Phase-Shift Definition for Pion-Nucleon Scattering from Chiral Perturbation Theory

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Chiral Perturbation Theory is considered as a very precise method when applied to pion-nucleon scattering near threshold and in the unphysical region since in these cases the pion momenta are small. In this framework, third order calculation yields a non-relativistic scattering amplitude with nine free parameters. From the fact that the resulting partial-wave amplitudes do not respect elastic unitarity relation, one has that the phase-shift definition is ambiguous. In this article, we present the comparison of the model with experimental data for two different phase-shift formulas and we conclude that the results are very sensitive to phase-shift definition.

1 Introduction

Although QCD is widely accepted as the fundamental gauge theory underlying the strong interactions, we still lack the analytical tools for *ab initio* descriptions of low-energy properties and processes. However, new techniques have been developed to extend the results of the current-algebra days and *systematically* explore corrections to the soft-pion predictions based on symmetry properties of QCD Green functions.

The starting point is a theorem by Weinberg stating that a perturbative description in terms of the most general effective Lagrangian containing all possible terms compatible with assumed symmetry principles yields the most general S matrix consistent with the fundamental principles of quantum field theory and the assumed symmetry principles [1].

The method to go beyond the soft-pion predictions is called chiral perturbation theory (ChPT) [2] and describes the dynamics of Goldstone bosons in the framework of an effective field theory, which provides a systematic method for discussing the consequences of the global flavor symmetries of QCD at low energies.

This method has been sucessfully applied to meson decay and meson-meson scattering, with the help of Weinberg's power counting scheme. This scheme establishes that any given diagram behaves as E^D, where $D \geq 2$ is determined by the structure of the vertices and the topology of the diagram in question. For a given value of D, Weinberg's formula unambiguously determines to which order in the momentum and quark mass expansion the Lagrangian needs to be known.

Chiral Perturbation Theory faces problems when baryons are to be introduced in the formalism. This comes about because the nucleon mass is not small even in the chiral limit and thus the characteristic parameter $m_\pi/4\pi f_\pi$ does not control the low energy expansion any more. A method called Heavy Baryon Chiral Perturbation Theory (HBChPT) was invented to allow one to introduce static baryons in ChPT formalism [3] and to describe the dynamics of baryons at low energies: static properties such as masses or magnetic moments, form factors, or, eventually, more complicated processes, such as pion-nucleon scattering, Compton scattering, pion photoproduction etc.

In this paper we analyse low energy pion-nucleon phase-shifts derived from HBChPT. Working at order $\mathcal{O}(p^3)$, the resulting amplitude[4] depends on *nine* free parameters which can be adjusted to fit the $S-$ and $P-$ partial-wave phase-shifts to the experimental data[5]. It is known that elastic unitarity is violated at that order of the calculation, so that phase-shift definition is arbitrary. From our present analysis we conclude that the fits are very sensitive to the phase-shift definition. In section II we present the method of HBChPT applied to pion-nucleon scattering and the resulting amplitudes. In section III we present the fitting procedure and the conclusions.

2 HBChPT and pion-nucleon scattering

We are interested in the baryon-to-baryon transition amplitude in the presence of external fields

$$\mathcal{F}(\vec{p}\,', \vec{p}; v, a, s, p) = \langle \vec{p}\,'; \mathrm{out}|\vec{p}; \mathrm{in}\rangle^{\mathrm{c}}_{v,a,s,p}, \quad \vec{p} \neq \vec{p}\,',$$

determined by the Lagrangian

$$\mathcal{L} = \mathcal{L}^0_{\rm QCD} + \mathcal{L}_{\rm ext} = \mathcal{L}^0_{\rm QCD} + \bar{q}\gamma_\mu(v^\mu + \gamma_5 a^\mu)q - \bar{q}(s - i\gamma_5 p)q.$$

The functional $\mathcal{F}$ consists of connected diagrams only. For example, the matrix elements of the axial-vector currents A (or similarly for vector currents V) between one-baryon states is given by

$$\langle \vec{p}\,'|A^{\mu,a}(x)|\vec{p}\rangle = \left.\frac{\delta}{i\delta a^a_\mu(x)}\mathcal{F}(\vec{p}\,',\vec{p};v,a,s,p)\right|_{v=0,a=0,s=M,p=0},$$

where $M = \mathrm{diag}(m_u, m_d, m_s)$ denotes the quark-mass matrix and

$$V^{\mu,a}(x) = \bar{q}(x)\gamma^\mu \frac{\lambda^a}{2} q(x), \quad A^{\mu,a}(x) = \bar{q}(x)\gamma^\mu\gamma_5 \frac{\lambda^a}{2} q(x).$$

The fields entering the Lagrangian are assumed to transform under irreducible representations of the subgroup H which leaves the vacuum invariant whereas the symmetry group G of the Hamiltonian or Lagrangian is nonlinearly realized.

