On Wishart distribution

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Abstract

This paper proposes a unified approach that enables the Wishart distribution to be studied simultaneously in the real, complex, quaternion and octonion cases. In particular, the noncentral generalised Wishart distribution, the joint density of the eigenvalues and the distribution of the maximum eigenvalue are obtained for real normed division algebras.

1 Introduction

Many results first described in statistical theory are then found in real cases, and the version for complex cases is subsequently studied. This has been described in various papers, see Bravais (1846)(cited by Wooding (1956)) and Wooding (1956); James (1964, Sections 4 and 8); Muirhead (1982) and Ratnarajah *et al.* (2005), among many other examples.

Using some concepts and results derived from abstract algebra, it is possible to propose a unified means of addressing not only real and complex cases but also the quaternion and octonion cases. Part of this approach has been used for some time in random matrix theory, see Edelman and Rao (2005) and Forrester (2009).

For the sake of completeness, in the present study the case of octonions is considered, but it should be noted that many results for the octonion case can only be conjectured, because there remain many unresolved theoretical problems in this respect, see Dray and Manogue (1999). Furthermore, the relevance of the octonion case for understanding the real world has yet to be clarified, see Baez (2002).

The rest of this paper is structured as follows: Section 2 reviews some definitions and notation on real normed division algebras. Some results for Jacobians and generalised hypergeometric functions, together with an extension of one of the basic properties of zonal polynomials (which is also valid for Jack polynomials) are also given. Section 3 then derives the noncentral generalised Wishart distribution, and as a corollary the noncentral inverse

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Key words. Jacobians; Jack polynomials; generalised hypergeometric functions; non singular Wishart distribution; non central distributions; real, complex, quaternion and octonion random matrices.
2000 Mathematical Subject Classification. Primary 60E05, 62E15; secondary 15A52

generalised Wishart distribution is obtained. In Section 4, we obtain the joint density function of the eigenvalues and the distribution of the maximum eigenvalue, the latter under a matrix multivariate normal distribution. All these results are obtained for real normed division algebras.

2 Preliminary results

Let us introduce some notation and results that will be useful.

2.1 Notation and real normed division algebras

A detailed discussion of real normed division algebras may be found in Baez (2002). For convenience, we shall introduce some notations, although in general we adhere to standard notations.

For our purposes, a **vector space** is always a finite-dimensional module over the field of real numbers. An **algebra** $\mathfrak{F}$ is a vector space that is equipped with a bilinear map $m : \mathfrak{F} \times \mathfrak{F} \to \mathfrak{F}$ termed *multiplication* and a nonzero element $1 \in \mathfrak{F}$ termed the *unit* such that $m(1, a) = m(a, 1) = 1$. As usual, we abbreviate $m(a, b) = ab$ as ab. We do not assume $\mathfrak{F}$ associative. Given an algebra, we freely think of real numbers as elements of this algebra via the map $\omega \mapsto \omega 1$.

An algebra $\mathfrak{F}$ is a **division algebra** if given $a, b \in \mathfrak{F}$ with $ab = 0$, then either $a = 0$ or $b = 0$. Equivalently, $\mathfrak{F}$ is a division algebra if the operation of left and right multiplications by any nonzero element is invertible. A **normed division algebra** is an algebra $\mathfrak{F}$ that is also a normed vector space with $||ab|| = ||a||||b||$. This implies that $\mathfrak{F}$ is a division algebra and that $||1|| = 1$.

There are exactly four normed division algebras: real numbers ($\Re$), complex numbers ($\mathfrak{C}$), quaternions ($\mathfrak{H}$) and octonions ($\mathfrak{O}$), see Baez (2002). We take into account that $\Re$, $\mathfrak{C}$, $\mathfrak{H}$ and $\mathfrak{O}$ are the only normed division algebras; moreover, they are the only alternative division algebras, and all division algebras have a real dimension of $1, 2, 4$ or 8, which is denoted by β, see Baez (2002, Theorems 1, 2 and 3). In other branches of mathematics, the parameter $\alpha = 2/\beta$ is used, see Edelman and Rao (2005).

Let $\mathcal{L}^{\beta}_{m,n}$ be the linear space of all $n \times m$ matrices of rank $m \leq n$ over $\mathfrak{F}$ with m distinct positive singular values, where $\mathfrak{F}$ denotes a *real finite-dimensional normed division algebra*. Let $\mathfrak{F}^{n\times m}$ be the set of all $n \times m$ matrices over $\mathfrak{F}$. The dimension of $\mathfrak{F}^{n\times m}$ over $\Re$ is βmn. Let $\mathbf{A} \in \mathfrak{F}^{n\times m}$, then $\mathbf{A}^* = \overline{\mathbf{A}}^T$ denotes the usual conjugate transpose.

The set of matrices $\mathbf{H}_1 \in \mathfrak{F}^{n\times m}$ such that $\mathbf{H}_1^*\mathbf{H}_1 = \mathbf{I}_m$ is a manifold denoted $\mathcal{V}^{\beta}_{m,n}$, is termed the *Stiefel manifold* ($\mathbf{H}_1$ is also known as *semi-orthogonal* ($\beta = 1$), *semi-unitary* ($\beta = 2$), *semi-symplectic* ($\beta = 4$) and *semi-exceptional type* ($\beta = 8$) matrices, see Dray and Manogue (1999)). The dimension of $\mathcal{V}^{\beta}_{m,n}$ over $\Re$ is $[\beta mn - m(m-1)\beta/2 - m]$. In particular, $\mathcal{V}^{\beta}_{m,m}$ with dimension over $\Re$, $[m(m+1)\beta/2 - m]$, is the maximal compact subgroup $\mathfrak{U}^{\beta}(m)$ of $\mathcal{L}^{\beta}_{m,m}$ and consists of all matrices $\mathbf{H} \in \mathfrak{F}^{m\times m}$ such that $\mathbf{H}^*\mathbf{H} = \mathbf{I}_m$. Therefore, $\mathfrak{U}^{\beta}(m)$ is the *real orthogonal group* $\mathcal{O}(m)$ ($\beta = 1$), the *unitary group* $\mathcal{U}(m)$ ($\beta = 2$), *compact symplectic group* $\mathcal{S}p(m)$ ($\beta = 4$) or *exceptional type matrices* $\mathcal{O}o(m)$ ($\beta = 8$), for $\mathfrak{F} = \Re$, $\mathfrak{C}$, $\mathfrak{H}$ or $\mathfrak{O}$, respectively.

