A further generalization of random self-decomposability

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Abstract

The notion of random self-decomposability is generalized further. The notion is then extended to non-negative integer-valued distributions.
Keywords: self-decomposability, random self-decomposability, random infinite divisibility, geometric infinite divisibility, Harris infinite divisibility, geometric distribution, Harris distribution, Laplace transform, characteristic function.

1. Introduction

Recently the notion of random self-decomposability (RSD) has been introduced by Kozubowski and Podgórski [4] generalizing SD. They showed that if a CF is RSD then it is both SD and geometrically infinitely divisible (GID). Satheesh and Sandhya [9] generalized this notion to Harris-RSD ($HRSD$) and showed that if a CF is $HRSD$ then it is both SD and Harris-ID (HID). With this nomenclature RSD is geometric-RSD ($GRSD$). Here we explore further generalizations of $HRSD$ *viz.* $\mathcal{N}RSD$ and φRSD, motivated by the elegent Proposition 2.3 in Kozubowski and Podgórski [4].

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We need the notion of $\mathcal{N}$-infinitely divisible ($\mathcal{N}ID$) laws here. Let φ be a Laplace transform (LT) that is also a standard solution to the Poincare equation, $\varphi(t) = P(\varphi(\theta t)), \theta \in \Theta$ where P is a probability generating function (PGF) (*see* Gnedenko and Korolev, [3], p.140).

Definition 1.1. Let φ be a standard solution to the Poincare equation and N_θ a positive integer-valued random variable ($r.v.$) having finite mean with PGF $P_\theta(s) = \varphi(\frac{1}{\theta}\varphi^{-1}(s))$, $\theta \in \Theta \subseteq (0,1)$. A characteristic function (CF) $f(t)$ is $\mathcal{N}ID$ if for each $\theta \in \Theta$ there exists a CF $f_\theta(t)$ that is independent of N_θ such that $f(t) = P_\theta(f_\theta(t))$, for all $t \in \mathbf{R}$.

Theorem 1.1. *(Gnedenko and Korolev, 1996, Theorem 4.6.3 on p.147) [3] Let φ be a standard solution to the Poincare equation. A CF $f(t)$ is $\mathcal{N}ID$ iff it admits the representation $f(t) = \varphi(-\log h(t))$ where $h(t)$ is a CF that is ID. $f(t)$ is $\mathcal{N}$stable if $h(t)$ is stable (p.151, [3]).*

In the next section we describe $\mathcal{N}RSD$ laws and its discrete analogue in Section 3. In Section 4 we describe φRSD laws and its discrete analogue.

2. $\mathcal{N}RSD$ distributions

Definition 2.1. A CF $f(t)$ is $\mathcal{N}RSD$ if for each $c \in (0,1]$ and each $\theta \in [0,1)$

$$f_{c,\theta}(t) = f_c(t).f_\theta(ct) \tag{1}$$

is a CF, where $f_c(t)$ and $f_\theta(t)$ are given by

$$f_c(t) = \frac{f(t)}{f(ct)} \tag{2}$$

$$f_\theta(t) = \varphi\{\theta\varphi^{-1}(f(t))\}, \tag{3}$$

φ being a standard solution to the Poincare equation.

We now notice that the discussion leading to conceiving and proving Proposition 2.3 in Kozubowski and Podgórski [4] holds in this generalization as well. When $c = 1$ equation (1) becomes

$$f_{1,\theta}(t) = f_\theta(t) = \varphi\{\theta\varphi^{-1}(f(t))\} \tag{4}$$

Or

$$f(t) = \varphi\{\frac{1}{\theta}\varphi^{-1}(f_\theta(t))\} \tag{5}$$

for each $\theta \in [0, 1)$. That is $f(t)$ is $\mathcal{N}ID$ and hence has no real zeroes. On the other hand since $\varphi(0) = 1$, when $\theta = 0$ equation (1) implies

$$f_{c,0}(t) = f_c(t) = \frac{f(t)}{f(ct)} \tag{6}$$

is a CF for each $c \in (0, 1]$. That is $f(t)$ is SD.

Conversely, if $f(t)$ is SD then for each $c \in (0, 1]$ the function $f_c(t)$ in (2) is a genuine CF and similarly if $f(t)$ is $\mathcal{N}ID$ then for each $\theta \in [0, 1)$ the function $f_\theta(t)$ in (5) also is a genuine CF. Consequently (1) is a well defined CF.

