Limits on Preserving Quantum Coherence using Multi-Pulse Control

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We explore the physical limits of pulsed dynamical decoupling methods for decoherence control as determined by finite timing resources. By focusing on a decohering qubit controlled by arbitrary sequences of ideal instantaneous pulses, we establish non-perturbative quantitative upper bounds to the achievable coherence for specified maximum pulsing rate and spectral bandwidth, and introduce numerically optimized control sequences that saturate the performance bound subject to these constraints. As a byproduct, our analysis rigorously rules out the existence of fault-tolerance thresholds for purely open-loop unitary control of generic open quantum systems.

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Building on the discovery of spin-echo and multiple-pulse techniques in nuclear magnetic resonance [1], dynamical decoupling (DD) methods for open quantum systems [2] have become a versatile tool for decoherence control in quantum engineering and fault-tolerant quantum computation. DD involves "open loop" (feedback-free) quantum control based on the application of a time-dependent Hamiltonian which, in the simplest setting, effects a pre-determined sequence of unitary operations (pulses) drawn from a basic repertoire. Physically, DD relies on the ability to access control time scales that are short relative to the correlation time scale of the interaction to be removed. The reduction in decoherence is achieved *perturbatively*, by ensuring that sufficiently high orders of the error-inducing Hamiltonian are removed. Recently, a number of increasingly powerful pulsed DD schemes have been proposed and validated in the laboratory. Uhrig DD (UDD) sequences [3], for instance, perturbatively cancel pure dephasing in a single qubit up to an arbitrarily high order n while using a minimal number (n) of pulses, paving the way to further optimization for given sequence duration [4, 5] and/or specific noise environments [6], to nearly-optimal protocols for generic single-qubit decoherence [7]. Experimentally, UDD has been employed to prolong coherence time in systems ranging from trapped ions [4, 5] and atomic ensembles [8] to spin-based devices [9], and to enhance contrast in magnetic resonance imaging of tissue [10].

In a realistic DD setting, the achievable performance is inevitably influenced by errors both due to limited control (resulting in a variety of pulse imperfections) as well as deviations from the intended decoherence model. Some of these limitations may be regarded as non-fundamental in nature: for instance, it is conceivable that both model uncertainty and pulse non-idealities be largely removed by more accurate system identification and control design. We argue, however, that even in the most favorable DD scenario, an intrinsic constraint is implied by the fact that the rate at which control operations may be effected is necessarily finite – as resulting from a "minimum switching time" $\tau_{\min}$ for the available control modulation. Our goal in what follows is to rigorously quantify the ultimate performance limits to preserving quantum coherence using DD as arising from the sole constraint of *finite timing resources*.

We focus on the paradigmatic case of a single qubit undergoing pure dephasing due to either a quantum bosonic bath at equilibrium or classical (Gaussian) noise, and controlled through a sequence of instantaneous π pulses. While representing an adequate idealization of realistic decoherence control settings (including [4, 5, 8, 10]), such a scenario has the advantage of being exactly solvable analytically [2, 3], enabling rigorous conclusions to be established. Our main result is a *non-perturbative* lower bound for the minimum decoherence error achievable by *any* DD sequence subject to a timing constraint $\tau_{\min}$, for noise spectra characterized by a spectral bandwidth ω_c. We argue that UDD sequences saturate the bound in functional form but not in absolute sense, and show how to generate "bandwidth-adapted" DD sequences that achieve optimum performance over a desired storage time while respecting the pulse-rate constraint. Conceptually, our analysis highlights connections between DD theory and complex analysis of polynomials, and provides further insight into the fundamental capabilities and limitations of open-loop non-dissipative quantum control.

Control setting.— Our target system is a single qubit whose dephasing dynamics in the quantum regime is described by a diagonal spin-boson Hamiltonian of the form $H = H_S \otimes I_B + H_{SB} + I_S \otimes H_B$, with $H_S = \omega S_z$ and

$$H_{SB} = S_z \otimes \sum_k (g_k b_k + g_k^* b_k^\dagger), \quad H_B = \sum_k \omega_k b_k^\dagger b_k.$$

Here, $I_{S(B)}$ denote the identity operator acting on the system (bath), $S_z = \hbar\sigma_z/2$ is the spin operator along the quantization axis, whereas b_k ($b_k^\dagger$) are canonical annihilation (creation) operators for the kth bosonic mode, characterized by a frequency ω_k and coupling strength g_k. Under the assumption that the bath is (initially) at thermal equilibrium at temperature $1/(k_B\beta)$, its influence on the qubit dynamics is encapsulated by the spectral density function $I(\omega) \equiv \sum_k |g_k|^2\delta(\omega-\omega_k)$. Typically, in the continuum limit, $I(\omega)$ has a power-law behavior in the infrared, $I(\omega) \sim \alpha\omega^s$ as $\omega \to 0$, and decays to zero sufficiently fast beyond an ultraviolet cutoff ω_c [11].

