Proposal of a two-qutrit contextuality test free of the finite precision and compatibility loopholes

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It has been argued that any test of quantum contextuality is nullified by the fact that perfect orthogonality and perfect compatibility cannot be achieved in finite precision experiments. We introduce an experimentally testable two-qutrit violation of an inequality for noncontextual theories in which orthogonality and compatibility are guaranteed by the fact that measurements are performed on separated qutrits. The inequality is a direct translation of the basic building block of Kochen and Specker's proof of quantum contextuality for a qutrit, despite inequality's proof be completely independent of this diagram.

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Introduction—Contextuality is a fascinating property of nature: The *result* of an experiment may depend on other compatible experiments that may be performed. This is surprising because the experiment gives the same result when repeated after any number of compatible experiments. It is also surprising because the *probability* of obtaining any particular result does not depend on which compatible experiments might be performed (probabilities are noncontextual). Nature's contextuality was pointed out by the discovery that some predictions of quantum mechanics (QM) cannot be reproduced by any noncontextual theory, and that this conflict occurs for any state of any system with three or more distinguishable states [1]. Quantum contextuality has deep consequences like the impossibility of describing nature by classical theories with bounded speed of information [2] or bounded density of memory [3], or the possibility of device-independent eternally secure communications [4].

In 1999, a debate on the physical impact and experimental testability of the Kochen-Specker theorem started in the pages of Physical Review Letters. Meyer [5] and Kent [6] pointed out that finite precision measurement nullifies the physical content of the Kochen-Specker theorem. More than twenty papers have been published since then supporting (for instance, [7, 8]) or criticizing (for instance, [9, 10]) this conclusion. More recently, a series of experimental tests with ions [11], neutrons [12], photons [13, 14], and nuclear magnetic resonance systems [15] have been questioned due to a variant of Meyer and Kent's objection called the compatibility loophole [11, 16].

In this Letter we present a proposal for a type of experiments that (we hope) will definitely close the debate on the physical relevance and experimental testability of the Kochen-Specker theorem, and stimulate a new generation of experiments.

The simplest physical system exhibiting contextuality is a quantum three-state system or qutrit. A simple way to prove contextuality on a qutrit is the following [17, 18]. Suppose the qutrit is initially in the state

$$|i\rangle = \frac{1}{\sqrt{3}}(|0\rangle + |1\rangle + |2\rangle), \tag{1}$$

where $|0\rangle$, $|1\rangle$, and $|2\rangle$ are three orthogonal states. Then, one measures whether the system is in the state

$$|f\rangle = \frac{1}{\sqrt{3}}(|0\rangle - |1\rangle + |2\rangle) \tag{2}$$

and obtains the result 1 (representing "yes"). According to QM, this result occurs with probability

$$|\langle f|i\rangle|^2 = \frac{1}{9}. \tag{3}$$

Now suppose that, instead of the projection onto $|f\rangle$ one would have measured the observable

$$H_0 = a_0|a_0\rangle\langle a_0| + b_0|b_0\rangle\langle b_0| + c_0|c_0\rangle\langle c_0|, \tag{4}$$

where $|a_0\rangle$, $|b_0\rangle$, and $|c_0\rangle$ are the following orthogonal states:

$$|a_0\rangle = \frac{1}{\sqrt{2}}(|1\rangle - |2\rangle), \tag{5a}$$

$$|b_0\rangle = \frac{1}{\sqrt{2}}(|1\rangle + |2\rangle), \tag{5b}$$

$$|c_0\rangle = |0\rangle. \tag{5c}$$

Then, according to QM, the result a_0 can never be obtained since $\langle a_0|i\rangle = 0$. In addition, in any noncontextual theory in agreement with QM, the result of H_0 cannot be b_0 since $\langle f|b_0\rangle = 0$. Therefore, in any noncontextual theory in agreement with QM, the initial state $|i\rangle$ together

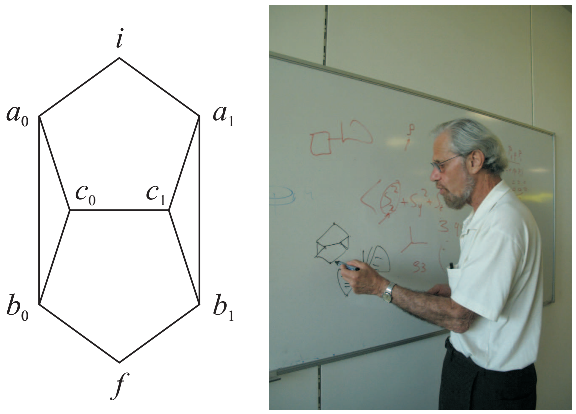

FIG. 1. Building block of Kochen and Specker's proof of quantum contextuality (left). Vertices represent propositions; two of them are joined when the propositions are compatible and both cannot be true. Kochen drawing the block in Zürich in 2009 (right).

