New measures for entangled states

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Abstract

We propose new measures that distinguish and classify entangled states. The measures are algebraic invariants of linear maps associated with the states. Considering qubits as well as higher spin systems, we obtained complete entanglement classifications for cases that were either unsolved or only conjectured in the literature.

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I. INTRODUCTION

Ever since the formulation of the EPR paradox [1], the phenomenon of quantum entanglement has generated an ever increasing stream of research. Although applications of entangled states are already advanced and extensive, fundamental understanding of the phenomenon is nevertheless still incomplete. Such an understanding requires the description of qualitative features of entanglement (which are well-known and simple to formulate) and its quantitative features (which are very complex and not completely understood). The latter are usually described in terms of entanglement measures that can distinguish entangled states, and finding these is very difficult. Since distinction implies classification, this leads to the problem of entanglement classification, which is a prominent unsolved problem in quantum information theory.

Entanglement has been studied mostly for subsystems of spin one half. Quantum computation favors this case since a qubit with its two degrees of freedom represents the fundamental unit of quantum information. However, as we do not yet know on which quantum systems large practical quantum computers will ultimately be based, higher spin quantum states should be studied as well. Such a study would be much advanced by understanding the classification of entangled states for the general case of subsystems of arbitrary spins.

The classical theory of invariants [2] provides the standard method of finding entanglement measures. Variants of this method are used in most known cases of partial or complete classification; see, for example, [3–16]. We propose a new method of entanglement classification that uses only basic linear algebra [17] and whose measures are discrete algebraic invariants complementing the known continuous measures.

The following brief review of key properties of entangled states motivates the use of algebraic invariants as new entanglement measures. In the main text, we fully develop these ideas and apply them to several cases of three entangled higher spin systems, only some of which are known in the literature.

We first observe that the phenomenon of entanglement is a consequence of the superposition principle and the tensor product postulate. By the superposition principle, the state space of a physical system is a vector space, so that any linear combination of state vectors is such as well. By the tensor product postulate, the state space of a system consisting of several subsystems is a subspace of the tensor product of the state spaces of all the subsystems. The principle and postulate together imply that a state vector of the system is a linear combination of tensor products of state vectors of the subsystems.

Consequently, the information provided by a given state of the system can be divided into two parts: (1) a list of contributing states of the subsystems and (2) the manner in which these are combined. The former and the latter describe macro and micro properties of the states, respectively, and entanglement characterizes the latter. For given states of the subsystems (the first part is fixed), there are various ways in which linear combinations of their tensor products can be formed (the second part varies). In the simplest case, only one state from each subsystem contributes, and there is only one term in the linear combination, which results in no entanglement. As numbers of contributing states from each subsystem increase, new ways to form linear combinations become available. If new added states are linearly independent from those already included, this results in states that have, in general, higher degrees of entanglement. The process rapidly becomes complicated for a large number of subsystems because of the combinatorial nature of the procedure. Exploring all resulting possibilities and partitioning states into corresponding classes formed by related states is the goal of entanglement classification.

We now proceed with explicit details of our method and demonstrate its use with several examples. A more general, comprehensive, and detailed discussion of the method—including a theorem on a correspondence between entanglement classes and algebraic invariants—and its application to more complicated examples can be found in [18].

II. METHOD

Let S be a system that consists of subsystems $S_1, \ldots, S_n$, and assume that state spaces of $S, S_1, \ldots, S_n$ are finite dimensional vector spaces $V, V_1, \ldots, V_n$, respectively. (We consider vector spaces over either $\mathbb{R}$ or $\mathbb{C}$; for either choice, all our results are the same.) The space V is a subspace of $V_1 \otimes \cdots \otimes V_n$, where a specific choice of V is determined by the nature of the subsystems. In a particularly important case of identical subsystems, V is determined by a permutation symmetry acting on the subsystems. We consider here only the simplest case where $V = V_1 \otimes \cdots \otimes V_n$.

Entanglement properties of $v \in V$ are determined by specific ways in which v is formed from elements of $V_1, \ldots, V_n$. From this point of view, the simplest elements of V are decomposable vectors. Any decomposable vector $v \in V$ can be written in the factorizable form $v = v_1 \otimes \cdots \otimes v_n$, where $v_i \in V_i$, and an important property of such a vector v is that each subsystem S_i is in a definite state v_i. Although decomposable vectors comprise only a small part of V, they span all of it. This simple property of tensor products leads to remarkable complications and plays the central role in our classification of entangled states.