DD over an evolution interval $[0, T]$ is achieved by applying a train of n instantaneous π pulses (each implementing a Pauli σ_x operator) at times $\{t_j\}$, where $0 < t_1 < \ldots < t_n < T$, and we also let $t_0 \equiv 0$ and $t_{n+1} \equiv T$. While keeping the number of pulses n to a minimum may be desirable for various practical reasons, neither n nor the resulting sequence duration need to be constrained *a priori*. An arbitrary long duration T may, in fact, be needed for quantum memory. In contrast, infinite pulse rates are fundamentally impossible, due to the existence, in general, of a *finite minimum switching time* $\tau_{\min} > 0$ that lower-bounds the smallest control time scale achievable by any sequence:

$$\tau \equiv \min_{j\in\{0,\ldots,n\}} (t_{j+1} - t_j) \geq \tau_{\min}. \quad (1)$$

If the system is initially in a nontrivial coherent superposition of S_z eigenstates, its purity in the presence of DD will decay with a factor of $\exp(-2\chi_{\{t_j\}})$ [2, 3], where

$$\chi_{\{t_j\}} = \int_0^\infty \lambda(\omega)\,|f_{\{\tilde{t}_j\}}(\omega)|^2 d\omega, \quad \tilde{t}_j \equiv \frac{t_j}{\tau}, \quad (2)$$

$$f_{\{\tilde{t}_j\}}(\omega) = \sum_{j=0}^{n} (-1)^j (e^{i\tilde{t}_j\omega\tau} - e^{i\tilde{t}_{j+1}\omega\tau}), \quad (3)$$

and the "spectral measure" $\lambda(\omega) \equiv 2\coth(\beta\omega/2)I(\omega)/\omega^2$. Notice that in terms of the rescaled pulse times $\tilde{t}_j$, the minimum switching time requirement of Eq. (1) reduces to $\tilde{t}_{j+1} - \tilde{t}_j \geq 1$ [12]. Physically, Eqs. (2)-(3) also describe the purity decay resulting from pure dephasing dynamics in the (semi)classical limit, as due to stochastic fluctuations of the qubit energy splitting and experimentally investigated in [4, 5, 8]. In this case, $H_{SB} \equiv 0$ and $H_S = [\omega + \xi(t)]S_z$, where $\xi(t)$ is a Gaussian random variable with a power spectrum $S(\omega)$ [13]. In order to evaluate $\chi_{\{t_j\}}$, it suffices to redefine the spectral measure as $\lambda(\omega) = S(\omega)/2\pi\omega^2$. We shall refer to the central quantity $\chi_{\{t_j\}}$ as the *decoupling error*. The objective of DD is to minimize $\chi_{\{t_j\}}$. Our main problem then directly ties to the following: Given the fundamental constraint of Eq. (1), what is a lower bound on $\chi_{\{t_j\}}$?

Non-perturbative performance bound.— A lower bound on $\chi_{\{t_j\}}$ can be obtained by restricting the integral in Eq. (2) to a finite range $[0, \omega_c]$, with a tight bound ensuing if ω_c coincides with the spectral cutoff in either $I(\omega)$ or $S(\omega)$. We separate the dependencies of $\chi_{\{t_j\}}$ upon the timings and the spectral measure $\lambda(\omega)$ by applying Cauchy's inequality to the functions $\lambda^{1/2}|f|$ and $\lambda^{-1/2}$:

$$\chi_{\{t_j\}} \geq \frac{1}{M_{\{\lambda\}}}\Big(\int_0^{\omega_c} |f_{\{\tilde{t}_j\}}(\omega)|d\omega\Big)^2, \; M_{\{\lambda\}} \equiv \int_0^{\omega_c} \frac{d\omega}{\lambda(\omega)}. \quad (4)$$

Thus, the integral $\int_0^{\omega_c} |f_{\{\tilde{t}_j\}}(\omega)|d\omega$, which is the L_1-norm of the "filter function" $f_{\{\tilde{t}_j\}}$ over $[0, \omega_c]$, determines a worst-case lower bound on $\chi_{\{t_j\}}$ for *all* spectral densities $\lambda(\omega)$ for which the integral defining $M_{\{\lambda\}}$ is finite.