with the positive probability (3) imply $H_0 = c_0$, whenever the yes answer should be found for $|f\rangle$. Similarly, if one would have measured

$$H_1 = a_1|a_1\rangle\langle a_1| + b_1|b_1\rangle\langle b_1| + c_1|c_1\rangle\langle c_1|, \tag{6}$$

where

$$|a_1\rangle = \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle), \tag{7a}$$

$$|b_1\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle), \tag{7b}$$

$$|c_1\rangle = |2\rangle. \tag{7c}$$

then, the same reasoning leads to the conclusion that, in any noncontextual theory in agreement with QM, $H_1 = c_1$.

However, $H_0 = c_0$ and $H_1 = c_1$ cannot happen simultaneously since $\langle c_1|c_0\rangle = 0$. Therefore, there is no noncontextual assignment of results to the six propositions $|a_0\rangle\langle a_0|, \ldots, |c_1\rangle\langle c_1|$ in agreement with the predictions of QM for a qutrit prepared in the state $|i\rangle$ and postselected in the state $|f\rangle$. Indeed, the orthogonality relations between these eight states constitutes the building block of the Kochen-Specker state-independent proof of quantum contextuality [1, 19]. See Fig. 1.

What if perfect orthogonality cannot be experimentally achieved? Then the previous argument vanishes. For instance, the block of Fig. 1 cannot be constructed if the unit vectors are restricted to vectors with rational components [5] or to vectors that do not have any orthogonal vector [7], although in both cases these vectors are dense in the set of unit vectors and therefore are undistinguishable from the set of all unit vectors by experiments with finite precision [5–10]. A related problem affecting some recent experimental violations [11–13] of noncontextual inequalities [20, 21] involving sequential measurements on the same physical system is the fact that these inequalities are based on the assumption that the sequential measurements are perfectly compatible, something which does not occur in actual experiments, so to conclude contextuality extra assumptions are needed [11, 16].

Here we avoid these extra assumptions by measuring one of the observables in one qutrit and the second observable on a distant qutrit. Spatial separation provides a physical basis to the assumption that both measurements are not only approximately but perfectly compatible. The combination of Kochen-Specker configurations and correlated systems is a standard way to show contextuality and nonlocality for composite systems [17, 22–24].

For this purpose we have to derive an inequality valid for any noncontextual theory based on the observables of the block of Kochen and Specker and involving correlations between measurements on two different qutrits A and B.

Lemma: Consider Q_i as dichotomic observables, H_i as trichotomic observables, with labels A and B corresponding to the respective party. Then, for any noncontextual theory, the following inequality holds:

$$\mathcal{K} \leq 0, \tag{8a}$$

where

$$\begin{aligned}\mathcal{K} = \sum_{i\neq j=0}^{1} & P(Q_i^A = 1, Q_j^B = 1) - P(Q_0^A = 1, H_i^B = a_i) \\ & - P(H_i^A = a_i, Q_0^B = 1) - P(Q_1^A = 1, H_i^B = b_i) \\ & - P(H_i^A = b_i, Q_1^B = 1) - P(H_i^A = a_i, H_i^B = b_i) \\ & - P(H_i^A = b_i, H_i^B = a_i) - P(H_0^A = a_0, H_0^B = c_0) \\ & - P(H_0^A = c_0, H_0^B = a_0) - P(H_0^A = b_0, H_0^B = c_0) \\ & - P(H_0^A = c_0, H_0^B = b_0) - P(H_i^A = c_i, H_j^B = c_j),\end{aligned} \tag{8b}$$

where, $P(Q_0^A = 1, H_0^B = a_0)$ denotes the joint probability of obtaining the results 1 and a_0 for Q_0^A and H_0^B, respectively.

