mathrm{g}}^{\mathrm{eff}} + N(u_{\mathrm{c}} + v_{\mathrm{c}})/2))^2} + \right.$$

$$\left. 2(u-v)^2 n_{s_1}^{\mathrm{O}} \frac{\gamma_{s_1}^{\mathrm{O}} + \gamma_{s_2}^{\mathrm{S}} + \gamma_{s_3}^{\mathrm{S}}}{(\gamma_{s_1}^{\mathrm{O}} + \gamma_{s_2}^{\mathrm{S}} + \gamma_{s_3}^{\mathrm{S}})^2 + (\hbar^{-1}(\epsilon_{s_1} - \epsilon_{s_2} - \epsilon_{s_3} + 3\lambda + \epsilon_{\mathrm{g}}^{\mathrm{eff}} + N(u_{\mathrm{c}} + v_{\mathrm{c}})/2))^2} \right] \ . \tag{71}$$

D. Estimate of the Decoherence Rate

To estimate the decoherence rate for our BEC quantum superposition, let us consider one of the terms in (68), namely $\dot{\rho} = 16v^2 d_3[J_z,[J_z,\rho]]$, which arises form the Δ_3 term. This term corresponds to elastic two-body collisions between condensate and non-condensate atoms, and induces phase decoherence. To this end we first make an estimation for the coefficient $\alpha_{s_1^*,s_2}$ that enters in d_3. We assume that we have a big harmonic trap plus the dip located in its center. We take as the wave function for the ground state a gaussian $g(\vec{x}) = (m\omega_{\mathrm{dip}}/(\pi\hbar))^{3/2} \exp(-m\omega_{\mathrm{dip}}r^2/\hbar)$, where ω_{dip} is the frequency of the optical trapping potential in the dip. In principle, we should use thermal wave packets for the wave functions of the non-condensed particles [22]. However, due to the strong localization of the condensate in the dip, we can approximate the non-condensed particles as plane waves. Then

$$\alpha_{s_1^*,s_2} = \frac{1}{4} \int d^3x\, g^2(\vec{x}) \frac{e^{i\vec{k}_1 \cdot \vec{x}}}{\sqrt{V}} \frac{e^{-i\vec{k}_2 \cdot \vec{x}}}{\sqrt{V}} = \frac{1}{4V} \exp\left(-\frac{\hbar}{4m\omega_{\mathrm{dip}}}|\vec{k}_1 - \vec{k}_2|^2 \right) \tag{72}$$

On the other hand, the decay rate γ is much smaller than w. This lets us replace the last factor in d_3 by $\pi\delta(\hbar^{-1}(\epsilon_{s_1} - \epsilon_{s_2}))$, where we have used the identity $\lim_{\gamma \to 0} \frac{\gamma}{\gamma^2 + x^2} = \pi\delta(x)$. Finally, replacing the sum over momenta by the continuum expression $\sum_{\vec{k}} \to V \int d^3k = V \int d\Omega dk k^2$, and using the free dispersion relation $\epsilon_{\vec{k}} = \hbar^2 k^2/2m$, we find

$$d_3 = -\frac{\pi m}{16\hbar^3} \int dk_1 k_1^3 z e^{-\beta\lambda} e^{-\beta\hbar^2 k_1^2/2m} \int d\Omega_1 d\Omega_2 e^{-\frac{\hbar k_1^2}{m\omega_{\mathrm{dip}}}(1-\cos\alpha)} \tag{73}$$

where α is the angle between the two vectors $\vec{k}_1$ and $\vec{k}_2$ corresponding to the two scattered non-condensed particles. Since the condensate in the dip is very localized, its typical linear dimension being $l_{\mathrm{dip}} = \sqrt{\hbar/m\omega_{\mathrm{dip}}}$, then $l_{\mathrm{dip}} \ll l_{\mathrm{thermal}}$, where $l_{\mathrm{thermal}} = \sqrt{\hbar^2/2mk_{\mathrm{B}}T}$ is the thermal de Broglie wavelength. This means that in the above expression the first (thermal) exponential factor dominates, and we can approximate d_3 by

$$d_3 \approx -\frac{\pi m(4\pi)^2}{16\hbar^3} \int dk_1 k_1^3 z e^{-\beta\lambda} e^{-\beta\hbar^2 k_1^2/2m} = -\frac{\pi^2 m^2}{4\hbar^4} \rho_{\mathrm{E}} v_T \tag{74}$$

where $v_T = \sqrt{2k_{\mathrm{B}}T/m}$ is the thermal velocity of non-condensed particles, and $\rho_{\mathrm{E}} = N_{\mathrm{E}}/V = V^{-1} \sum_{\vec{k}} z e^{-\beta\lambda} e^{-\beta\hbar^2 k_1^2/2m}$ is the density of non-condensed particles in the antisymmetric states.