The physical observables are invariant under field transformations, so that we choose Ψ to denote the nucleon field and U the SU(2) matrix containing the pion fields. We denote $u^2(x) = U(x)$, and define the nonlinear realization: $u'^2 = Ru^2L^+$ and $\Psi' = u'^{-1}R\Psi$. where R and L are matrices of $SU(2)_L \otimes SU(2)_R$ group.

The local character of the transformation implies that we need to introduce a covariant derivative by adding the connection

$$\Gamma_\mu = \frac{1}{2}\left[u^\dagger(\partial_\mu - ir_\mu)u + u(\partial_\mu - il_\mu)u^\dagger\right],$$

to the ordinary derivative. At $\mathcal{O}(p)$ there exists another Hermitian building block, the so-called vielbein

$$u_\mu \equiv i\left[u^\dagger(\partial_\mu - ir_\mu)u - u(\partial_\mu - il_\mu)u^\dagger\right],$$

which under parity transforms as an axial vector. The most general Lagrangian with the smallest number of derivatives is given by

$$\mathcal{L}^{(1)}_{\pi N} = \bar{\Psi}\left(i\not{D} - m + \frac{g_{\scriptscriptstyle A}}{2}\gamma^\mu\gamma_5 u_\mu\right)\Psi.$$

At this point we introduce the basic formalism. The transition amplitude for $\pi^a(q) + N(p) \to \pi^b(q') + N(p')$ scattering is given by

$$T^{ab} = \bar{u}(p')\left(A^{ab}(s,t,u) + \frac{1}{2}(q' + q'')\,B^{ab}(s,t,u)\right)u(p).$$

The nonrelativistic reduction is

$$T \simeq \chi'^+\left[A + \left(E_\pi + \frac{\vec{q}^{\,2} + \vec{q}^{\,'}\cdot\vec{q}}{2m}\right)B + i\frac{\vec{\sigma}\cdot\vec{q}^{\,'}\times\vec{q}}{2m}B\right]\chi.$$

The decomposition $A^{ab} = \delta^{ab}A^+ + \frac{1}{2}[\tau^b,\tau^a]A^-$, and similarly for B, obeys the following crossing properties: $A^\pm(s,t,u) = \pm A^\pm(u,t,s)$ and $B^\pm(s,t,u) = \mp B^\pm(u,t,s)$. The total isospin amplitudes are $A_{1/2} = A^+ + 2A^-$ and $A_{3/2} = A^+ - A^-$ and similarly for B.

For elastic scattering one must have

$$\mathrm{Im}\, f^\pm_{I\,\ell}(s) = |\vec{q}|\,|f^\pm_{I\,\ell}(s)|^2, \tag{1}$$

where $f^\pm_{I\,\ell}$ are the partial wave amplitudes constructed from the Dirac amplitudes.

Let us turn to the tree-level calculation to the πN scattering amplitude. Using the approximation

$$u \simeq 1 + i\frac{\vec{\tau}\cdot\vec{\phi}}{2F_0}, u_\mu \simeq -\frac{\vec{\tau}\cdot\partial_\mu\vec{\phi}}{F_0}, \Gamma_\mu \simeq \frac{i}{4F_0^2}\vec{\tau}\cdot\vec{\phi}\times\partial_\mu\vec{\phi},$$

we get the Lagrangian

$$\mathcal{L} = -\frac{1}{2}\frac{g_{\scriptscriptstyle A}}{F_0}\bar{\Psi}\gamma^\mu\gamma_5\tau^b\partial_\mu\phi^b\Psi - \frac{1}{4F_0^2}\bar{\Psi}\gamma^\mu\epsilon_{cde}\tau^c\phi^d\partial_\mu\phi^e\Psi.$$

From the nucleon pole contribution we obtain the resulting amplitudes

$$A^+ = \frac{g^2_{\scriptscriptstyle A}}{F_0^2}\,m,\ A^- = 0,\ B^+ = -\frac{g^2_{\scriptscriptstyle A} m}{F_0^2}\frac{\nu}{\nu^2 - \nu_B^2}$$

$$\text{and}\ \ B^- = \frac{1}{2F_0^2} - \frac{g^2_{\scriptscriptstyle A}}{2F_0^2} - \frac{g^2_{\scriptscriptstyle A} m}{F_0^2}\frac{\nu}{\nu^2 - \nu_B^2}.$$

We are going to show that this corresponds to the leading order heavy-baryon result.