Proof: Each probability in (8) measures the frequency of a set of hidden variable states. In our scenario there are four dichotomic observables, Q_0^A, Q_1^A, Q_0^B, and Q_1^B, and four trichotomic observables, H_0^A, H_1^A, H_0^B, and H_1^B. Therefore, the number of hidden variable states is $2^4 3^4 = 1296$. $P(Q_0^A = 1, Q_1^B = 1)$ is the sum of the frequencies of 324 of these states, and similarly $P(Q_1^A = 1, Q_0^B = 1)$. There are 81 states appearing both in $P(Q_0^A = 1, Q_1^B = 1)$ and $P(Q_1^A = 1, Q_0^B = 1)$, that is, there are 81 frequencies which appear with multiplicity 2, and 486 which appear with multiplicity 1 in $P(Q_0^A = 1, Q_1^B = 1) + P(Q_1^A = 1, Q_0^B = 1)$. $P(Q_0^A = 1, H_0^B = a_0)$, which enters with a minus sign in (8), contains 27 frequencies of the first type and 108 of the second. Every additional probability that is substracted contains frequencies present in $P(Q_0^A = 1, Q_1^B = 1) + P(Q_1^A = 1, Q_0^B = 1)$, and the sum of all of them contains all the frequencies in $P(Q_0^A=1,Q_1^B=1)+P(Q_1^A=1,Q_0^B=1)$ with the corresponding multiplicities. The inequality (8) is minimal in the sense that no term can be supressed from $\mathcal{K}$ without allowing for a noncontextual model for the probabilities. ■

Notice that inequality (8) also holds for any local theory. This means that, when the choice of measurement in one of the particles is spacelike separated from the result of the measurement on the other particle, then (8) is not only a noncontextual inequality but also a Bell inequality. In this case, its violation also proves the impossibility of local hidden-variable theories. Notice, however, that a test of contextuality using inequality (8) does not require spacelike separation since, in such test the role of spatial separation is only to provide a physical reason which guarantees that the experiments on qutrit A and the experiments on qutrit B are actually compatible to avoid the finite precision and compatibility loopholes of previous experiments.

Quantum violation.—The interesting feature of inequality (8) is that, for a two-qutrit system prepared in the state

$$|\psi\rangle=\frac{1}{\sqrt{3}}\left(|02\rangle-|11\rangle+|20\rangle\right), \tag{9}$$

and for the observables

$$Q_0^A=Q_1^B=|i\rangle\langle i|, \tag{10a}$$
$$Q_1^A=Q_0^B=|f\rangle\langle f|, \tag{10b}$$
$$H_0^A=a_0|a_0\rangle\langle a_0|+b_0|b_0\rangle\langle b_0|+c_0|c_0\rangle\langle c_0|, \tag{10c}$$
$$H_0^B=a_0|b_1\rangle\langle b_1|+b_0|a_1\rangle\langle a_1|+c_0|c_1\rangle\langle c_1|, \tag{10d}$$
$$H_1^A=a_1|a_1\rangle\langle a_1|+b_1|b_1\rangle\langle b_1|+c_1|c_1\rangle\langle c_1|, \tag{10e}$$
$$H_1^B=a_1|b_0\rangle\langle b_0|+b_1|a_0\rangle\langle a_0|+c_1|c_0\rangle\langle c_0|, \tag{10f}$$

where $|i\rangle$ is defined in (1), $|f\rangle$ is defined in (2), $|a_0\rangle$, $|b_0\rangle$, and $|c_0\rangle$ are defined in (5), and $|a_1\rangle$, $|b_1\rangle$, and $|c_1\rangle$ are defined in (7), the predictions of QM are

$$P(Q_i^A=1,Q_j^B=1)=\frac{1}{27}, \tag{11a}$$
$$P(Q_0^A=1,H_i^B=a_i)=P(H_i^A=a_i,Q_0^B=1)=0, \tag{11b}$$
$$P(Q_1^A=1,H_i^B=b_i)=P(H_i^A=b_i,Q_1^B=1)=0, \tag{11c}$$
$$P(H_i^A=a_i,H_i^B=b_i)=P(H_i^A=b_i,H_i^B=a_i)=0, \tag{11d}$$
$$P(H_0^A=a_0,H_0^B=c_0)=P(H_0^A=c_0,H_0^B=a_0)=0, \tag{11e}$$
$$P(H_0^A=b_0,H_0^B=c_0)=P(H_0^A=c_0,H_0^B=b_0)=0, \tag{11f}$$
$$P(H_i^A=c_i,H_j^B=c_j)=0, \tag{11g}$$

where $i,j\in\{0,1\}$ and $i\neq j$. Therefore, according to QM,

$$\mathcal{K}_{\text{QM}}=\frac{2}{27}\approx 0.074, \tag{12}$$

violating inequality (8).