Interestingly, upon letting $e^{i\omega\tau} \equiv z \in \mathbb{C}$ in Eq. (3), the function $f_{\{\tilde{t}_j\}}(\omega)$ takes the form of a complex "polynomial" $P_{\{\tilde{t}_j\}}(z)$ with non-integer exponents. Such *Müntz polynomials* have been studied in the mathematical literature, and a plethora of results (and conjectures) exist on their associated norm inequalities, zeroes, and multiplicities [14]. The (now resolved) Littlewood conjecture in harmonic analysis [15] may be invoked, in particular, to lower-bound the L_1-norm of $f_{\{\tilde{t}_j\}}$:

$$\chi_{\{t_j\}} \geq \frac{\omega_c^2}{4\pi^2 M_{\{\lambda\}}} C(\log n)^2, \quad \text{if } \omega_c\tau > 2\pi, \quad (5)$$

with $C = O(1)$. Also note that, regardless of $\omega_c\tau$, an upper bound follows immediately from Eq. (2): $\chi_{\{t_j\}} \leq m_{\{\lambda\}} n^2$, where $m_{\{\lambda\}} \equiv \int_0^\infty \lambda(\omega)d\omega$. Eq. (5) implies that in the 'slow-control' regime where $\omega_c\tau > 2\pi$, the DD error worsens when more pulses are applied, and coherence may be best preserved by doing nothing. This reinforces how sufficiently fast modulation time scales are essential for achieving decoherence reduction, as we discuss next.

The 'fast-control' regime ($\omega_c\tau < 2\pi$) is implicit in perturbative DD treatments, where the filter function $f_{\{\tilde{t}_j\}}(\omega)$ is chosen to have a Taylor series that starts at $(\omega\tau)^m$, so that $\chi_{\{t_j\}}$ remains small for sufficiently small values of $\omega_c\tau$. While this perturbative approach has been used for designing efficient DD schemes, it cannot lead to a lower bound on the attainable DD error in the presence of a timing constraint. Consider for example UDD$_n$ sequences, in which case $t_j = T\sin^2[\pi j/(2n+2)]$ for $j = 1, \cdots, n$, and $\tau \equiv t_1$. If τ is kept fixed, increasing n is only possible at the expense of lengthening the total duration as $T(n) = \mathcal{O}[\tau n^2]$. Irrespective of the fact that perturbatively the error scales as $\mathcal{O}[(\omega_c\tau)^n]$ it carries a prefactor that grows too fast with n, eventually causing the perturbative description to break down [16, 17]. While not accessible from perturbation theory, a zero lower bound on the DD error would imply that even with a *fixed* $\tau_{\min}$, arbitrarily high DD accuracy would be achievable by using a sufficiently long sequence. In analogy with the accuracy threshold theorem in fault-tolerant quantum computation [18] such a zero lower bound could thus signify a threshold-like behavior for DD.

Despite the analogy, the lower bound on $\chi_{\{t_j\}}$ is strictly positive for spectral measures of compact support. The L_1-norm integral of $f_{\{\tilde{t}_j\}}$ can be mapped to the size of the corresponding Müntz polynomial $P_{\{\tilde{t}_j\}}(z)$ over an arc of the unit circle of length $\omega_c\tau$. Theorem 2.2 in [19], in conjunction with Eq. (4), then implies:

$$\chi_{\{t_j\}} \geq \frac{1}{M_{\{\lambda\}}\tau^2}\, c e^{-a/(\omega_c\tau)}, \quad \text{if } \omega_c\tau < 2\pi, \quad (6)$$

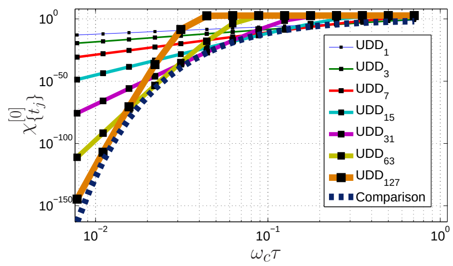

FIG. 1: (Color online) Decoupling error for UDD_n sequences vs. $\omega_c\tau$, for a "flat" spectral measure $\lambda^{[0]}(\omega) \equiv \Theta(\omega - \omega_c)$. The comparison curve denotes the general lower bound, Eq. (6), evaluated for $a = 3$, $c = 1/2$, chosen to approximate a fit.