We denote by $\mathfrak{S}^{\beta}_m$ the real vector space of all $\mathbf{S} \in \mathfrak{F}^{m\times m}$ such that $\mathbf{S} = \mathbf{S}^*$. Let $\mathfrak{P}^{\beta}_m$ be the *cone of positive definite matrices* $\mathbf{S} \in \mathfrak{F}^{m\times m}$; then $\mathfrak{P}^{\beta}_m$ is an open subset of $\mathfrak{S}^{\beta}_m$. Over $\Re$, $\mathfrak{S}^{\beta}_m$ consist of *symmetric* matrices; over $\mathfrak{C}$, *Hermitian* matrices; over $\mathfrak{H}$, *quaternionic Hermitian* matrices (also termed *self-dual matrices*) and over $\mathfrak{O}$, *octonionic Hermitian* matrices. Generically, the elements of $\mathfrak{S}^{\beta}_m$ are termed **Hermitian matrices**, irrespective of the nature of $\mathfrak{F}$. The dimension of $\mathfrak{S}^{\beta}_m$ over $\Re$ is $[m(m-1)\beta + 2]/2$.

Let $\mathfrak{D}_m^\beta$ be the *diagonal subgroup* of $\mathcal{L}_{m,m}^\beta$ consisting of all $\mathbf{D} \in \mathfrak{F}^{m\times m}$, $\mathbf{D} = \operatorname{diag}(d_1, \dots, d_m)$.

For any matrix $\mathbf{X} \in \mathfrak{F}^{n\times m}$, $d\mathbf{X}$ denotes the *matrix of differentials* (dx_{ij}). Finally, we define the *measure* or volume element $(d\mathbf{X})$ when $\mathbf{X} \in \mathfrak{F}^{m\times n}$, $\mathfrak{S}_m^\beta$, $\mathfrak{D}_m^\beta$ or $\mathcal{V}_{m,n}^\beta$, see Dimitriu (2002).

If $\mathbf{X} \in \mathfrak{F}^{n\times m}$ then $(d\mathbf{X})$ (the Lebesgue measure in $\mathfrak{F}^{n\times m}$) denotes the exterior product of the βmn functionally independent variables

$$(d\mathbf{X}) = \bigwedge_{i=1}^{n}\bigwedge_{j=1}^{m} dx_{ij} \quad \text{ where } \quad dx_{ij} = \bigwedge_{k=1}^{\beta} dx_{ij}^{(k)}.$$

If $\mathbf{S} \in \mathfrak{S}_m^\beta$ (or $\mathbf{S} \in \mathfrak{T}_L^\beta(m)$) then $(d\mathbf{S})$ (the Lebesgue measure in $\mathfrak{S}_m^\beta$ or in $\mathfrak{T}_L^\beta(m)$) denotes the exterior product of the $m(m+1)\beta/2$ functionally independent variables (or denotes the exterior product of the $m(m-1)\beta/2+n$ functionally independent variables, if $s_{ii} \in \Re$ for all $i = 1, \dots, m$)

$$(d\mathbf{S}) = \begin{cases} \displaystyle\bigwedge_{i\leq j}^{m}\bigwedge_{k=1}^{\beta} ds_{ij}^{(k)}, & \\ \displaystyle\bigwedge_{i=1}^{m} ds_{ii} \bigwedge_{i<j}^{m}\bigwedge_{k=1}^{\beta} ds_{ij}^{(k)}, & \text{if } s_{ii} \in \Re. \end{cases}$$

The context generally establishes the conditions on the elements of $\mathbf{S}$, that is, if $s_{ij} \in \Re$, $\in \mathfrak{C}$, $\in \mathfrak{H}$ or $\in \mathfrak{O}$. It is considered that

$$(d\mathbf{S}) = \bigwedge_{i\leq j}^{m}\bigwedge_{k=1}^{\beta} ds_{ij}^{(k)} \equiv \bigwedge_{i=1}^{m} ds_{ii} \bigwedge_{i<j}^{m}\bigwedge_{k=1}^{\beta} ds_{ij}^{(k)}.$$

Observe, too, that for the Lebesgue measure $(d\mathbf{S})$ defined thus, it is required that $\mathbf{S} \in \mathfrak{P}_m^\beta$, that is, $\mathbf{S}$ must be a non singular Hermitian matrix (Hermitian definite positive matrix).

If $\mathbf{\Lambda} \in \mathfrak{D}_m^\beta$ then $(d\mathbf{\Lambda})$ (the Legesgue measure in $\mathfrak{D}_m^\beta$) denotes the exterior product of the βm functionally independent variables

$$(d\mathbf{\Lambda}) = \bigwedge_{i=1}^{n}\bigwedge_{k=1}^{\beta} d\lambda_i^{(k)}.$$

If $\mathbf{H}_1 \in \mathcal{V}_{m,n}^\beta$ then

$$(\mathbf{H}_1^* d\mathbf{H}_1) = \bigwedge_{i=1}^{n}\bigwedge_{j=i+1}^{m} \mathbf{h}_j^* d\mathbf{h}_i.$$

where $\mathbf{H} = (\mathbf{H}_1|\mathbf{H}_2) = (\mathbf{h}_1, \dots, \mathbf{h}_m|\mathbf{h}_{m+1}, \dots, \mathbf{h}_n) \in \mathfrak{U}^\beta(m)$. It can be proved that this differential form does not depend on the choice of the $\mathbf{H}_2$ matrix. When $m = 1$; $\mathcal{V}_{1,n}^\beta$ defines the unit sphere in $\mathfrak{F}^n$. This is, of course, an $(n-1)\beta$- dimensional surface in $\mathfrak{F}^n$. When $m = n$ and denoting $\mathbf{H}_1$ by $\mathbf{H}$, $(\mathbf{H}^* d\mathbf{H})$ is termed the *Haar measure* on $\mathfrak{U}^\beta(m)$.

