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Photonic excess noise and wave localization

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This is a theory for the effect of localization on the super-Poissonian noise of radiation propagating through an absorbing disordered waveguide. Localization suppresses both the mean photon current $\bar{I}$ and the noise power P, but the Fano factor $P/\bar{I}$ is found to remain unaffected. For strong absorption the Fano factor has the universal value $1+\frac{3}{2}f$ (with f the Bose-Einstein function), regardless of whether the waveguide is long or short compared to the localization length.

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The coherent radiation from a laser has Poisson statistics [1,2]. Its noise power $P_{\rm Poisson}$ equals the mean current $\bar{I}$ (in units of photons per second). Elastic scattering has no effect on the noise, because the radiation remains in a coherent state. The coherent state is degraded by absorption, resulting in an excess noise $P-P_{\rm Poisson}>0$ [3]. The Fano factor $P/P_{\rm Poisson}$ deviates from unity by an amount proportional to the Bose-Einstein function f. It is a small effect ($f\sim 10^{-2}$ at room temperature for infrared frequencies), but of interest because of its fundamental origin: The excess noise is required to preserve the canonical commutation relations of the electromagnetic field in an absorbing dielectric [4–6].

The interference of multiply scattered waves may lead to localization [7]. Localization suppresses both the mean current and the fluctuations — on top of the suppression due to absorption. Localization is readily observed in a waveguide geometry [8], where it sets in once the length L of the waveguide becomes longer than the localization length $\xi\simeq Nl$ (with l the mean free path and N the number of propagating modes). Typically, ξ is much larger than the absorption length $\xi_{\rm a}$, so that localization and absorption coexist. The interplay of absorption and localization has been studied previously for the mean current [9–12]. Here we go beyond these studies to include the current fluctuations.

It is instructive to contrast the super-Poissonian photonic noise with the sub-Poissonian electronic analogue. In the case of electrical conduction through a disordered wire, the (zero-temperature) noise power is smaller than the Poisson value as a result of Fermi-Dirac statistics. The reduction is a factor 1/3 in the absence of localization [13,14]. The effect of localization is to restore Poisson statistics, so that the Fano factor increases from 1/3 to 1 when L becomes larger than ξ. What we will show in this paper is that the photonic excess noise responds entirely differently to localization: Although localization suppresses P and $\bar{I}$, the Fano factor remains unaffected, equal to the value $1+\frac{3}{2}f$ obtained in the absence of localization [15,16].

Let us begin our analysis with a more precise formulation of the problem. The noise power

$$P=\int_{-\infty}^{\infty}dt\,\overline{\delta I(0)\delta I(t)} \tag{1}$$

quantifies the size of the time-dependent fluctuations of the photon current $I(t)=\bar{I}+\delta I(t)$. (The bar $\overline{\cdots}$ indicates an average over many measurements on the same system.) For a Poisson process, the power $P_{\rm Poisson}=\bar{I}$ equals the mean current and the Fano factor $\mathcal{F}=P/P_{\rm Poisson}$ equals unity. We consider monochromatic radiation (frequency ω_0) incident in a single mode (labeled m_0) on a waveguide containing a disordered medium (at temperature T). (See Fig. 1.) The incident radiation has Fano factor $\mathcal{F}_{\rm in}$. We wish to know how the Fano factor changes as the radiation propagates through the waveguide.

FIG. 1. Monochromatic radiation (thick arrow) is incident on a disordered absorbing medium (shaded), embedded in a waveguide. The transmitted radiation is measured by a photodetector.

Starting point of our investigation is a formula that relates the Fano factor to the scattering matrix of the medium [15],

$$\mathcal{F}=1+[t^\dagger t]_{m_0m_0}(\mathcal{F}_{\rm in}-1)+2f(\omega_0,T)\frac{[t^\dagger(\mathbb{1}-rr^\dagger-tt^\dagger)t]_{m_0m_0}}{[t^\dagger t]_{m_0m_0}}. \tag{2}$$

(We have assumed detection with quantum efficiency 1 in a narrow frequency interval around ω_0.) The function $f(\omega,T)=[\exp(\hbar\omega/kT)-1]^{-1}$ is the Bose-Einstein function. The transmission matrix t and the reflection matrix r are $N\times N$ matrices, with N the number of propagating modes at frequency ω_0. The term proportional to f in

Eq. (2) is the excess noise. For a unitary scattering matrix, $rr^\dagger + tt^\dagger$ equals the unit matrix $\mathbb{1}$, hence the excess noise vanishes.

In what follows we will assume that the incident radiation is in a coherent state, so that $\mathcal{F}_{\rm in} = 1$ and the deviation of $\mathcal{F}$ from unity is entirely due to the excess noise. Since the Bose-Einstein function at room temperature is negligibly small at optical frequencies, one would need to use the coherent radiation from an infrared or microwave laser. Alternatively, one could use a non-coherent source and extract the excess noise contribution by subtracting the noise at low temperature from that at room temperature.