Given the expression for d_3, we can give an estimate for the decoherence rate. In the $|J,M\rangle$ basis, with $J = N/2$ and $J_z|J,M\rangle = (1/2)(N_A - N_B)|J,M\rangle$, the master equation reads

$$\dot{\rho}(M,M') \approx 16v^2 d_3(M-M')^2 \rho(M,M') = 16\pi^3 \gamma(M-M')^2 \rho(M,M') \tag{75}$$

For the quantum superposition state $|N,0\rangle + |0,N\rangle$, $M - M' = N$, so our final estimate for the decoherence rate is

$$t_{\text{dec}}^{-1} \approx 16\pi^3 \left(4\pi a^2 \frac{N_{\text{E}}}{V} v_T \right) N^2 \tag{76}$$

The only $O(z)$ contribution to the amplitude decoherence can come from a term proportional to d_4. However, under closer inspection d_4 turns out to be $O(z^2)$. d_4 comes from a depletion/growth inelastic process such that a noncondensate particle in the initial state of s_1 collides with a condensate particle and as a result they both end in noncondensate modes s_2 and s_3. For this to happen s_1 must have a sufficiently high energy to overcome a gap between the condensate mode and the environmental modes. Thus in this case $n_{s_1}^O = O(z^2)$ and not $O(z)$.

V. DISCUSSION

The aim of this paper was to discuss the "longevity" of Schrödinger cats in BEC's. We have shown that while in the standard traps decoherence rates are significant enough to prevent long-lived macroscopic superpositions of internal states of the condensate, the strategy of trap engineering and symmetrization of the environment will be able to deal with that issue.

What remains to be considered is how one can generate such macroscopic quantum superposition, and how one can detect it. The issue of generation was already touched upon in Refs. [7,8]. We have little to add to this. However, in the context of the Gordon and Savage proposal, it is fairly clear that the time needed to generate the cat state would have to be short compared to the decoherence time. If our estimates of Eq.(1) are correct, symmetrization procedure appears necessary for the success of such schemes.

Detection of Schrödinger cat states is perhaps a more challenging subject. In principle, states of the form $(|N, 0\rangle + |0, N\rangle)/\sqrt{2}$ have a character of GHZ states, and one could envision performing measurements analogous to those suggested in [23] and carried out in [24], where a 4-atom entangled state was studied. However, this sort of parity-check strategy, appropriate for $N \leq 10$, is likely to fail when N is larger, or when (as would be the case for the "quasi-squeezed" states anticipated here [17]) N is not even well defined.

A strong circumstantial evidence can be nevertheless obtained from two measurements. The first one would consist of a preparation of the cat state, and of a measurement of the internal states of the atoms. It is expected that in each instance all (to within the experimental error) would turn up to be in either A or B states. However, averaged over many runs, the number of either of these two alternatives would be approximately equal. Decoherence in which the environment also "monitors" the internal state of the atoms in the A versus B basis would not influence this prediction. We need to check separately whether the cat state was indeed coherent. To do this, one could evolve the system "backwards". However, this is not really necessary. For, as Gordon and Savage point out, when, in their scheme, we let the system evolve unitarily for more or less twice the time needed for the generation of the cat state, it will approximately return to the initial configuration. Thus, we can acquire strong evidence of the coherence of the cat provided that this unitary return to the initial configuration can be experimentally confirmed.

These are admittedly rather vague ideas, which serve more as a "proof of principle" rather than as a blueprint for an experiment. Nevertheless, they may, we hope, encourage more concrete investigation of such issues with a specific experiment in mind.

VI. ACKNOWLEDGEMENTS

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