The surface area or volume of the Stiefel manifold $\mathcal{V}_{m,n}^\beta$ is

$$\operatorname{Vol}(\mathcal{V}_{m,n}^\beta) = \int_{\mathbf{H}_1\in\mathcal{V}_{m,n}^\beta} (\mathbf{H}_1^* d\mathbf{H}_1) = \frac{2^m \pi^{mn\beta/2}}{\Gamma_m^\beta[n\beta/2]}, \tag{1}$$

where $\Gamma_m^\beta[a]$ denotes the multivariate Gamma function for the space $\mathfrak{S}_m^\beta$, and is defined by

$$\begin{aligned} \Gamma_m^\beta[a] &= \int_{\mathbf{A}\in\mathfrak{P}_m^\beta} \operatorname{etr}\{-\mathbf{A}\}|\mathbf{A}|^{a-(m-1)\beta/2-1}(d\mathbf{A}) \\ &= \pi^{m(m-1)\beta/4}\prod_{i=1}^{m}\Gamma[a-(i-1)\beta/2], \end{aligned}$$

where $\text{etr}(\cdot) = \exp(\text{tr}(\cdot))$, $|\cdot|$ denotes the determinant and $\text{Re}(a) > (m-1)\beta/2$, see Gross and Richards (1987).

2.2 Jacobians and and some results on integration

First, we summarise diverse Jacobians in terms of the β parameter, some based on the work of Dimitriu (2002), while other results are proposed as extensions of real, complex or quaternion cases, see James (1954), James (1964), Khatri (1965), Metha (1991), Ratnarajah *et al.* (2005) and Li and Xue (2009).

Lemma 2.1 (Singular value decomposition, SVD)**.** *Let* $\mathbf{X} \in \mathcal{L}^{\beta}_{m,n}$*, such that* $\mathbf{X} = \mathbf{V}_1\mathbf{D}\mathbf{W}^*$ *with* $\mathbf{V}_1 \in \mathcal{V}^{\beta}_{m,n}$*,* $\mathbf{W} \in \mathfrak{U}^{\beta}(m)$ *and* $\mathbf{D} = \text{diag}(d_1, \cdots, d_m) \in \mathfrak{D}^1_m$*,* $d_1 > \cdots > d_m > 0$*. Then*

$$(d\mathbf{X}) = 2^{-m}\pi^{\tau}\prod_{i=1}^{m} d_i^{\beta(n-m+1)-1}\prod_{i<j}^{m}(d_i^2 - d_j^2)^{\beta}(d\mathbf{D})(\mathbf{V}_1^* d\mathbf{V}_1)(\mathbf{W}^* d\mathbf{W}), \tag{2}$$

where

$$\tau = \begin{cases} 0, & \beta = 1; \\ -m, & \beta = 2; \\ -2m, & \beta = 4; \\ -4m, & \beta = 8. \end{cases}$$

Lemma 2.2 (Spectral decomposition)**.** *Let* $\mathbf{S} \in \mathfrak{P}^{\beta}_m$*. Then the spectral decomposition can be written as* $\mathbf{S} = \mathbf{W}\mathbf{\Lambda}\mathbf{W}^*$*, where* $\mathbf{W} \in \mathfrak{U}^{\beta}(m)$ *and* $\mathbf{\Lambda} = \text{diag}(\lambda_1, \ldots, \lambda_m) \in \mathfrak{D}^1_m$*, with* $\lambda_1 > \cdots > \lambda_m > 0$*. Then*

$$(d\mathbf{S}) = 2^{-m}\pi^{\tau}\prod_{i<j}^{m}(\lambda_i - \lambda_j)^{\beta}(d\mathbf{\Lambda})(\mathbf{W}^* d\mathbf{W}), \tag{3}$$

where τ *is defined in Lemma 2.1.*

Lemma 2.3. *Let* $\mathbf{X} \in \mathcal{L}^{\beta}_{m,n}$*, and* $\mathbf{S} = \mathbf{X}^*\mathbf{X} \in \mathfrak{P}^{\beta}_m$*. Then*

$$(d\mathbf{X}) = 2^{-m}|\mathbf{S}|^{\beta(n-m+1)/2-1}(d\mathbf{S})(\mathbf{V}_1^* d\mathbf{V}_1). \tag{4}$$

Theorem 2.1. *Let* $\mathbf{S} \in \mathfrak{P}^{\beta}_m$*. Then ignoring the sign, if* $\mathbf{Y} = \mathbf{S}^{-1}$

$$(d\mathbf{Y}) = |\mathbf{S}|^{-\beta(m-1)-2}(d\mathbf{S}). \tag{5}$$

Consider the following property of Jack polynomials.

