Diversity and Arbitrage in a Regulatory Breakup Model

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Abstract

In 1999 Robert Fernholz observed an inconsistency between the normative assumption of existence of an equivalent martingale measure (EMM) and the empirical reality of diversity in equity markets. We explore a method of imposing diversity on market models by a type of antitrust regulation that is compatible with EMMs. The regulatory procedure breaks up companies that become too large, while holding the total number of companies constant by imposing a simultaneous merge of other companies. As an example, regulation is imposed on a market model in which diversity is maintained via a log-pole in the drift of the largest company.

1 Introduction

What does the empirical phenomenon of diversity in equity markets imply about investment opportunities in those markets? The answer depends on the mechanism by which diversity is maintained.

The notion of diversity, the condition that no company's capitalization (shares multiplied by stock price) may approach that of the entire market, was introduced by Robert Fernholz in a 1999 paper [9] and 2002 book [11] (see also a recent review with Ioannis Karatzas [13]). He made the observation that one of the most useful tools of financial mathematics, the equivalent martingale measure (EMM), implies for a large class of models something grossly inconsistent with real markets: lack of diversity [10]. Historically, the major world stock markets have been diverse, and they should be expected to remain so as long as they are subject to a form of antitrust regulation that prevents concentration of capital into a single company.

Fernholz demonstrated that under common assumptions of financial market modeling that diverse market models necessarily admit strong relative arbitrage with respect to the market portfolio. Portfolio A is a strong relative arbitrage with respect to portfolio B by horizon T if A strictly outperforms B at time T with probability one. A sufficient set of assumptions are: capitalizations are modeled by Itô processes that pay no dividends, trading may occur in continuous time with no transaction costs, and the covariance process of the log capitalizations is uniformly elliptic. Importantly, the relative arbitrage portfolios of Fernholz do not depend on the parameters of the market, and therefore do not require estimation of these parameters to construct in practice. They are long-only portfolios (no short sales) built from portfolio generating functions [10–14], requiring

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only the weights of the market portfolio as input. If, additionally, the covariance process is bounded from above uniformly in time, then no equivalent local martingale measure (ELMM) is possible for such models. Therefore the fundamental theorem of asset pricing [3, 5] implies that they admit a free lunch with vanishing risk (FLVR).

To make the case that the argument above pertains to the existence of (approximate) relative arbitrage in real markets, dividends must be taken into account. Dividends provide a means for large companies to slow their growth in terms of capitalization while still generating competitive total returns (stock return + dividend return) for their shareholders. An exploratory statistical analysis [8] of the dividends paid by companies traded on U.S. equity exchanges from 1967-1996 suggests that this factor was insufficient to prevent relative arbitrage with respect to the U.S. market portfolio over this period, before accounting for transaction costs.

It is not easy to formulate diverse Itô process models (however see [23] for a clever probabilistic construction utilizing a non-equivalent measure change). Almost all market models commonly used in the literature, including geometric Brownian motion, are not diverse and therefore do not accurately model reality. Any diverse Itô process model with uniformly elliptic and uniformly bounded covariance must have the characteristic that the difference in the rate of expected return of the largest company, compared to some other company, diverges to $-\infty$ as the largest approaches a relative size cap [14]. Some possible economic rationale to support this type of model includes: difficulties in achieving high return on investment for very largely capitalized companies, and the cost of antitrust suits brought against such companies.

Since the onset of antitrust regulation in the U.S. in the late 19th century, there have been two main regulatory methods of dealing with companies which get too large: antitrust suits or fines, and antitrust breakup. The latter is rarely used, with some notable examples being the breakups of Standard Oil (1911) [30] and AT&T (1982) [31]. Suits or fines are used much more often than breakups to discipline companies that are deemed to be dominating their market in an unfair manner. Recent examples in Europe include Microsoft [22] in 2004 (€497 million) and Intel [16] in 2009 (€1.06 billion) being fined by the European Union for anticompetitive practices. Models in which diversity is maintained via the rate of expected return of any company diverging to $-\infty$ as that company's relative size becomes very large can be interpreted as continuous-path approximations of the case where suits or fines are used to regulate big companies. Models in which regulatory breakup is the primary means of maintaining market diversity have not been well-studied from a mathematical point of view in the financial mathematics literature. They are the subject of this paper.