Nondecomposable vectors are vectors that cannot be written in the factorizable form, and for such states, we cannot say in which state each subsystem is. The simplest example of a nondecomposable vector in V is $v + v'$, where $v = v_1 \otimes \cdots \otimes v_n$, $v' = v'_1 \otimes \cdots \otimes v'_n$ and $v_i \in V_i$, $v'_i \in V_i$ are such that there are at least two linearly independent pairs of vectors among the pairs $\{(v_i, v'_i)\}$ for each i. (The EPR state for $n = 2$ and the GHZ state [19] for $n = 3$ are such examples.) It seems plausible (and will be proved later) that linear combinations with a larger number of terms and a smaller number of linear relations among vectors in each tensor product represent states with larger degrees of entanglement. To define degrees of entanglement, we proceed as follows.

We first note that a degrees of entanglement is an algebraic invariant: a quantity that depends only on properties of spaces and does not depend on properties of individual vectors. Here such invariants can appear only as dimensions of linear subspaces of V, and only subspaces linearly depending on v can participate in classification of entangled states. To find the required subspaces, we note that any linear subspace can be defined using an appropriate linear map. Specifically, vector spaces W and W' together with a linear map $f : W \to W'$ define two associated fundamental subspaces: the kernel and image of the map,

$$\begin{aligned}\ker f &= \{w \in W \colon f(w) = 0\} \subset W,\\ \operatorname{im} f &= \{w' \in W' \colon w' = f(w), w \in W\} \subset W'.\end{aligned}$$

Introducing the transpose map $f' \colon W' \to W$, we find that the kernels and images of the maps f and f' are related. In particular, the dimensions of their kernels satisfy [17]

$$\dim W - \dim \ker f = \dim W' - \dim \ker f'.$$

Thus, if both maps are used to classify subspaces, then it suffices to consider only their kernels, for example.

Since we seek a map $f(v)$ that is linear in v, we have to choose

$$f(v)\colon W \to W', \quad f(v)(w) = v \otimes w^*,$$

where W, W' are such that $W \otimes W' = V$ and w^* is the dual of w. The kernel $K(v) = \ker f(v)$ and the invariant $k(v) = \dim K(v)$ describe an entanglement property of v associated with a particular choice of (W, W'). For $w \in K(v)$, the equation $v \otimes w^* = 0$ implies the general form of v,

$$v = \sum_{i=1}^{\dim W - k(v)} w_i \otimes w'_i, \quad \{w_i\} \subset W, \quad \{w'_i\} \subset W',$$

where

$$\dim \operatorname{span}(\{w_i\}) = \dim \operatorname{span}(\{w'_i\}) = \dim W - k(v),$$

and the dimension of the span of a set of vectors is the number of its linearly independent elements. The general form of v is a convenient computational tool that allows us to group states with identical entanglement properties.

To complete the formulation of the method, we need to explore all combinatorial possibilities present in the problem. To this end, we choose all possible pairs of spaces (W, W') such that $W \otimes W' = V$, and for each such choice we find the corresponding map $f(v)$, the kernel $K(v)$, and the invariant $k(v)$ for each $v \in V$. The resulting set of the kernels $\{K(v)\}$ determines the entanglement class of v. Note, however, that the set of the invariants $\{k(v)\}$ does not suffice to specify the class uniquely, but the set of dimensions of all possible intersections of elements of $\{K(v)\}$ does. Now choosing the smallest subset of independent invariants among such a set, we arrive at the complete set of degrees of entanglement of elements of V. Finally, by examining possible values of the invariants, we find the set of entanglement classes of V.

III. EXAMPLES

Our classification method works for arbitrary finite n and $D = (\dim V_i)_{1 \le i \le n}$. As illustrative examples, we consider entanglement for the case $n = 2$ for arbitrary $D = (d_1, d_2)$ and the case $n = 3$ for $D = (2, 2, d)$ and $D = (2, 3, d)$, where d is arbitrary and corresponds to spin $\frac{1}{2}(d-1)$.

TABLE I: Maps and associated quantities for $n = 2$, $D = (d_1, d_2)$.

$f(v)$	W	W'	$K(v)$	$k(v)$
$f_1(v)$	V_1	V_2	$K_1(v)$	$k_1(v) \in \{d_1 - m, \ldots, d_1\}$
$f_2(v)$	V_2	V_1	$K_2(v)$	$k_2(v) \in \{d_2 - m, \ldots, d_2\}$

We first consider the case $n = 2$. Maps and associated quantities are given in Table I, where we use the notation $m = \min\{d_1, d_2\}$. The invariants $k_1(v)$, $k_2(v)$ describe properties of v that are related to the partition of the system S into subsystems (S_1, S_2), (S_2, S_1), respectively. Since the partitions (S_1, S_2), (S_2, S_1) are equivalent, there is a relation between $k_1(v)$, $k_2(v)$; specifically $d_1 - k_1(v) = d_2 - k_2(v)$ for each $v \in V$. All values of the invariants given in Table I and satisfying this relation are allowed. This results in the complete set of entanglement classes $\{C_l\}_{0 \le l \le m}$ and general forms of their elements as given in Table II.