Lemma 2.4. *If* $\mathbf{X} \in \mathfrak{L}^{\beta}_{m,n}$*, then*

$$\int_{\mathbf{H}_1 \in \mathcal{V}^{\beta}_{m,n}} (\text{tr}(\mathbf{X}\mathbf{H}_1))^{2k}(d\mathbf{H}_1) = \sum_{\kappa}\frac{\left(\frac{1}{2}\right)_k}{[\beta n/2]^{\beta}_{\kappa}}C^{\beta}_{\kappa}(\mathbf{X}\mathbf{X}^*), \tag{6}$$

where $C^{\beta}_{\kappa}(\mathbf{B})$ *are the Jack polynomials of weight* κ *of* $\mathbf{B} \in \mathfrak{S}^{\beta}_m$ *corresponding to the partition* $\kappa = (k_1, \ldots k_m)$ *of* k*,* $k_1 \geq \cdots \geq k_m \geq 0$ *with* $\sum_{i=1}^m k_i = k$*, see Sawyer (1997), Gross and Richards (1987); and* $[a]^{\beta}_{\kappa}$ *denotes the generalised Pochhammer symbol of weight* κ*, defined as*

$$[a]^{\beta}_{\kappa} = \prod_{i=1}^{m}(a - (i-1)\beta/2)_{k_1},$$

where $\Re(a) > (m-1)\beta/2 - k_m$ *and* $(a)_i = a(a+1)\cdots(a+i-1)$*.*

Proof. See Díaz-García and Gutiérrez-Jáimez (2010). □ □

Now, we utilise the complexification $\mathfrak{S}_m^{\beta,\mathfrak{C}} = \mathfrak{S}_m^{\beta} + i\mathfrak{S}_m^{\beta}$ of $\mathfrak{S}_m^{\beta}$. That is, $\mathfrak{S}_m^{\beta,\mathfrak{C}}$ consist of all matrices $\mathbf{X} \in (\mathfrak{F}^{\mathfrak{C}})^{m\times m}$ of the form $\mathbf{Z} = \mathbf{X} + i\mathbf{Y}$, with $\mathbf{X}, \mathbf{Y} \in \mathfrak{S}_m^{\beta}$. We refer to $\mathbf{X} = \mathrm{Re}(\mathbf{Z})$ and $\mathbf{Y} = \mathrm{Im}(\mathbf{Z})$ as the *real and imaginary parts* of $\mathbf{Z}$, respectively. The *generalised right half-plane* $\mathbf{\Phi} = \mathfrak{P}_m^{\beta} + i\mathfrak{S}_m^{\beta}$ in $\mathfrak{S}_m^{\beta,\mathfrak{C}}$ consists of all $\mathbf{Z} \in \mathfrak{S}_m^{\beta,\mathfrak{C}}$ such that $\mathrm{Re}(\mathbf{Z}) \in \mathfrak{P}_m^{\beta}$, see Gross and Richards (1987, p. 801). Also, as in Davis (1980), consider the following notation,

$$\sum_{k,l=0}^{\infty}\sum_{\kappa,\delta;\phi\in\kappa\cdot\delta} \equiv \sum_{k=0}^{\infty}\sum_{l=0}^{\infty}\sum_{\kappa}\sum_{\delta}\sum_{\phi\in\kappa\cdot\delta}.$$

$C_{\phi}^{[\beta]\kappa,\delta}(\mathbf{A},\mathbf{B})$ denotes the invariant polynomials, which are defined in Chikuse and Davis (1979) and Davis (1980) in the real case. Díaz-García (2009) studied these invariant polynomials and many of their basic properties for real normed division algebras.

Theorem 2.2. *Let $\mathbf{\Delta} \in \mathbf{\Phi}$ then*

$$\int_{\mathbf{0}<\mathbf{X}<\mathbf{\Delta}} |\mathbf{X}|^{a-(m-1)\beta/2-1}\,\mathrm{etr}\{-\mathbf{X}\mathbf{A}\}\,{}_pF_q^{\beta}(a_1,\dots,a_p;b_1,\dots,b_q;\mathbf{B}\mathbf{X})(d\mathbf{X})$$

$$= \frac{\Gamma_m^{\beta}[a]\Gamma_m^{\beta}[(m-1)\beta/2+1]}{\Gamma_m^{\beta}[a+(m-1)\beta/2+1]}|\mathbf{\Delta}|^{a}$$
$$\times\sum_{k,l=0}^{\infty}\sum_{\phi\in\kappa\cdot\delta}\frac{[a_1]_{\kappa}^{\beta},\dots,[a_p]_{\kappa}^{\beta}}{k!l![b_1]_{\kappa}^{\beta},\dots,[a_q]_{\kappa}^{\beta}}\frac{\theta_{\phi}^{[\beta]\kappa,\delta}C_{\phi}^{[\beta]\kappa,\delta}(-\mathbf{A}\mathbf{\Delta},\mathbf{B}\mathbf{\Delta})}{[a+(m-1)\beta/2+1]_{\phi}^{\beta}}$$

where $\theta_{\phi}^{[\beta]\kappa,\delta}$ is defined in Díaz-García (2009, eq. (52)), see also Davis (1980). Also, ${}_qF_p$ denotes the hypergeometric function defined in terms of Jack polynomials, see Gross and Richards (1987) and Koev and Edelman (2006).

Proof. This follows immediately, expanding ${}_pF_q^{\beta}$ in terms of Jack polynomials and using Díaz-García (2009, eq. (5.38)). □

3 Wishart distribution

Recall that $\mathbf{X} \in \mathfrak{L}_{m,n}^{\beta}$ has a matrix multivariate elliptically contoured distribution for real normed division algebras if its density, with respect to the Lebesgue measure, is given by (see Díaz-García and Gutiérrez-Jáimez (2009)):

$$f_{\mathbf{X}}(\mathbf{X}) = \frac{C^{\beta}(m,n)}{|\mathbf{\Sigma}|^{\beta n/2}|\mathbf{\Theta}|^{\beta m/2}}h\left\{\mathrm{tr}\left[\mathbf{\Sigma}^{-1}(\mathbf{X}-\boldsymbol{\mu})^{*}\mathbf{\Theta}^{-1}(\mathbf{X}-\boldsymbol{\mu})\right]\right\},$$

where $\boldsymbol{\mu} \in \mathfrak{L}_{m,n}^{\beta}$, $\mathbf{\Sigma} \in \mathfrak{P}_m^{\beta}$, $\mathbf{\Theta} \in \mathfrak{P}_m^{\beta}$. The function $h : \mathfrak{F} \to [0,\infty)$ is termed the generator function, and it is such that $\int_{\mathfrak{P}_1^{\beta}} u^{\beta nm-1}h(u^2)du < \infty$ and

$$C^{\beta}(m,n) = \frac{\Gamma[\beta mn/2]}{2\pi^{\beta mn/2}}\left\{\int_{\mathfrak{P}_1^{\beta}} u^{\beta nm-1}h(u^2)du\right\}$$