The heavy-baryon formulation of ChPT consists of an expansion in terms of small external momentum. Clearly it cannot simply be the four-momentum of the initial and final nucleons, because their energy components are not small. Instead, one separates an external nucleon four-momentum into a large piece of the order of the nucleon mass and a small residual component.

Let us introduce the so-called velocity-dependent fields. Using v^μ with the properties $v^2 = 1$, $v^0 \geq 1$ and the projectors $2P_\pm = 1 \pm v$, we define

$$\mathcal{N}_v \equiv e^{imv\cdot x}P_{v+}\Psi,\ \text{and}\ \mathcal{H}_v \equiv e^{imv\cdot x}P_{v-}\Psi,$$

so that $\Psi(x) = e^{-imv\cdot x}\left[\mathcal{N}_v(x) + \mathcal{H}_v(x)\right]$. The fields $\mathcal{N}_v$ and $\mathcal{H}_v$ are often called the light and heavy components of the field Ψ.

The lowest-order Lagrangian is

$$\mathcal{L}^{(1)}_{\pi N} = \bar{\mathcal{N}_v}\left[iv\cdot D + g_{\scriptscriptstyle A} S_v\cdot u\right]\mathcal{N}_v,$$

where the spin matrix $S^\mu = \frac{i}{2}\gamma_5\sigma^{\mu\nu}v_\nu$ satisfies $v\cdot S = 0$. When comparing to the relativistic Lagrangian, one sees that the nucleon mass has disappeared from the leading-order Lagrangian. It only shows up in higher orders as powers of $1/m$. In the power counting scheme $\mathcal{L}^{(1)}$ counts as $\mathcal{O}(q)$, because the covariant derivative D_μ and the chiral vielbein u_μ both count as $\mathcal{O}(q)$. The four-momenta of the initial and final nucleons are written as $p = mv + k$ and $p' = mv + k'$, respectively, with $v\cdot k = 0 = v\cdot k'$ to leading order in $1/m$.

Using the expansions of the connection and the vielbein, the lowest order relevant part of the interaction Lagrangian is

$$\mathcal{L}_{\rm int} = -\frac{g_A}{F_0}\bar{\mathcal{N}} S^\mu \vec{\tau}\cdot\partial_\mu\vec{\phi}\mathcal{N} - \frac{1}{4F_0^2} v^\mu \bar{\mathcal{N}}\vec{\tau}\cdot\vec{\phi}\times\partial_\mu\vec{\phi}\mathcal{N}.$$

The corresponding Feynman rules for the vertices, for a single incoming pion with momentum q and isospin a and for an incoming pion with q, a and an outgoing pion with q', b, are, respectively:

$$-\frac{g_A}{F_0} S_v \cdot q\tau^a \text{ and } \frac{v\cdot(q+q')}{4F_0^2}\epsilon_{abc}\tau^c$$

the last gives rise to a contact term. The results for direct channel nucleon pole term effectively reduces to that of a two-component theory as in the Foldy-Wouthuysen transformation.

By performing a nonrelativistic reduction, one verifies that, at leading order in m, the relativistic Lagrangian and the heavy-baryon Lagrangian indeed generate the same πN scattering amplitude. This equivalence follows from an expansion of the functions $A^{(+)}$ and $B^{(+)}$ because both contain terms of order m.