When a company is fined money, this directly and adversely affects the value of the company, so the risk of antitrust fines is a mark against investing in large companies. In contrast, the key mathematical feature of a corporate breakup with regards to investment is that capital need not be removed from the system. That is, when a company is broken into parts, no net value needs to be lost. Indeed, from a regulator's perspective, avoidance of monopolies maintains the viability of an industry's innovation and growth prospects. Although it need not be the case in practice, for simplicity, we make the modeling assumption that total market values of companies, as well as the portfolio values of investors, are *conserved* at each regulatory breakup. The conservation of portfolio value implies that the capital gains process from investment in equity is not the stochastic integral of the trading strategy (shares of equity) with respect to the stock capitalization process. Instead, a net capitalization process, with the finite number of regulatory jumps removed, plays the role of integrator.

Another assumption we make is that the number of companies remains constant. This may seem inconsistent with the breakup of companies, but in our typical example of regulation we balance the number of companies in the economy by also requiring that two companies merge into a new company at the same time as regulation splits a company into two. This is imposed mainly for mathematical tractability. It isolates the effect of regulation on diversity and arbitrage without delving into the mathematics of models in which the number of stocks is a stochastic process.

As an application, we examine a regulated form of a log-pole market model and show that diversity and equivalent martingale measures coexist in this case, and additionally that there are no relative arbitrages. Furthermore, the regulated form satisfies the notion of "sufficient intrinsic volatility" of the market, a more general sufficient condition for relative arbitrage in unregulated models [12]. These results do not contradict the work of Fernholz et al., because in our model it is the regulated capitalization process that is diverse and the net capitalization process (which has regulatory jumps removed) that has an EMM.

This paper is organized as follows. Section 2 defines the class of premodels for the regulation procedure, the admissible trading strategies, portfolios, and the notion of diversity. In section 3 we introduce the regulatory procedure, including defining the regulatory mapping and the triggering mechanism for regulation. A heuristic implementation of regulation is provided and the economic rationale for the modeling assumptions is discussed. In section 4 the regulated market is formally constructed and the net capitalization process is introduced as the appropriate integrator for trading strategies in the regulated market. Our exemplar of regulation, the split-merge rule, is also introduced, which essentially splits the biggest company and forces the smallest two companies to merge at a regulatory event. Section 5 reviews some well-known mathematics of arbitrage, proves a fundamental theorem of asset pricing for regulated models, and reviews the results of Fernholz et al. regarding arbitrage and diversity in unregulated models. Section 6 applies the regulatory procedure to geometric Brownian motion and to a log-pole market model to illustrate the compatibility of diversity and EMMs in regulated models. Section 7 presents some concluding remarks and directions for future research.

2 Premodel

We first introduce the class of models which we will consider for regulation. We also define the set of trading strategies that are admissible for discussions of arbitrage, and the notion of a portfolio for discussions of relative arbitrage.

The stock capitalization process $\widetilde{X} = (\widetilde{X}_{1,t}, \ldots, \widetilde{X}_{n,t})'_{t\geq 0}$ represents the capitalizations (number of shares multiplied by stock price) of the $n \geq 2$ companies which are traded on an exchange, where the notation A' denotes the transpose of the matrix A. The stock capitalizations are each assumed to be almost surely (a.s.) strictly positive for all time, with $\widetilde{X}$ taking values in the open, connected, conic set $O^x \subseteq \mathbb{R}^n_{++} := (0,\infty)^n$. We use the notation "$\boxdot$" to denote the Hadamard entrywise product in order to write many of the equations of financial mathematics more concisely. For $k \times l$ matrices Q, R, we have $[Q \boxdot R]_{i,j} := (Q_{i,j})(R_{i,j}),\ 1 \leq i \leq k,\ 1 \leq j \leq l$. The dynamics of $\widetilde{X}$ are determined by the stochastic differential equation (SDE)