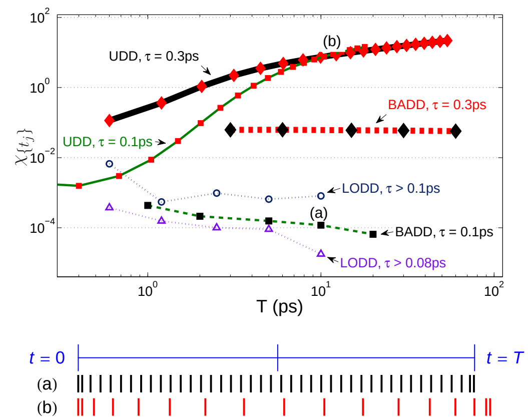

FIG. 2: (Color online) Decoupling error for BADD (dashed), LODD (dotted), and UDD (solid) sequences vs. total duration T with the minimum interval τ's indicated, for a dephasing exciton qubit operating at temperature 77K (see text). The search space for BADD and LODD covers up to $n = 100$ pulses, whereas for UDD $n \leq 20$. A comparison between the pulse timing patterns for the BADD and UDD sequences corresponding to $T \approx 10$, $\tau = 0.1$ps (points (a) and (b) in the main graph) is also shown. Notice how the intervals of BADD sequence (a) are compressed at the endpoints, but become effectively uniform mid-sequence. This resembles the (analytical) interpolated DD protocol identified in Ref. 17.

where c and a are numeric constants independent of τ, ω_c, and $\{t_j\}$. That the bound in Eq. (6) *cannot* be obtained by perturbative methods is manifest from the fact that it describes an essential singularity in $\omega_c\tau$.

Achieving the performance bound.— The minimum switching time $\tau_{\min}$ enters the above lower bound naturally, whereas both the total duration T and pulse number n are markedly absent from it. The fact that the strength of DD is ultimately gauged upon $\tau_{\min}$ and not T identifies the control rate as the key resource that DD leverages for dynamically removing errors: *if* the bound can be achieved, it should be possible to do so irrespective of how long T, provided that n is uncontrained. Interestingly, the error associated with UDD sequences, χ_n^{UDD}, can saturate the fundamental limit in Eq. (6) in functional form although *not* necessarily in *absolute* sense (see also Fig. 1). This follows from noting that an upper bound to χ_n^{UDD} in the presence of a hard spectral cutoff may be obtained from an upper bound to $|f_n^{\mathrm{UDD}}(\omega)|$, by tailoring n to the bandwidth, $n \equiv n_0 \approx 1/(e^2\omega_c\tau)$ (see Remark 2.6 in [19]). This yields:

$$\chi_n^{\mathrm{UDD}} \leq m_{\{\lambda\}} \cdot \max_{\omega\in[0,\omega_c]} |f_{n_0}^{\mathrm{UDD}}(\omega)|^2 \leq \frac{m_{\{\lambda\}}}{\omega_c\tau} c' e^{-a'/(\omega_c\tau)},$$

where $c' = 2/(\pi e^2)$, $a' = 2/e^2$, and a similar functional form as in Eq. (6) is manifest. With $\tau \equiv t_1 \equiv \tau_{\min}$ fixed, the duration T of the "tailored UDD_n" sequences scales as $\mathcal{O}[1/(\omega_c^2\tau_{\min})]$, and the longest allowed τ-value that results in coherence improvement scales as $1/(n\omega_c)$. Thus, UDD provides no guarantee that the error reaches its absolute minimum and accessing the required τ becomes increasingly harder as T grows. This motivates searching for DD sequences that can operate *beyond* the perturbative regime and retain their efficacy over the broadest range possible, up to $1/(n\omega_c) \lesssim \tau \lesssim 1/\omega_c$.

Various optimized DD strategies have been recently investigated for the qubit-dephasing setting under consideration. In "locally optimized" (LO) DD [4], in particular, optimal pulse timings are determined via direct minimization of the error $\chi_{\{t_j\}}$ for a fixed target storage time T, whereas in "optimized noise filtration" (OF) DD only the integral of the filter function is minimized [5] (see also [6] for a noise-adapted perturbative approach). While such optimized pulse sequences can access regimes where perturbative DD approaches are not efficient, they focus on matching the total sequence duration T as the fundamental constraint. However, this may fail to produce a satisfactory control solution if the timing constraint imposed by Eq. (1) is significant.