TABLE II: The entanglement classes, their algebraic invariants, and general forms of their representative elements for $n = 2$, $D = (d_1, d_2)$. Each expression $[j_1, j_2]$ stands for $u_{1,j_1} \otimes u_{2,j_2}$. For each $l \in \{1, \ldots, m\}$, $\{u_{i,j}\}_{1 \le j \le l}$ is a set of any linearly independent elements of V_i. (For example, the EPR state is $v = [1,1] + [2,2] = u_{1,1} \otimes u_{2,1} + u_{1,2} \otimes u_{2,2}$ in this notation.)

	$k_1(v)$	$k_2(v)$	v
C_0	d_1	d_2	0
C_1	$d_1 - 1$	$d_2 - 1$	$[1,1]$
...	......	......	....
C_l	$d_1 - l$	$d_2 - l$	$\sum_{j=1}^{l} [j,j]$
...	......	......	....
C_m	$d_1 - m$	$d_2 - m$	$\sum_{j=1}^{m} [j,j]$

Next we consider the case $n = 3$. Maps and associated quantities are given in Table III, where we use the notation $m_{1,2} = \min\{d_1 d_2, d_3\}$, $m_{1,3} = \min\{d_1 d_3, d_2\}$, $m_{2,3} = \min\{d_2 d_3, d_1\}$. The invariants $k_1(v)$, $k_2(v)$, $k_3(v)$, $k_{1,2}(v)$, $k_{1,3}(v)$, $k_{2,3}(v)$ describe properties of v that are related to the partition of the system S into subsystems $(S_1, S_2 \cup S_3)$, $(S_2, S_1 \cup S_3)$, $(S_3, S_1 \cup S_2)$, $(S_1 \cup S_2, S_3)$, $(S_1 \cup S_3, S_2)$, $(S_2 \cup S_3, S_1)$, respectively. Similarly to the case of two spaces, the partitions in each group in $((S_1, S_2 \cup S_3), (S_2 \cup S_3, S_1))$, $((S_2, S_1 \cup S_3), (S_1 \cup S_3, S_2))$, $((S_3, S_1 \cup S_2), (S_1 \cup S_2, S_3))$ are equivalent and there are relations

$$
\begin{aligned}
d_1 - k_1(v) &= d_2 d_3 - k_{2,3}(v), \\
d_2 - k_2(v) &= d_1 d_3 - k_{1,3}(v), \\
d_3 - k_3(v) &= d_1 d_2 - k_{1,2}(v)
\end{aligned}
$$

for each $v \in V$.

TABLE III: Maps and associated quantities for $n = 3$, $D = (d_1, d_2, d_3)$.

$f(v)$	W	W'	$K(v)$	$k(v)$
$f_1(v)$	V_1	$V_2 \otimes V_3$	$K_1(v)$	$k_1(v) \in \{d_1 - m_1, \ldots, d_1\}$
$f_2(v)$	V_2	$V_1 \otimes V_3$	$K_2(v)$	$k_2(v) \in \{d_2 - m_2, \ldots, d_2\}$
$f_3(v)$	V_3	$V_1 \otimes V_2$	$K_3(v)$	$k_3(v) \in \{d_3 - m_3, \ldots, d_3\}$
$f_{1,2}(v)$	$V_1 \otimes V_2$	V_3	$K_{1,2}(v)$	$k_{1,2}(v) \in \{d_1 d_2 - m_{1,2}, \ldots, d_1 d_2\}$
$f_{1,3}(v)$	$V_1 \otimes V_3$	V_2	$K_{1,3}(v)$	$k_{1,3}(v) \in \{d_1 d_3 - m_{1,3}, \ldots, d_1 d_3\}$
$f_{2,3}(v)$	$V_2 \otimes V_3$	V_1	$K_{2,3}(v)$	$k_{2,3}(v) \in \{d_2 d_3 - m_{2,3}, \ldots, d_2 d_3\}$

When classifying states for $n = 3$, we face two difficulties not present in the case $n = 2$. First, not all values of the invariants given in Table III and satisfying the above relations are allowed. Second, states with the same values of the invariants $k_1(v)$, $k_2(v)$, $k_3(v)$ can be distinguished with the help of an additional subspace of V,

$$K_{1,2,3}(v) = (K_{1,2}(v) \otimes V_3) \cap (K_{1,3}(v) \otimes V_2) \cap (K_{2,3}(v) \otimes V_1).$$

The invariant $k_{1,2,3}(v) = \dim K_{1,2,3}(v)$ describes a property of v that is related to the partition of the system S into subsystems (S_1, S_2, S_3), and $k_{1,2,3}(v)$ is irreducible in the sense that it cannot be expressed in terms of the invariants $k_1(v)$, $k_2(v)$, $k_3(v)$.