$$d\widetilde{X}_t = \widetilde{X}_t \boxdot \left(b(\widetilde{X}_t)dt + \sigma(\widetilde{X}_t)dW_t \right), \tag{2.1}$$

$$\widetilde{X}_0 = x_0 \in O^x, \tag{2.2}$$

for which $(\Omega, \mathcal{F}, \mathbb{F}, \widetilde{X}, W, P)$ is a solution, where W is an n-dimensional Brownian motion. The functions $b(\cdot)$ and $\sigma(\cdot)$ are locally bounded Borel functions, and for notational ease we will often refer to $b(\widetilde{X})$ and $\sigma(\widetilde{X})$ as $\widetilde{b}$ and $\widetilde{\sigma}$, respectively. We also require that the SDE (2.1) satisfies strong existence and pathwise uniqueness for any initial $x_0 \in O^x$, and that $P(\widetilde{X}_t \in O^x,\ \forall t \geq 0) = 1$. We only consider volatility matrices $\sigma(x) \in \mathbb{R}^{n \times n}$ having full rank n, $\forall x \in O^x$, which guarantees that no stock's risk can be completely hedged over any time interval by investment in the other stocks. We assume that $\mathcal{F}$ and $\mathcal{F}_0$ contain $\mathcal{N}$, the P-null sets, and confine attention to the augmented Brownian filtration $\mathbb{F} = \mathbb{F}^W := \{\mathcal{F}_t^W\}_{0 \le t < \infty}$, where $\mathcal{F}_t^W := \sigma\left(\{W_s\}_{0 \le s \le t}\right) \bigvee \mathcal{N}$.

The process $\widetilde{B}$ represents a money market account, for which we impose that a.s. $\widetilde{B} \equiv 1$, corresponding to zero interest rate. This assumption causes no important loss of generality, because if instead $\widetilde{B}$ is a semimartingale with strictly positive paths, then it can be used as a numéraire to discount all of the assets, including itself. Then $\widetilde{X}$ would be taken to be a model of discounted capitalizations (see e.g. section 10.3 of [1]).

Our other standing assumptions are that capitalizations are exogenously determined, no dividends are paid, markets are perfectly liquid, trading is frictionless (no transaction costs) and may occur in arbitrary quantities, and there are no taxes.

2.1 Investment in the Premodel

We define a *shares process* $(\widetilde{H}_t^B, \widetilde{H}_t')'_{t \ge 0}$, $\widetilde{H}_t' := (\widetilde{H}_{1,t}, \dots \widetilde{H}_{n,t})$ to be an adapted process representing the number of shares of money market and each stock, respectively, held at time t. Note that since $\widetilde{X}$ is a stock capitalization process, the number of shares outstanding of each company has effectively been normalized to one, and so $\widetilde{H}$ is with respect to this one share. The *wealth process* associated with shares process $(\widetilde{H}^B, \widetilde{H})$ is defined as

$$\widetilde{V}_t^{(\widetilde{H}^B, \widetilde{H})} := \widetilde{H}_t^B + \widetilde{H}_t' \widetilde{X}_t.$$

We call $\widetilde{H}$ a *trading strategy* and follow Delbaen and Schachermayer's [5] definition of admissible trading strategies.

Definition 2.1. *Admissible trading strategies* are predictable processes $\widetilde{H}$ such that

(i) $\widetilde{H}$ is $\widetilde{X}$-integrable, that is, the stochastic integral $\widetilde{H} \cdot \widetilde{X} = (\widetilde{H} \cdot \widetilde{X})_{t \ge 0} := \left(\int_0^t \widetilde{H}_s d\widetilde{X}_s\right)_{t \ge 0}$ is well-defined in the sense of stochastic integration theory for semimartingales.