Such a distribution is denoted by $\mathbf{X} \sim \mathcal{E}_{n\times m}^{\beta}(\boldsymbol{\mu},\mathbf{\Sigma},\mathbf{\Theta},h)$, for the real case see Fang and Zhang (1990) and Gupta, and Varga (1993) and Micheas *et al.* (2006) for the complex case. Observe that this class of matrix multivariate distributions includes normal, contaminated normal, Pearson type II and VI, Kotz, Jensen-Logistic, power exponential and Bessel distributions, among others; these distributions have tails that are more or less weighted, and/or present a greater or smaller degree of kurtosis than the normal distribution.

Theorem 3.1. *Let $\mathbf{S} = \mathbf{X}^*\mathbf{\Theta}^{-1}\mathbf{X} \in \mathfrak{P}_m^\beta$. $\mathbf{S}$ is said to have a generalised Wishart distribution for a real normed division algebra, this fact being denoted as $\mathbf{S} \sim \mathcal{GW}_m^\beta(n, \mathbf{\Sigma}, \mathbf{\Omega}, h)$. Moreover, its density function is*

$$\frac{\pi^{\beta mn/2} C^\beta(m,n)}{\Gamma_m^\beta[\beta n/2] |\mathbf{\Sigma}|^{\beta n/2}} |\mathbf{S}|^{\beta(n-m+1)/2-1} \sum_{k=0}^{\infty} \frac{h^{(2k)}\left(\operatorname{tr} \mathbf{\Sigma}^{-1}\mathbf{S} + \mathbf{\Omega}\right)}{k!} \frac{C_\kappa^\beta\left(\mathbf{\Omega}\mathbf{\Sigma}^{-1}\mathbf{S}\right)}{[\beta n/2]_\kappa^\beta} \tag{7}$$

where $\mathbf{\Omega} = \mathbf{\Sigma}^{-1}\boldsymbol{\mu}^*\mathbf{\Theta}^{-1}\boldsymbol{\mu}$ and $h^{(j)}(\cdot)$ is the jth derivative of h with respect to $v = \operatorname{tr} \mathbf{\Sigma}^{-1}\mathbf{S}$.

Proof. Let $\mathbf{S} = \mathbf{X}^*\mathbf{\Theta}^{-1}\mathbf{X} = \mathbf{Y}^*\mathbf{Y}$, where

$$\mathbf{Y} = \mathbf{\Theta}^{-1/2}\mathbf{X} \sim \mathcal{E}_{n\times m}^\beta(\mathbf{\Theta}^{-1/2}\boldsymbol{\mu}, \mathbf{\Sigma}, \mathbf{I}_m, h),$$

with $(\mathbf{\Theta}^{1/2})^2 = \mathbf{\Theta}$, and so

$$f_{\mathbf{Y}}(\mathbf{Y}) = \frac{C^\beta(m,n)}{|\mathbf{\Sigma}|^{\beta n/2}} h\left[\operatorname{tr} \mathbf{\Sigma}^{-1}(\mathbf{Y}-\boldsymbol{\mu})^*(\mathbf{Y}-\boldsymbol{\mu})\right]$$

Let us now consider the singular value decomposition of matrix $\mathbf{Y} = \mathbf{V}_1\mathbf{D}\mathbf{W}^*$. Then by Lemma 2.1, the joint density function of $\mathbf{V}_1$, $\mathbf{D}$ and $\mathbf{W}$ is

$$\begin{aligned} &\frac{2^{-m}C^\beta(m,n)\pi^\tau \displaystyle\prod_{i=1}^m d_i^{\beta(n-m+1)-1} \prod_{i<j}^m (d_i^2 - d_j^2)^\beta}{|\mathbf{\Sigma}|^{\beta n/2}} \\ &\qquad \times h\left[\operatorname{tr}\left(\mathbf{\Sigma}^{-1}\mathbf{W}\mathbf{D}^2\mathbf{W}^* + \mathbf{\Omega}\right) + \operatorname{tr}\left(-2\boldsymbol{\mu}_{\mathbf{Y}}\mathbf{\Sigma}^{-1}\mathbf{W}\mathbf{D}\mathbf{V}_1^*\right)\right] \\ &\qquad \times (\mathbf{W}^*d\mathbf{W})(d\mathbf{D})(\mathbf{V}_1^*d\mathbf{V}_1), \end{aligned} \tag{8}$$

where $\boldsymbol{\mu}_{\mathbf{Y}} = \mathbf{\Theta}^{-1/2}\boldsymbol{\mu}$ and $\mathbf{\Sigma}^{-1}\boldsymbol{\mu}^*\mathbf{\Theta}^{-1}\boldsymbol{\mu}$. Let us now assume that h can be expanded in series of power, that is

$$h(v+a) = \sum_{k=0}^{\infty} \frac{h^{(k)}(a)v^k}{k!}.$$

Hence, considering only h in (8)

$$= \sum_{k=0}^{\infty} \frac{h^{(k)}\left[\operatorname{tr}\left(\mathbf{\Sigma}^{-1}\mathbf{W}\mathbf{D}^2\mathbf{W}^* + \mathbf{\Omega}\right)\right]}{k!} \left(\operatorname{tr}\left(-2\boldsymbol{\mu}_{\mathbf{Y}}\mathbf{\Sigma}^{-1}\mathbf{W}\mathbf{D}\mathbf{V}_1^*\right)\right)^k.$$