TABLE IV: The entanglement classes, their algebraic invariants, and general forms of their representative elements for $n = 3$, $D = (2, 2, d)$. Classes for which any of the invariants in $(k_3(v), k_{1,2,3}(v))$ are negative should be discarded. Classes within a horizontal block are added each time d increases by 1, so that there are 7, 9, 10 classes for $d = 2$, $d = 3$, $d \geq 4$, respectively. Each expression $[j_1, j_2, j_3]$ stands for $u_{1,j_1} \otimes u_{2,j_2} \otimes u_{3,j_3}$, where $\{u_{i,j}\}$ is a set of any linearly independent elements of V_i. (For example, for $d = 2$, the GHZ state is $v = [1, 1, 1] + [2, 2, 2] = u_{1,1} \otimes u_{2,1} \otimes u_{3,1} + u_{1,2} \otimes u_{2,2} \otimes u_{3,2}$ in this notation.)

	$k_1(v)$	$k_2(v)$	$k_3(v)$	$k_{1,2,3}(v)$	v
C_0	2	2	d	$4d$	0
C_1	1	1	$d-1$	$3d-2$	$[1,1,1]$
C_2	0	0	$d-1$	$3d-3$	$[1,1,1]+[2,2,1]$
C_3	0	1	$d-2$	$2d-1$	$[1,1,1]+[2,1,2]$
C_4	1	0	$d-2$	$2d-1$	$[1,1,1]+[1,2,2]$
C_5	0	0	$d-2$	$2d-3$	$[1,1,1]+[1,2,2]+[2,1,2]$
C_6	0	0	$d-2$	$2d-4$	$[1,1,1]+[2,2,2]$
C_7	0	0	$d-3$	$d-2$	$[1,1,1]+[1,2,2]+[2,2,3]$
C_8	0	0	$d-3$	$d-3$	$[1,1,1]+[1,2,2]+[2,1,2]+[2,2,3]$
C_9	0	0	$d-4$	0	$[1,1,1]+[1,2,2]+[2,1,3]+[2,2,4]$

As our main computational device, we use the general forms of v that follow from the equation $v \otimes w^* = 0$ for three cases $w \in K_1(v)$, $w \in K_2(v)$, $w \in K_3(v)$. The consistency of the resulting forms leads to restrictions on possible values of invariants and consequently to the complete set of equivalent classes.

Although it is possible to perform these computations for any D, we give the results only for the cases $D = (2, 2, d)$ and $D = (2, 3, d)$, where d is arbitrary; other cases are similarly treated. Due to the special role played by V_3 in these examples, it is convenient to proceed by first considering each possible value of $k_3(v)$, then finding allowed values of $k_1(v)$, $k_2(v)$, and finally those of $k_{1,2,3}(v)$. As a result, we obtain the complete set of entanglement classes and general forms of their reperesentative elements as given in Tables IV and V. For each class in these tables, there are several possible general forms of reperesentative elements related by certain permutations; see [18] for details.

IV. CONCLUSIONS

The superposition principle and the tensor product postulate in quantum mechanics give rise to the phenomenon of entanglement. As a result, a state vector of a system consisting

TABLE V: The entanglement classes, their algebraic invariants, and general forms of their representative elements for $n = 3$, $D = (2, 3, d)$. Classes for which any of the invariants in $(k_3(v), k_{1,2,3}(v))$ are negative should be discarded. Classes within a horizontal block are added each time d increases by 1, so that there are 9, 17, 23, 25, 26 classes for $d = 2$, $d = 3$, $d = 4$, $d = 5$, $d \geq 6$, respectively. Each expression $[j_1, j_2, j_3]$ stands for $u_{1,j_1} \otimes u_{2,j_2} \otimes u_{3,j_3}$, where $\{u_{i,j}\}$ is a set of any linearly independent elements of V_i.