(ii) There is a constant R such that

$$(\widetilde{H} \cdot \widetilde{X})_t \ge -R, \quad \text{a.s.}, \forall t \ge 0. \tag{2.3}$$

The second restriction is designed to rule out "doubling strategies" (see p.8 of [18]) and represent the realistic constraint that credit lines are limited.

We only consider *self-financing* wealth processes, which are defined as those $\widetilde{V}^{(\widetilde{H}^B, \widetilde{H})}$ that satisfy

$$\widetilde{V}_t^{(\widetilde{H}^B, \widetilde{H})} = \widetilde{V}_0^{(\widetilde{H}^B, \widetilde{H})} + (\widetilde{H} \cdot \widetilde{X})_t, \quad \forall t \ge 0,$$

for $\widetilde{X}$-integrable trading strategy $\widetilde{H}$. For any admissible trading strategy $\widetilde{H}$ and initial wealth $\widetilde{V}_0^{(\widetilde{H}^B, \widetilde{H})} = w \in \mathbb{R}$, there exists an adapted choice of $\widetilde{H}^B$, namely $\widetilde{H}_t^B = w + (\widetilde{H} \cdot \widetilde{X})_t - \widetilde{H}_t' \widetilde{X}_t$ that makes the wealth process self-financing. Therefore, in analysis of self-financing wealth processes whose trading strategies are admissible, we are free to forget $\widetilde{H}^B$ and focus on the admissible trading strategies. Henceforth we assume that all trading strategies are admissible, all wealth processes are self-financing, and refer to $\widetilde{V}^{(\widetilde{H}^B, \widetilde{H})}$ as $\widetilde{V}^{w, \widetilde{H}}$, where $\widetilde{V}_0^{w, \widetilde{H}} = w \in \mathbb{R}$.

It will also be useful in the context of relative arbitrage to develop the notion of a portfolio, *á la* Fernholz and Karatzas [13].

Definition 2.2. A *portfolio* $\widetilde{\pi}$ is an $\mathbb{F}$-progressively measurable n-dimensional process bounded uniformly in (t, ω), with values in the set

$$\bigcup_{\kappa \in \mathbb{N}} \big\{(\pi_1, \dots \pi_n) \in \mathbb{R}^n \mid \pi_1^2 + \dots + \pi_n^2 \le \kappa^2, \ \sum_{i=1}^{n} \pi_i = 1\big\}. \tag{2.4}$$

A *long-only portfolio* $\widetilde{\pi}$ is a portfolio that takes values in the unit simplex

$$\Delta^n := \big\{(\pi_1, \dots \pi_n) \in \mathbb{R}^n \mid \pi_1 \ge 0, \dots \pi_n \ge 0, \ \sum_{i=1}^{n} \pi_i = 1\big\}.$$

A portfolio $\widetilde{\pi}$ represents the fractional amount of an investor's wealth invested in each stock. In contrast to a trading strategy, no borrowing from or lending to the money market is allowed when investment occurs via a portfolio. The uniform boundedness requirement makes a portfolio somewhat more restrictive than a trading strategy, in particular ruling out "suicide strategies" of the type in section 1.2 of Karatzas and Shreve [18]. This condition may be dropped in favor of more general integrability conditions, for example see D. Fernholz and Karatzas [7], but we do not pursue this here.

For $w \in \mathbb{R}_{++}$, the wealth process $\widetilde{V}^{w,\widetilde{\pi}}$ corresponding to a portfolio is defined to be the solution to

$$\begin{aligned} d\widetilde{V}_t^{w,\widetilde{\pi}} &= \widetilde{V}_t^{w,\widetilde{\pi}} \sum_{i=1}^{n} \widetilde{\pi}_{i,t} \frac{d\widetilde{X}_{i,t}}{\widetilde{X}_{i,t}}, \\ &= (\widetilde{V}_t^{w,\widetilde{\pi}}) \widetilde{\pi}_t' \left[\widetilde{b}_t dt + \widetilde{\sigma}_t d\widetilde{W}_t\right], \end{aligned} \tag{2.5}$$