And from Lemma 2.4 noting that (6) is zero for all odd k,

$$\begin{aligned} &\int_{\mathbf{H}\in\mathcal{V}_{m,n}^\beta} \left[\operatorname{tr}\left(-2\operatorname{tr}\boldsymbol{\mu}_{\mathbf{Y}}^*\mathbf{\Sigma}^{-1}\mathbf{W}\mathbf{D}\mathbf{V}_1^*\right)\right]^{2k} (\mathbf{V}_1^*d\mathbf{V}_1) \\ &\qquad\qquad = \frac{2^m\pi^{\beta mn/2}}{\Gamma_m[\beta n/2]} \sum_\kappa \frac{\left(\frac{1}{2}\right)_k 4^k}{[\beta n/2]_\kappa^\beta} C_\kappa\left(\mathbf{\Omega}\mathbf{\Sigma}^{-1}\mathbf{W}\mathbf{D}^2\mathbf{W}^*\right). \end{aligned}$$

Observing that $\left(\frac{1}{2}\right)_k 4^k/(2k)! = 1/k!$, the joint density function of $\mathbf{D}$ and $\mathbf{W}$ is

$$\frac{\pi^{\beta mn/2+\tau} C^\beta(m,n) \prod_{i=1}^m d_i^{\beta(n-m+1)-1} \prod_{i<j}^m (d_i^2-d_j^2)^\beta}{\Gamma_m[\beta n/2] |\mathbf{\Sigma}|^{\beta n/2}}$$
$$\times \sum_{k=0}^\infty \frac{h^{(2k)}\left[\operatorname{tr}\left(\mathbf{\Sigma}^{-1}\mathbf{W}\mathbf{D}^2\mathbf{W}^* + \mathbf{\Omega}\right)\right]}{k!} \sum_\kappa \frac{C_\kappa\left(\mathbf{\Omega}\mathbf{\Sigma}^{-1}\mathbf{W}\mathbf{D}^2\mathbf{W}^*\right)}{[\beta n/2]_\kappa^\beta}$$
$$\times (\mathbf{W}^* d\mathbf{W})(d\mathbf{D}).$$

Finally, let $\mathbf{S} = \mathbf{Y}^*\mathbf{Y} = \mathbf{W}\mathbf{D}^2\mathbf{W}^*$. The desired result is obtained from lemmas 2.2 and 2.3, noting that $(d\mathbf{D}) = 2^{-m}|\mathbf{D}^2|^{-1/2}(d\mathbf{D}^2)$ and $\prod_{i=1}^m d_i^2 = |\mathbf{S}|$. □ □

Distribution (7) was found by Díaz-García and Gutiérrez-Jáimez (2009) for the real case and for the general central case by Díaz-García and Gutiérrez-Jáimez (2009).

Corollary 3.1. *Assume that* $\mathbf{X}$ *is a matrix multivariate normal distribution for real normed division algebras. Then* $\mathbf{S} = \mathbf{X}^*\mathbf{\Theta}^{-1}\mathbf{X}$ *has a Wishart distribution for real normed division algebras and is denoted as* $\mathbf{S} \sim \mathcal{W}_m^\beta(n, \mathbf{\Sigma}, \mathbf{\Omega})$. *Moreover its density is*

$$\frac{1}{(2/\beta)^{\beta mn/2}\Gamma_m^\beta[\beta n/2]|\mathbf{\Sigma}|^{\beta n/2}} |\mathbf{S}|^{\beta(n-m+1)/2-1} \operatorname{etr}\{-\beta\left(\mathbf{\Sigma}^{-1}\mathbf{S}+\mathbf{\Omega}\right)/2\}$$
$$\times {}_0F_1^\beta\left(\beta n/2; \beta^2\mathbf{\Omega}\mathbf{\Sigma}^{-1}\mathbf{S}/4\right). \qquad (9)$$

Proof. This follows from (7) taking into account that for the normal case, $h(u) = \exp\{-\beta u/2\}$ and $C^\beta(m,n) = (2\pi/\beta)^{-\beta mn/2}$. □ □

This result has been found by Herz (1955) and James (1961) for the real case; by James (1964), Khatri (1965) and Ratnarajah *et al.* (2005) for the complex case and by Li and Xue (2009) for the central quaternion case, among other authors. The general central case of (9) was found by Díaz-García and Gutiérrez-Jáimez (2009).

From Theorems 2.1 and 3.1 it is straightforward to obtain the distribution of $\mathbf{S}^{-1}$, termed the inverse generalised Wishart distribution.

Corollary 3.2. *In Theorem 3.1 we define* $\mathbf{W} = \mathbf{S}^{-1}$. *Then its density function is*

$$\frac{\pi^{\beta mn/2}C^\beta(m,n)|\mathbf{W}|^{\beta(n+m+1)/2-3}}{\Gamma_m^\beta[\beta n/2]|\mathbf{\Sigma}|^{\beta n/2}} \sum_{k=0}^\infty \frac{h^{(2k)}\left(\operatorname{tr}\mathbf{\Sigma}^{-1}\mathbf{W}^{-1}+\mathbf{\Omega}\right)}{k!} \frac{C_\kappa^\beta\left(\mathbf{\Omega}\mathbf{\Sigma}^{-1}\mathbf{W}^{-1}\right)}{[\beta n/2]_\kappa^\beta} \qquad (10)$$

The density (10) was found by Ip, Wong, and Liu (2007) in the real case.

4 Eigenvalue densities

In this section we find the general joint density function of the eigenvalues of $\mathbf{S}$ and the density of λ_{max} for the normal case.