Remark 2.1 It may be noted that for the CF $f(t)$ to be SD we only require that (a result due to Biggins and Shanbhag *s*ee Fosum [2]) (2) holds for all c in some left neighbourhood of 1. Thus we may simplify the requirement here as: A CF $f(t)$ is $\mathcal{N}RSD$ if for each $c \in (a, 1]$, and each $\theta \in [0, 1)$ (1) holds, where $0 < a < 1$.

Remark 2.2 In fact we may have apparently still weaker requirement in describing CFs that are $\mathcal{N}RSD$ as follows. A CF $f(t)$ is $\mathcal{N}RSD$ if for each $c \in (a, 1)$, and each $\theta \in (0, 1)$ (1) holds, where $0 < a < 1$. Now letting $c \uparrow 1$ we have $f(t)$ is $\mathcal{N}ID$. On the other hand letting $\theta \downarrow 0$ we have $f(t)$ is SD since $lim_{\theta\downarrow 0} f_\theta(t) = 1$, *see e.g* Gnedenko and Korolev [3], page 149.

Example 2.1 For the LT $\varphi(s) = (1+s)^{-\alpha}, \alpha > 0$, $\varphi(\varphi^{-1}(s)/p)$ is a PGF of a non-degenerate distribution only if $\alpha = \frac{1}{k}, k \geq 1$ integer, see Example 1 in Bunge [1] or Corollary 4.5 in Satheesh *et al.* [6]. This PGF is that of Harris distribution (Satheesh et al. [7]) and the corresponding $\mathcal{N}RSD$ distribution is $HRSD$. When $k = 1$ above, we have $GRSD$ (RSD distributions of Kozubowski and Podgórski [4]).

Example 2.2 Invoking Theorem 1.1 when $\varphi(s)$ is SD and $\log h(t) = -\lambda|t|^\alpha$ we have, for each $c \in (a, 1]$

$$f(t) = \varphi(|t|^\alpha) = \varphi(c|t|^\alpha).\varphi_c(|t|^\alpha). \tag{7}$$

That is $f(t)$ is both SD and $\mathcal{N}$-strictly stable. Thus we have a good collection of CFs that are both SD and HID and thus $HRSD$. Kozubowski and Podgórski [4]) present examples of a variety of CFs $h(t)$ that are stable.

3. Discrete analogue of $\mathcal{N}RSD$ distributions

Steutel and van Harn [10] had described discrete SD (DSD) distributions. Satheesh and Sandhya [9] have described $DHRSD$, discrete analogue of $HRSD$ distributions. We now introduce discrete $\mathcal{N}RSD$ ($D\mathcal{N}RSD$) distributions.

Definition 3.1. (Satheesh *et al.* [7]) Let φ be a standard solution to the Poincare equation and N_θ a positive integer-valued *r.v.* having finite mean with PGF $P_\theta(s) = \varphi(\frac{1}{\theta}\varphi^{-1}(s))$, $\theta \in \Theta \subseteq (0, 1)$. A PGF $P(s)$ is $D\mathcal{N}ID$ if for each $\theta \in \Theta$ there exists a PGF $Q_\theta(s)$ that is independent of N_θ such that $P(s) = P_\theta(Q_\theta(s))$, for all $|s| \leq 1$.

Theorem 3.1. *(Satheesh et al. [7]) Let φ be a standard solution to the Poincare equation. A PGF $P(s)$ is $D\mathcal{N}ID$ iff it admits the representation $P(s) = \varphi(-\log R(s))$ where $R(s)$ is a PGF that is DID.*

Definition 3.2. A PGF $P(s)$ is $D\mathcal{N}RSD$ if for each $c \in (0,1]$ and each $\theta \in [0,1)$

$$P_{c,\theta}(s) = P_c(s).Q_\theta(1 - c + cs) \tag{8}$$

is a PGF, where $P_c(s)$ and $Q_\theta(s)$ are given by

$$P_c(s) = \frac{P(s)}{P(1 - c + cs)} \tag{9}$$

$$Q_\theta(s) = \varphi\{\theta\varphi^{-1}(P(s))\}, \tag{10}$$

φ being a standard solution to the Poincare equation.