So far we have concentrated on the leading-order, m-independent, heavy-baryon Lagrangian. However, it is clear that the Lagrangian also generates terms of higher order in $1/m$ and additional new chiral structures of higher orders in momentum. The $1/m$ correction resulting from the leading Lagrangian is

$$\begin{aligned}\mathcal{L}^{(2)} \;=\; & \frac{1}{2m}\bar{\mathcal{N}}\Bigg[(v\cdot D)^2 - D^2 - ig_A\{S\cdot D, v\cdot u\} - \\ & \frac{g_A^2}{4}(v\cdot u)^2 + \frac{1}{2}\epsilon^{\mu\nu\rho\sigma}v_\rho S_\sigma\,(iu_\mu u_\nu)\Bigg]\mathcal{N}.\end{aligned}$$

At $\mathcal{O}(q^2)$ the heavy-baryon Lagrangian contains another contribution which may be obtained as the projection of the relativistic Lagrangian onto the light components,

$$\bar{\mathcal{N}}\left[c_1 {\rm tr}(\chi_+) + c_2(v\cdot u)^2 + c_3 u\cdot u + c_4[S_v^\mu, S_v^\nu]u_\mu u_\nu\right]\mathcal{N},$$

where $\chi_\pm = u^\dagger\chi u^\dagger \pm u\chi^\dagger u$ and $\chi = 2Bm_q$.

The Lagrangian at order $\mathcal{O}(q^3)$ is computed in the tree approximation. Its various constants get renormalized in order to absorb the divergences of one loop calculation.

$$\begin{aligned}\bar{\Psi}\Bigg\{ & \frac{d_1+d_2}{4m}\Big([u_\mu,[D_\nu,u^\mu]]D^\nu + [u_\mu,[D^\mu,u_\nu]]D^\nu\Big) \\ & + \frac{d_3}{12m^3}[u_\mu,[D_\nu,u_\lambda]]D^\mu D^\nu D^\lambda + \frac{d_5}{2m}[\chi_-,u_\mu]D^\mu \\ & + \frac{d_{14}}{4m}\sigma^{\mu\nu}\langle[D_\lambda,u_\mu]u_\nu\rangle D^\lambda + \frac{d_{15}}{4m}\sigma^{\mu\nu}\langle u_\mu[D_\nu,u_\lambda]\rangle D^\lambda \\ & + \frac{d_{16}}{2}\gamma^\mu\gamma_5\langle\chi_+\rangle u_\mu + \frac{d_{18}}{2}\gamma^\mu\gamma_5[D_\mu,\chi_-]\Bigg\}\Psi.\end{aligned}$$

3 Fit Procedure and Conclusions

The full one-loop amplitude to order q^3 is obtained after mass and coupling constant renormalization, and the pertinent formulas can be found in refs.[6, 7]. The final result can be separated into the tree, loop and counterterm contributions. The last one introduces *nine* parameters that can be used to fit the amplitudes to the experimental data[5].

Now we arrive to the main point of this exercise. As we mentionned before, for a given isospin I, the phase shifts $\delta^I_{l\pm}(s)$ can be extracted from the partial waves via

$$f^I_{l\pm}(s) = \frac{1}{2i|\vec{q}|}\Big[\exp(2i\delta^I_{l\pm}) - 1\Big]. \qquad (2)$$

For vanishing inelasticity, the phase shifts are real. Moreover, from the heavy baryon approach, the imaginary parts stemming from the loop contributions at order q^3 fulfill perturbative unitarity. In terms of the partial wave amplitudes this reads

$$\mathrm{Im}\, f_l^{I(3)}(s) = |\vec{q}_\pi|\left(\mathrm{Re}\, f_l^{I(1)}(s)\right)^2, \qquad (3)$$

Usually, in effective field theory aplications[8], one defines the phase-shifts by

$$\delta^I_{l\pm}(s) = \arctan\big(|\vec{q}|\,\mathrm{Re}\, f^I_{l\pm}(s)\big). \qquad (4)$$

Alternatively, we will adopt the definition

$$\delta^I_{l\pm}(s) = \arctan\frac{\mathrm{Im}\, f^I_{l\pm}(s)}{\mathrm{Re}\, f^I_{l\pm}(s)}. \qquad (5)$$

Our proposal is to study the sensivity of the fit procedure to the phase-shift definition. The main motivation is that some low energy constants play important role in the theoretical evaluation of many physical constants. If the fitting procedure was unambiguous, one could have, by phase-shift fit, a powerful method to access several LECs values. Our strategy was to fix *nine* free parameters in order to fit the HBChPT amplitudes to *six* S- and P-wave experimental phase-shifts. In order to do that we use the definition (4) for low values (below 200 MeV) of the pion three–momentum in the lab system, called $\vec{q}_\pi$, with norm q_π,

$$q_\pi = \sqrt{\frac{1}{4m^2}\big(s - M_\pi^2 - m^{\,2}\big)^2 - M_\pi^2}\,.$$

Once fixed the LECs values, we plot the resulting partial-wave phase-shifts from the definition (2) as well.