Discussion.—There is a one-to-one correspondence between the Kochen-Specker building block of Fig. 1 and inequality (8). This correspondence is illustrated in Table I. However, the important point is that, in spite of this one-to-one correspondence, inequality (8) has been proven from the assumption of noncontextuality (and can also be proven from the assumption of locality), without any reference to Fig. 1, and without additional assumptions like infinite precision measurements or non-testable compatibility. Therefore, inequality (8) is totally independent of QM and provides a tool to experimentally test contextuality without additional assumptions, and, at the same time, it constitutes a translation of the basic block of the proof in [1].

Any state-independent proof of the Kochen-Specker theorem contains several blocks similar to the one in Fig. 1, in which if the system is prepared in a state $|i\rangle$, then, in any noncontextual theory, one will never obtain the state $|f\rangle$ [25]. An important point is that the method to convert the block in Fig. 1 into an experimentally testable nocontextual inequality among correlations between two systems can be applied to *any* block in any proof of the KS theorem in any finite dimension $d>2$ (for an almost exhaustive list of possible blocks in any dimension, see [25, 26]).

Testing the violation of inequality (8) is very demanding experimentally. It requires testing 20 joint probabilities and, assuming that one obtains $P(Q_i^A=1,Q_j^B=1)\approx\frac{1}{27}$ and ϵ for the other 18 probabilities, then one must have $\epsilon<\frac{1}{243}\approx 0.0041$ in order to observe a violation of inequality (8). The importance of this result is that it explicitly shows that there is no fundamental obstacle to observe Kochen-Specker contextuality on (pairs of) qutrits. No additional assumptions are needed to deal with the fact that measurements have a finite preci-

TABLE I. Correspondence between the geometrical relations in Fig. 1 and the experimentally testable probabilities in inequality (8); $j\in\{0,1\}$.

Fig. 1	Inequality (8)
$f\not\perp i$	$P(Q_0^A=1,Q_1^B=1),P(Q_1^A=1,Q_0^B=1)$
$i\perp a_j$	$P(Q_0^A=1,H_j^B=a_j),P(H_j^A=a_j,Q_0^B=1)$
$f\perp b_j$	$P(Q_1^A=1,H_j^B=b_j),P(H_j^A=b_j,Q_1^B=1)$
$a_j\perp b_j$	$P(H_j^A=a_j,H_j^B=b_j),P(H_j^A=b_j,H_j^B=a_j)$
$a_j\perp c_j$	$P(H_j^A=a_j,H_j^B=c_j),P(H_j^A=c_j,H_j^B=a_j)$
$b_j\perp c_j$	$P(H_j^A=b_j,H_j^B=c_j),P(H_j^A=c_j,H_j^B=b_j)$
$c_0\perp c_1$	$P(H_0^A=c_0,H_1^B=c_1),P(H_1^A=c_1,H_0^B=c_0)$

sion and compatibility between sequential measurements is not perfect.

Two features distinguishes this proposal: It tests the quantum contextuality of the simplest physical system that can exhibit contextuality (qutrits), rather than on a system with more distinguishable states as in previous experiments [11–13]. It is free of the finite precision and compatibility loopholes of these previous experiments.

Using the same approach, one can obtain a noncontextual two-qutrit inequality for the complete (state-independent) proof of Kochen and Specker [1] or any other state-independent proof for a qutrit [27]. This can be done by adding all the inequalities corresponding to the composing blocks, and then removing redundant probabilities. Unfortunately, the corresponding violations demand minimum values for ϵ of the same order required for the experiment proposed in this Letter. A more promising approach is to consider a two-ququart system (i.e., two four-state quantum systems) in a maximally entangled state and derive the inequality corresponding to the simplest (for any dimension) state-independent proof of contextuality which needs 18 yes-no tests [28] (instead of the 8 tests in Fig. 1) or its symmetrized version [27] using 24 yes-no tests which is more robust to imperfections (since it includes 16 critical 18-test proofs and 96 critical 20-test proofs [28]).

In this Letter we have established a one-to-one correspondence between *any* building block of *any* proof of the Kochen-Specker theorem and an experimental test of contextuality, free of the finite precision and compatibility loopholes. We hope that this proposal will definitely close the debate on the physical relevance and experimental testability of the Kochen-Specker theorem, and iduce a new generation of loophole-free experiments.

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