Theorem 4.1. *Assume that* $\mathbf{S} \sim \mathcal{GW}_m^\beta(n, \mathbf{\Sigma}, \mathbf{\Omega}, h)$. *Then the joint density of eigenvalues* $\lambda_1, \dots, \lambda_m > 0$, *of* $\mathbf{S}$ *is*

$$\frac{\pi^{\beta(mn+m^2)/2+\tau} C^\beta(m,n) \displaystyle\prod_{i=1}^m \lambda_i^{\beta(n-m+1)/2-1} \prod_{i<j}^m (\lambda_i - \lambda_j)^\beta}{\Gamma_m^\beta[\beta n/2]\Gamma_m^\beta[\beta m/2]|\mathbf{\Sigma}|^{\beta n/2}}$$
$$\times \sum_{k,l=0}^\infty \sum_{\kappa,\delta;\phi\in\kappa\cdot\delta} \frac{h^{(2k+l)}(\operatorname{tr}\mathbf{\Omega})\theta_\phi^{[\beta]\kappa,\delta}}{k!l![\beta n/2]_\kappa^\beta} \frac{C_\phi^\beta(\mathbf{\Lambda})C_\phi^{[\beta]\kappa,\delta}(\mathbf{\Sigma}^{-1}, \mathbf{\Omega}\mathbf{\Sigma}^{-1})}{C_\phi^\beta(\mathbf{I})}. \tag{11}$$

Proof. Let $\mathbf{S} = \mathbf{W}\mathbf{\Lambda}\mathbf{W}^*$ the spectral decomposition of $\mathbf{S}$, where $\mathbf{W} \in \mathfrak{U}^\beta(m)$ and $\mathbf{\Lambda} = \operatorname{diag}(\lambda_1, \dots, \lambda_m) \in \mathfrak{D}^1(m)$, $\lambda_1 > \cdots > \lambda_m > 0$. Then by (7) and Lemma 2.2, the marginal density function of $\mathbf{\Lambda}$ is

$$\frac{2^{-m}\pi^{\beta mn/2+\tau} C^\beta(m,n) \displaystyle\prod_{i=1}^m \lambda_i^{\beta(n-m+1)/2-1} \prod_{i<j}^m (\lambda_i - \lambda_j)^\beta}{\Gamma_m^\beta[\beta n/2]|\mathbf{\Sigma}|^{\beta n/2}} \sum_{k=0}^\infty \frac{1}{k![\beta n/2]_\kappa^\beta}$$
$$\times \int_{\mathbf{W}\in\mathfrak{U}^\beta(m)} h^{(2k)}\left(\operatorname{tr}\mathbf{\Sigma}^{-1}\mathbf{W}\mathbf{\Lambda}\mathbf{W}^* + \mathbf{\Omega}\right) C_\kappa^\beta\left(\mathbf{\Omega}\mathbf{\Sigma}^{-1}\mathbf{W}\mathbf{\Lambda}\mathbf{W}^*\right)(\mathbf{W}^* d\mathbf{W}). \tag{12}$$

By denoting the integral in (12) by J and the expanding $h^{(2k)}$ into series of powers, we have

$$\begin{aligned}
J &= \sum_{l=0}^\infty \frac{h^{(2k+l)}(\operatorname{tr}\mathbf{\Omega})}{l!} \\
&\quad \times \int_{\mathbf{W}\in\mathfrak{U}^\beta(m)} \left(\operatorname{tr}\mathbf{\Sigma}^{-1}\mathbf{W}\mathbf{\Lambda}\mathbf{W}^*\right)^l C_\kappa^\beta\left(\mathbf{\Omega}\mathbf{\Sigma}^{-1}\mathbf{W}\mathbf{\Lambda}\mathbf{W}^*\right)(\mathbf{W}^* d\mathbf{W}) \\
&= \sum_{l=0}^\infty \sum_\delta \frac{h^{(2k+l)}(\operatorname{tr}\mathbf{\Omega})}{l!} \\
&\quad \times \int_{\mathbf{W}\in\mathfrak{U}^\beta(m)} C_\delta^\beta\left(\mathbf{\Sigma}^{-1}\mathbf{W}\mathbf{\Lambda}\mathbf{W}^*\right) C_\kappa^\beta\left(\mathbf{\Omega}\mathbf{\Sigma}^{-1}\mathbf{W}\mathbf{\Lambda}\mathbf{W}^*\right)(\mathbf{W}^* d\mathbf{W}) \\
&= \frac{2^m\pi^{\beta m^2/2}}{\Gamma_m^\beta[\beta m/2]} \\
&\quad \times \sum_{l=0}^\infty \sum_{\delta;\phi\in\kappa\cdot\delta} \frac{h^{(2k+l)}(\operatorname{tr}\mathbf{\Omega})\theta_\phi^{[\beta]\kappa,\delta}}{l!} \frac{C_\phi^\beta(\mathbf{\Lambda}) C_\phi^{[\beta]\kappa,\delta}\left(\mathbf{\Sigma^{-1}}, \mathbf{\Omega}\mathbf{\Sigma}^{-1}\right)}{C_\phi^\beta(\mathbf{I})}.
\end{aligned} \tag{13}$$

The last equality is obtained by applying Díaz-García (2009, eq. (5.1)). The desired result then follows by substituting (13) in (12). □ □

In the real case (11) was obtained by Díaz-García and Gutiérrez-Jáimez (2009).