The absorbing disordered waveguide is characterized by four length scales: the wavelength λ, the mean free path for scattering l, the absorption length $\xi_{\rm a}$, and the localization length $\xi = (N+1)l$. We assume the ordering of length scales $\lambda \ll l \ll \xi_{\rm a} \ll \xi$, which is the usual situation [8]. We ask for the average $\langle\mathcal{F}\rangle$ of the Fano factor, averaged over an ensemble of waveguides with different realizations of the disorder. For $L \gg \xi_{\rm a}$ we may neglect the matrix $tt^\dagger$ with respect to $\mathbb{1}$ in Eq. (2), so that the expression for the Fano factor (with $\mathcal{F}_{\rm in} = 1$) takes the form

$$\mathcal{F} = 1 + 2f(1 - C_1), \quad C_p \equiv \frac{[t^\dagger (rr^\dagger)^p t]_{m_0 m_0}}{[t^\dagger t]_{m_0 m_0}}. \qquad (3)$$

In the absence of localization, for $L \ll \xi$, one can simplify the calculation of $\langle\mathcal{F}\rangle$ by averaging separately the numerator and denominator in the coefficient C_1, since the sample-to-sample fluctuations are small. This diffusive regime was studied in Refs. [15,16]. Such a simplification is no longer possible in the localized regime and we should proceed differently.

We follow the general approach of Ref. [12], by considering the change in $\mathcal{F}$ upon attaching a short segment of length δL to one end of the waveguide. Transmission and reflection matrices are changed to leading order in δL according to

$$t \to t_{\delta L}(1 + r r_{\delta L})t, \quad r \to r'_{\delta L} + t_{\delta L}(1 + r r_{\delta L}) r t^{\rm T}_{\delta L}, \qquad (4)$$

where the superscript T indicates the transpose of a matrix. (Because of reciprocity the transmission matrix from left to right equals the transpose of the transmission matrix from right to left.) The transmission and reflection matrices $t_{\delta L}$, $r_{\delta L}$, $r'_{\delta L}$ of the short segment have zero mean and variances

$$\begin{aligned}\langle [r_{\delta L}]_{kl}[r_{\delta L}]^*_{mn}\rangle &= \langle [r'_{\delta L}]_{kl}[r'_{\delta L}]^*_{mn}\rangle \\ &= (\delta_{km}\delta_{ln} + \delta_{kn}\delta_{lm})\delta L/\xi, \qquad (5a)\\ \langle [t_{\delta L}]_{kl}[t_{\delta L}]^*_{mn}\rangle &= N^{-1}\delta_{km}\delta_{ln}(1 - \delta L/l - \delta L/l_{\rm a}), \qquad (5b)\end{aligned}$$

where $l_{\rm a} = 2\xi_{\rm a}^2/l$ is the ballistic absorption length. All covariances vanish. Substituting Eq. (4) into Eq. (3) and averaging we find the evolution equation

$$\begin{aligned}\xi\frac{d\langle C_1\rangle}{dL} &= -2\langle C_1\rho_1\rangle + \langle\rho_2\rangle \\ &\quad - \frac{\xi l}{\xi_{\rm a}^2}\langle C_1\rangle + 1 + 2\langle C_2 - C_1\rangle - \langle C_1^2\rangle \\ &\quad - 4\,{\rm Re}\,\big\langle [t^\dagger t]^{-2}_{m_0m_0}[t^\dagger r t^*]_{m_0m_0}[t^{\rm T} r^\dagger r r^\dagger t]_{m_0m_0}\big\rangle \\ &\quad + 2\big\langle (1 + C_1)[t^\dagger t]^{-2}_{m_0m_0}\big|[t^\dagger r t^*]_{m_0m_0}\big|^2\big\rangle, \qquad (6)\end{aligned}$$

where we have defined $\rho_p = {\rm tr}\,(1 - rr^\dagger)^p$.

For $L \gg \xi_{\rm a}$ we may replace the average of the product $\langle C_1\rho_1\rangle$ by the product of averages $\langle C_1\rangle\langle\rho_1\rangle$, because [12] statistical correlations with traces that involve reflection matrices only are of relative order $\xi_{\rm a}/\xi$ — which we have assumed to be $\ll 1$. The moments of the reflection matrix are given for $L \gg \xi_{\rm a}$ by [17]

$$\langle\rho_p\rangle = \frac{\Gamma(p - 1/2)}{\sqrt{\pi}\,\Gamma(p)}\,\frac{\xi}{\xi_{\rm a}}, \qquad (7)$$

hence they are $\gg 1$ and also $\gg \xi l/\xi_{\rm a}^2$. We may therefore neglect the terms in the second, third, and fourth line of Eq. (6). What remains is the differential equation

$$\xi\frac{d\langle C_1\rangle}{dL} = -2\langle C_1\rangle\langle\rho_1\rangle + \langle\rho_2\rangle, \qquad (8)$$

which for $L \gg \xi_{\rm a}$ has the solution

$$\langle C_1\rangle = \frac{\langle\rho_2\rangle}{2\langle\rho_1\rangle} = \frac{1}{4}. \qquad (9)$$

We conclude that the average Fano factor $\langle\mathcal{F}\rangle = 1 + 2f(1 - \langle C_1\rangle) \to 1 + \frac{3}{2}f$ for $L \gg \xi_{\rm a}$, regardless of whether L is small or large compared to ξ.