	$k_1(v)$	$k_2(v)$	$k_3(v)$	$k_{1,2,3}(v)$	v
C_0	2	3	d	$6d$	0
C_1	1	2	$d-1$	$5d-3$	$[1,1,1]$
C_2	0	1	$d-1$	$5d-5$	$[1,1,1]+[2,2,1]$
C_3	0	2	$d-2$	$4d-2$	$[1,1,1]+[2,1,2]$
C_4	1	1	$d-2$	$4d-3$	$[1,1,1]+[1,2,2]$
C_5	0	1	$d-2$	$4d-5$	$[1,1,1]+[1,2,2]+[2,1,2]$
C_6	0	1	$d-2$	$4d-6$	$[1,1,1]+[2,2,2]$
C_7	0	0	$d-2$	$4d-7$	$[1,1,1]+[1,2,2]+[2,3,1]$
C_8	0	0	$d-2$	$4d-8$	$[1,1,1]+[1,2,2]+[2,2,1]+[2,3,2]$
C_9	1	0	$d-3$	$3d-1$	$[1,1,1]+[1,2,2]+[1,3,3]$
C_{10}	0	1	$d-3$	$3d-4$	$[1,1,1]+[1,2,2]+[2,1,3]$
C_{11}	0	1	$d-3$	$3d-5$	$[1,1,1]+[1,2,2]+[2,1,2]+[2,2,3]$
C_{12}	0	0	$d-3$	$3d-5$	$[1,1,1]+[1,2,2]+[1,3,3]+[2,1,2]$
C_{13}	0	0	$d-3$	$3d-6$	$[1,1,1]+[1,2,2]+[2,3,3]$
C_{14}	0	0	$d-3$	$3d-7$	$[1,1,1]+[1,2,2]+[1,3,3]+[2,1,2]+[2,2,3]$
C_{15}	0	0	$d-3$	$3d-8$	$[1,1,1]+[1,2,2]+[2,1,3]+[2,3,1]$
C_{16}	0	0	$d-3$	$3d-9$	$[1,1,1]+[1,2,2]+[2,2,2]+[2,3,3]$
C_{17}	0	1	$d-4$	$2d-2$	$[1,1,1]+[1,2,2]+[2,1,3]+[2,2,4]$
C_{18}	0	0	$d-4$	$2d-3$	$[1,1,1]+[1,2,2]+[1,3,3]+[2,3,4]$
C_{19}	0	0	$d-4$	$2d-5$	$[1,1,1]+[1,2,2]+[1,3,3]+[2,2,4]+[2,3,1]$
C_{20}	0	0	$d-4$	$2d-6$	$[1,1,1]+[1,2,2]+[2,2,3]+[2,3,4]$
C_{21}	0	0	$d-4$	$2d-7$	$[1,1,1]+[1,2,2]+[1,3,3]+[2,2,3]+[2,3,4]$
C_{22}	0	0	$d-4$	$2d-8$	$[1,1,1]+[1,2,2]+[1,3,3]+[2,1,2]+[2,2,3]+[2,3,4]$
C_{23}	0	0	$d-5$	$d-3$	$[1,1,1]+[1,2,2]+[1,3,3]+[2,1,4]+[2,2,5]$
C_{24}	0	0	$d-5$	$d-5$	$[1,1,1]+[1,2,2]+[1,3,3]+[2,1,3]+[2,2,4]+[2,3,5]$
C_{25}	0	0	$d-6$	0	$[1,1,1]+[1,2,2]+[1,3,3]+[2,1,4]+[2,2,5]+[2,3,6]$

of several subsystems is a linear combination of tensor products of state vectors of the subsystems. The nature of the linear combination determines the entanglement of the state vector. More specifically, properties of algebraic structures associated with states can be used as entanglement measures.

We developed a method of classification of entangled states that uses linear maps to define degrees of entanglement. Our entanglement invariants are discrete measures, which should be contrasted with the standard continuous invariants.

For cases found in the literature, entanglement classifications obtained by using our method coincide with results obtained by other methods. We also obtained results for cases that were either unsolved or only conjectured in the literature. In particular, for the case $n = 3$, $D = (2, 2, d)$ for $d = 2, \dots, 5$, our method gives the same number of classes as classifications in [3], [4], [7], [9], while for $d > 5$, our method gives the same number of classes as the conjectured classification in [7], [9]. Our entanglement classes and representative elements for $D = (2, 3, d)$ are all new.

Although we considered here only some of the simpler cases of three subsystems, other cases are only slightly more complicated. In a further study [18], we consider a large selection of such cases and formulate a general conjecture about entanglement classes for all cases of three subsystems. Using our method, we also obtain entanglement classes of four qubits [18]. Furthermore, we believe the complete classification of entanglement of five qubits is now within reach and we plan to study it in the near future.

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