In order to guarantee that such a fundamental limitation is obeyed, we introduce optimized *bandwidth-adapted DD* (BADD) sequences where *both* the minimum switching time *and* the total time are constrained from the outset. Enforcing the additional timing constraint does not only bring the advantage of a compact space for numerical search, but it may also non-trivially alter the optimization process, resulting in different control solutions. We demonstrate the usefulness of BADD by focusing on the exciton qubit analyzed in Ref. 17, for which a spin-boson dephasing model with a supra-Ohmic spectral density is appropriate, $I^{\omega_c,s}(\omega) = \alpha\omega^s \exp(-\omega^2/\omega_c^2)$ with $s = 3$, $\alpha \approx 1.14 \times 10^{-26}\mathrm{s}^2$, $\omega_c \approx 3\,\mathrm{rad\,ps}^{-1}$, and the need to avoid unwanted excitation of higher-energy levels enforces a timing constraint $\tau_{\min} \approx 0.1$ ps. The results are summarized in Fig. 2. Beside indicating the inadequacy of perturbative UDD for $T \gtrsim 1$ ps, two main features emerge in comparing numerically optimized sequences. First, as predicted by Eq. (6), the minimum error achievable by BADD_n is mainly dictated by τ, largely independently of the total time T. Second, LODD performance

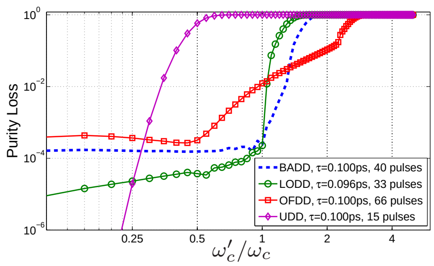

FIG. 3: (Color online) Purity loss, $1-e^{-2\chi_{\{t_j\}}}$, vs. actual over presumed cutoff, ω_c'/ω_c, for the supra-Ohmic spectral density ($s = 3$) corresponding to the exciton qubit. All sequences are adapted to $T \approx 10$ps, $\tau \approx 0.1$ps. Varying the "actual" power law of the noise to $s = 4$ and $s = 2$ resulted in a qualitatively similar behavior (data not shown).

is fairly sensitive to the timing constraint: for a fixed T (10 ps in Fig. 2), 'softening' the constraint selects LODD sequences that outperform BADD, the opposite behavior being seen if the tolerance on the intended τ is decreased. Thus, a BADD protocol effectively optimizes over a set of LODD sequences where the timing constraint is only approximately met, consistent with intuition.

In general, a tradeoff may be expected between the peak performance of a DD scheme and its robustness against uncertainty on the underlying spectral measure. Thus, sequences adapted to a presumed $\omega_c\tau_{\rm min}$ need not be adequate for the actual $\omega_c'\tau_{\rm min}$. Some illustrative results are depicted in Fig. 3 for sequences subject to the same timing constraint, but applied to a setting where $\omega_c' \neq \omega_c$. Clearly, a smaller cutoff ω_c' leads to smaller decoherence, but in a more pronounced way for perturbative UDD sequences. Expectedly, the knowledge of the spectral density explicitly assumed in generating BADD and LODD results in far better coherence compared to OFDD and UDD, especially when this knowledge is precise ($\omega_c'/\omega_c = 1$) or overestimates the cutoff. Comparatively, BADD sequences appear to be more robust than LODD sequences when the cutoff is underestimated.

Discussion.— Our mathematical description has relied on the solvability of the dephasing spin-boson model in the limit of instantaneous control pulses, however we expect similar timing-induced lower bounds to exist under more general conditions. In principle, non-Gaussian classical dephasing such as random telegraph noise could be addressed based on the exact solution presented in [20], whereas non-bosonic dephasing models of the form $H_{SB}+H_B \equiv S_z \otimes B_z + B_0$, could be tackled by matching the leading-order contributions in B_z with the bosonic case studied here. Note, however, that bounded timing resources do *not* prevent the DD accuracy bound to be zero in *special* cases – such as "monochromatic" or "non-dynamical" baths ($H_B = 0$), for both of which the length of the arc appearing in Eq. (6) vanishes. Similarly, "nilpotent" environments, where powers of the bath operators in H_{SB} and H_B vanish at some order, allow perturbative DD schemes to achieve perfect decoupling, as perturbation theory becomes exact. For more "adversarial" environments, where H_{SB} is not restricted to but includes single-axis decoherence, similar lower bounds must exist by inclusion. Elucidating the algebraic features responsible for a finite vs. zero performance bound remains an interesting open problem with implications for quantum error correction in general. As opposed to pulsed control scenarios, continuous-time modulation subject to finite energy/bandwidth constraints has also been explored for decoherence control [21]. Although, even for a purely dephasing qubit, finding the optimal modulation requires solving a non-linear integro-differential equation, it would be interesting to quantify the extent to which the extra freedom afforded by continuous controls may improve the achievable performance lower bounds.

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