The resulting values of the *nine* parameters fixed by a simultaneous fit of *six* low-energy phase-shifts are: c_1 = -1.57, c_2 = 3.00, c_3 = -6.05, c_4 = 3.55, $\bar{d}_1 + \bar{d}_2$ = 4.57, $\bar{d}_3$ = -4.67, $\bar{d}_5$ = 0.197, $\bar{d}_{14} - \bar{d}_{15}$ = -8.49 and $\bar{d}_{18}$ = 2.89.

We observe, by inspection of the figures, that the results are sensitive to phase-shift definition. We repeated the comparison with all model parameters kept equal to zero and we show the results in the same figures. We conclude that the results remain very sensitive with phase-shift definition.

We intend to perform a similar analysis using the amplitudes obtained by the unitarization program of current algebra [9]. We recall that those amplitudes have imaginary parts satisfying eq. (3).

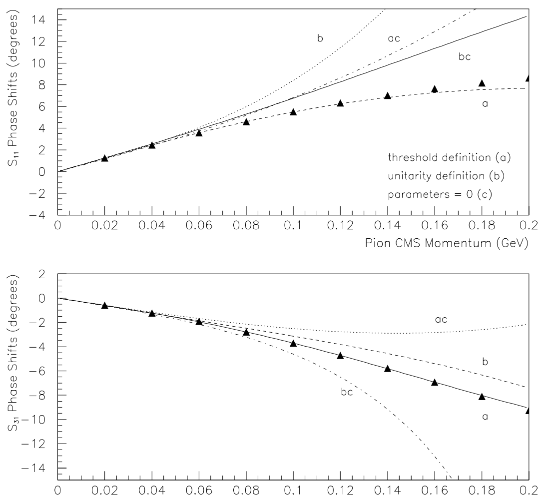

Figure 1. S-wave phase-shifts (in degrees) as functions of pion lab momentum (in GeV); with phase-shift definitons (4) (a with adjusted parameters and ac with vanishing ones) and (2) (respectively, b and bc, as for the previous definition).

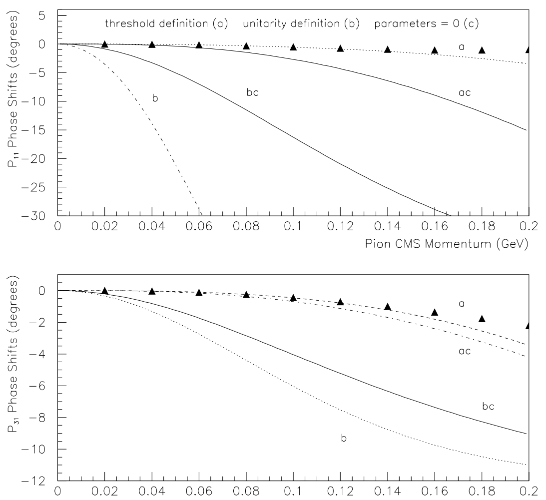

Figure 2. $J = 1/2$ P-wave phase-shifts (in degrees) as functions of pion lab momentum (in GeV); with phase-shift definitons (4) and (2); curve labels as in Fig. 1.

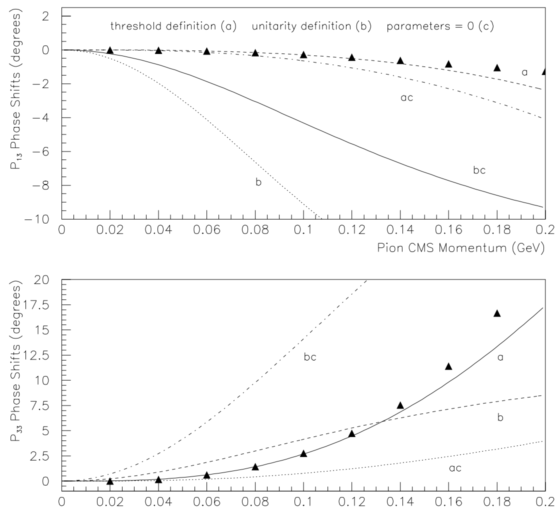

Figure 3. $J = 3/2$ P-wave phase-shifts (in degrees) as functions of pion lab momentum (in GeV); with phase-shift definitons (4) and (2); curve labels as in Fig. 1.

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