We may now proceed as in Section 2 describing the relation between DSD, $D\mathcal{N}ID$ and $D\mathcal{N}RSD$ distributions. Further, remarks similar to Remarks 2.1 and 2.2 are relevant here also and Examples on the lines of Example 2.1 nad 2.2 can also be discussed.

4. φRSD distributions

A further generalization of $\mathcal{N}RSD$ distributions is possible invoking the notion of φID law that generalizes $\mathcal{N}ID$ laws, see Satheesh [5] and Satheesh *et al.* [7], [8]) for its discrete analogue. We first describe the discrete case.

Definition 4.1. (Satheesh *et al.* [7]) Let φ be a LT. A PGF $P(s)$ is $D\varphi ID$ if there exists a sequence $\{\theta_n\} \downarrow 0$ as $n \to \infty$ and a sequence of PGFs $Q_n(s)$ such that

$$P(s) = lim_{n\to\infty}\varphi(\frac{1 - Q_n(s)}{\theta_n}). \tag{11}$$

Theorem 4.1. *(Satheesh et al. [8]) Let $\{Q_\theta(s), \theta \in \Theta\}$ be a family of PGFs and φ a LT. Then*

$$lim_{\theta\downarrow 0}\varphi(\frac{1-Q_\theta(s)}{\theta}) \tag{12}$$

exists and is $D\varphi ID$ iff there exists a PGF $R(s)$ that is DID such that

$$lim_{\theta\downarrow 0}\frac{1-Q_\theta(s)}{\theta} = -\log R(s) \tag{13}$$

Definition 4.2. A PGF $P(s)$ is $D\varphi RSD$ if for each $c \in (a, 1)$ and each $\theta \in (0, b), 0 < a, b < 1$

$$P_{c,\theta}(s) = P_c(s).Q_\theta(1 - c + cs) \tag{14}$$

is a PGF, where $P_c(s)$ and $Q_\theta(s)$ are given by

$$P_c(s) = \frac{P(s)}{P(1-c+cs)} \tag{15}$$

$$Q_\theta(s) = 1 - \theta\varphi^{-1}(P(s)). \tag{16}$$

The restriction of $\alpha = \frac{1}{k}, k \geq 1$ integer in Example 2.1 is not in this notion. We may now proceed as in Section 3 describing the relation between DSD, $D\varphi ID$ and $D\varphi RSD$ distributions. This has been possible since $lim_{\theta\downarrow 0}Q_\theta(t) = 1$. The case of φRSD follows on similar lines.

References

[1] Bunge, J (1996), Composition semi-groups and random stability, *Ann. Probab.*, **24**, 1476–1489.

[2] Fosum, E B (1995), A characterization of discrete self-decomposable distributions in terms of survival distributions and the self-decomposability of discrete logarithmic distribution, *Sankhya*, **57**, 317–341.

[3] Gnedenko, B V and Korolev, V Yu. (1996), Random Summation, limit Theorems and Applications, CRC Press, Boca Raton.

[4] Kozubowski, T J and Podgórski, K (2010), Random self-decomposability and autoregressive processes, *Statis. Probab. Lett.*, doi:10.1016/j.spl.2010.06.014.

[5] Satheesh, S (2004), Another look at random infinite divisibility, *Statist. Meth.*, **6**, 123–144.

[6] Satheesh, S; Nair, N U and Sandhya, E (2002), stability of random sums, *Stoch. Model. Appl.*, **5**, 17–26.

[7] Satheesh, S; Sandhya, E and Lovely, A T (2010a), Limit distributions of random sums of $\mathbf{Z}_+$-valued random variables, *Commu. Statist.-Theor. Meth.*, **39**, 1979–1984.

[8] Satheesh, S; Sandhya, E and Lovely, A T (2010b), Random infinite divisibility on $\mathbf{Z}_+$ and generalized INAR models, *ProbStat Forum*, **3**, 108–117.

[9] Satheesh, S and Sandhya, E (2010), A generalization of random self-decomposability, *submitted*, http://arxiv.org/abs/1009.5141v1.

[10] Steutel, F W and van Harn, K (1979), Discrete analogues of selfdecomposability and stability, *Ann. Probab.*, **7**, 893–899.