Corollary 4.1. *Assume that* $\mathbf{S} \sim \mathcal{W}_m^\beta(n, \mathbf{\Sigma}, \mathbf{\Omega})$. *Then the joint density of eigenvalues* $\lambda_1, \dots, \lambda_m > 0$, *of* $\mathbf{S}$ *is*

$$\frac{\pi^{\beta m^2/2+\tau} \displaystyle\prod_{i=1}^m \lambda_i^{\beta(n-m+1)/2-1} \prod_{i<j}^m (\lambda_i - \lambda_j)^\beta \operatorname{etr}\{-\beta\mathbf{\Omega}/2\}}{(2/\beta)^{\beta mn/2}\Gamma_m^\beta[\beta n/2]\Gamma_m^\beta[\beta m/2]|\mathbf{\Sigma}|^{\beta n/2}}$$
$$\times \sum_{k,l=0}^\infty \sum_{\kappa,\delta;\phi\in\kappa\cdot\delta} \frac{\theta_\phi^{[\beta]\kappa,\delta}}{k!l![\beta n/2]_\kappa^\beta} \frac{C_\phi^\beta(\mathbf{\Lambda})C_\phi^{[\beta]\kappa,\delta}(-\beta\mathbf{\Sigma}^{-1}/2, \beta^2\mathbf{\Omega}\mathbf{\Sigma}^{-1}/4)}{C_\phi^\beta(\mathbf{I})}$$

Proof. The proof follows from (11) taking into account that for the normal case, $h(u) = \exp\{-\beta u/2\}$ and $C^\beta(m,n) = (2\pi/\beta)^{-\beta mn/2}$ from where $h^{(2k+l)}(u) = (-\beta/2)^{2k+l}\exp\{-\beta u/2\}$ and observing that $C_\phi^{[\beta]\kappa,\delta}(a\mathbf{A}, b\mathbf{B}) = a^k b^l C_\phi^{[\beta]\kappa,\delta}(\mathbf{A},\mathbf{B})$, see Díaz-García (2009, eq. (5.8)). □ □

The result in Corollary 4.1 was obtained by Davis (1980) for the real case; by Ratnarajah *et al.* (2005) for the complex case, and by Li and Xue (2009) for the central quaternion case.

Theorem 4.2. *Let* $\mathbf{\Delta} \in \mathbf{\Phi}$ *and consider that* $\mathbf{S} \sim \mathcal{W}_m^\beta(n, \mathbf{\Sigma}, \mathbf{\Omega})$ *then the probability* $¶[\mathbf{S} < \mathbf{\Delta}]$ *is*

$$\frac{\Gamma_m^\beta[(m-1)\beta/2+1]|\mathbf{\Delta}|^{\beta n/2}\operatorname{etr}\{-\beta\mathbf{\Omega}/2\}}{(2/\beta)^{\beta mn/2}\Gamma_m^\beta[\beta(n+m-1)/2+1]|\mathbf{\Sigma}|^{\beta n/2}} \times \sum_{k,l=0}^{\infty}\sum_{\kappa,\delta;\phi\in\kappa\cdot\delta}\frac{[\beta n/2]_\phi^\beta}{k!l![\beta n/2]_\kappa^\beta}\frac{\theta_\phi^{[\beta]\kappa,\delta}C_\phi^{[\beta]\kappa,\delta}(-\beta\mathbf{\Sigma}^{-1}\mathbf{\Delta}/2, \beta^2\mathbf{\Omega}\mathbf{\Sigma}^{-1}\mathbf{\Delta}/4)}{[\beta(n+m-1)/2+1]_\phi^\beta}.$$

Proof. The proof follows from (9) and Theorem 2.2. □ □

Note that $\lambda_{\max} < y$ is equivalent to $\mathbf{S} < y\mathbf{I}$. Therefore, the distribution of $\lambda_{\max}$ is obtained by letting $\mathbf{\Delta} = y\mathbf{I}$ in Theorem 4.2, and hence:

Corollary 4.2. *Assume that* $\mathbf{S} \sim \mathcal{W}_m^\beta(n, \mathbf{\Sigma}, \mathbf{\Omega})$ *and let* $\lambda_{\max}$ *be the maximum eigenvalue of* $\mathbf{S}$. *Then* $¶[\lambda_{\max} < y]$ *is*

$$\frac{\Gamma_m^\beta[(m-1)\beta/2+1]y^{\beta mn/2}\operatorname{etr}\{-\beta\mathbf{\Omega}/2\}}{(2/\beta)^{\beta mn/2}\Gamma_m^\beta[\beta(n+m-1)/2+1]|\mathbf{\Sigma}|^{\beta n/2}} \times \sum_{k,l=0}^{\infty}\sum_{\kappa,\delta;\phi\in\kappa\cdot\delta}\frac{[\beta n/2]_\phi^\beta}{k!l![\beta n/2]_\kappa^\beta}\frac{\theta_\phi^{[\beta]\kappa,\delta}y^{k+l}C_\phi^{[\beta]\kappa,\delta}(-\beta\mathbf{\Sigma}^{-1}/2, \beta^2\mathbf{\Omega}\mathbf{\Sigma}^{-1}/4)}{[\beta(n+m-1)/2+1]_\phi^\beta}.$$

This latter result was found by Ratnarajah *et al.* (2005) for the complex case.

Conclusions

As shown in this paper, it is possible to take a general approach to the theory of distributions, for the real, complex quaternion and octonion cases simultaneously. However, any generalisation entails a cost, in this case the need to use some concepts, definitions and notation from abstract algebra. Thus, any reader interested in a particular case – real, complex, quaternions or octonions – need simply take the particular value of β in order to obtain the results desired.

In summary, the noncentral generalised Wishart distribution, the joint distribution of the eigenvalues and the maximum eigenvalue distribution are found under a unified approach that allows the simultaneous study of the real, complex, quaternion and octonion cases, generically termed distributions for real normed division algebras.

Acknowledgements

This research work was partially supported by CONACYT-México, Research Grant No. 81512 and IDI-Spain, Grants No. FQM2006-2271 and MTM2008-05785. This paper was written during J. A. Díaz-García's stay as a visiting professor at the Department of Statistics and O. R. of the University of Granada, Spain.

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