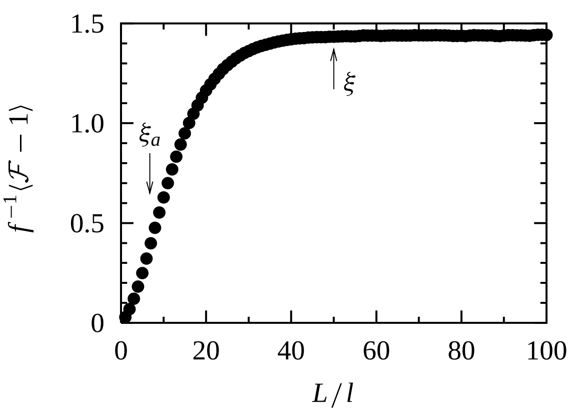

FIG. 2. Length dependence of the average Fano factor, computed from Eq. (2) with $\mathcal{F}_{\rm in} = 1$. The data points result from a numerical simulation for an absorbing disordered waveguide with $N = 50$ propagating modes. Arrows indicate the absorption length $\xi_{\rm a}$ and the localization length ξ. The average Fano factor is not affected by localization.

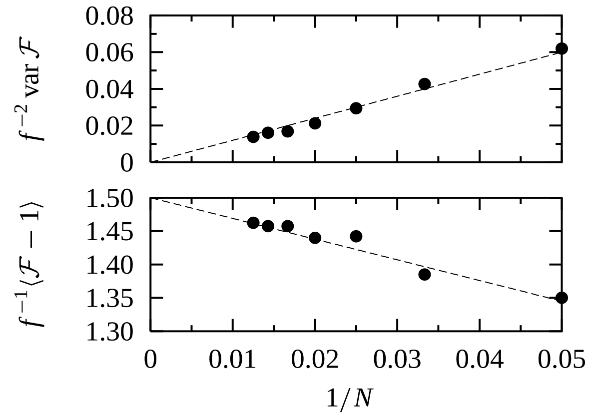

FIG. 3. Dependence of the average and variance of the Fano factor on the number N of propagating modes, for fixed length $L = 260\,l = 38.5\,\xi_{\rm a}$ of the waveguide. The length is larger than the localization length $\xi = (N+1)l$ for all data points. The dashed lines extrapolate to the theoretical expectation for $1/N \to 0$.

To support this analytical calculation we have carried out numerical simulations. The absorbing disordered waveguide is modeled by a two-dimensional square lattice (lattice constant a). The dielectric constant ε has a real part that fluctuates from site to site and a non-fluctuating imaginary part. The multiple scattering of a scalar wave Ψ is described by discretizing the Helmholtz equation $[\nabla^2 + (\omega_0/c)^2\varepsilon]\Psi = 0$ and computing the transmission and reflection matrices using the recursive Green function technique [18]. The mean free path $l = 20\,a$ and the absorption length $\xi_{\rm a} = 135\,a$ are determined from the average transmission probability $N^{-1}\langle{\rm tr}\,tt^\dagger\rangle = l/\xi_{\rm a}\sinh(L/\xi_{\rm a})$ in the diffusive regime [12]. Averages were performed over the $N/2$ modes m_0 near normal incidence and over some $10^2 - 10^3$ realizations of the disorder. Results are shown in Figs. 2 and 3.

The length dependence of the average Fano factor is plotted in Fig. 2, for $N = 50$ and L ranging from 0 to 2ξ. Clearly, localization has no effect. The limiting value of $f^{-1}\langle\mathcal{F}-1\rangle$ resulting from this simulation is slightly smaller than the value 3/2 predicted by the analytical theory for $N \gg 1$. The N-dependence of $\langle\mathcal{F}\rangle$ in the localized regime is shown in Fig. 3. A line through the data points extrapolates to the theoretical expectation $f^{-1}\langle\mathcal{F}-1\rangle \to 3/2$ for $N \to \infty$. Fig. 3 also shows the variance of the Fano factor. The variance extrapolates to 0 for $N \to \infty$, indicating that $\mathcal{F} = P/\bar{I}$ becomes self-averaging for large N. This is in contrast to P and $\bar{I}$ themselves, which fluctuate strongly in the localized regime.

In conclusion, we have demonstrated that localization of radiation in an absorbing disordered waveguide has no effect on the ratio of the excess noise and the mean current. In the limit of a large number of propagating modes, this ratio is self-averaging and takes on the universal value of 3/2 times the Bose-Einstein function. Observation of this photonic analogue of the universal 1/3 reduction of electronic shot noise presents an experimental challenge.

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