which by use of Itô's lemma can be verified to be

$$\widetilde{V}_t^{w,\widetilde{\pi}} = w \exp\left\{\int_0^t \widetilde{\gamma}_{\widetilde{\pi},s} ds + \int_0^t \widetilde{\pi}_s' \widetilde{\sigma}_s d\widetilde{W}_s\right\}, \quad \forall t \ge 0, \tag{2.6}$$

where

$$\widetilde{\gamma}_{\widetilde{\pi}} := \widetilde{\pi}' \widetilde{b} - \frac{1}{2} \widetilde{\pi}' \widetilde{a} \widetilde{\pi}$$

is called the *growth rate* of the portfolio $\widetilde{\pi}$ and

$$\widetilde{a} := \widetilde{\sigma} \widetilde{\sigma}'$$

is called the *covariance process* since

$$\widetilde{a}_{ij,t} = \frac{d}{dt} \left\langle \log \widetilde{X}_i, \log \widetilde{X}_j \right\rangle_t .$$

The definitions of the wealth process $\widetilde{V}^{w,\widetilde{\pi}}$ corresponding to a portfolio and $\widetilde{V}^{w,\widetilde{H}}$ corresponding to a trading strategy are consistent in the sense that any portfolio has an a.s. unique corresponding admissible trading strategy yielding the same wealth process from the same initial wealth. The corresponding trading strategy $H^{w,\widetilde{\pi}}$ can be obtained from

$$(\widetilde{V}^{w,\widetilde{\pi}}) \widetilde{\pi} = \widetilde{H}^{w,\widetilde{\pi}} \boxdot \widetilde{X}, \tag{2.7}$$

from which it follows that

$$d\widetilde{V}_t^{w,\widetilde{H}^{w,\widetilde{\pi}}} = (\widetilde{H}_t^{w,\widetilde{\pi}})' d\widetilde{X}_t, \\ = (\widetilde{V}^{w,\widetilde{\pi}})\widetilde{\pi}_t' \left[\widetilde{b}_t dt + \widetilde{\sigma}_t d\widetilde{W}_t\right],$$

in agreement with (2.5). By the form of (2.6), it can be seen that the uniform boundedness requirement of a portfolio implies that starting from strictly positive wealth and investing according to a portfolio $\widetilde{\pi}$, that wealth is a.s. strictly positive for all times in the future. This guarantees that $\widetilde{H}^{w,\widetilde{\pi}}$ is admissible since $\widetilde{H}^{w,\widetilde{\pi}} \cdot \widetilde{X}$ is uniformly bounded from below by $-w$.

The *market portfolio* $\widetilde{\mu}$ is of particular interest since "beating the market" is often a desirable goal for investors. The market portfolio is simply the relative capitalization of each company in the market with respect to the total:

$$\widetilde{M}_t := M(\widetilde{X}_t) := \sum_{i=1}^n \widetilde{X}_{i,t}, \qquad \widetilde{\mu}_{i,t} := \mu_i(\widetilde{X}_t) := \frac{\widetilde{X}_{i,t}}{\widetilde{M}_t}, \quad 1 \le i \le n.$$

Since the stock capitalization process $\widetilde{X}$ a.s. takes values in $O^x \subseteq \mathbb{R}^n_{++}$, then for $O^\mu := \mu(O^x)$, we have that a.s., $\forall t \ge 0$,

$$\widetilde{\mu}_t \in O^\mu \subseteq \mu(\mathbb{R}^n_{++}) = \Delta^n_+ := \left\{(\pi_1, \dots, \pi_n) \in \mathbb{R}^n \mid \pi_1 > 0, \dots, \pi_n > 0, \quad \sum_i^n \pi_i = 1\right\}.$$

The closure of a set $A \subseteq \mathbb{R}^n_{++}$ will be referred to as $\bar{A}$ and, unless otherwise stated, is taken with respect to the subspace topology of $\mathbb{R}^n_{++}$, and similarly for subsets of Δ^n_+. For example, $\bar{\mathbb{R}}^n_{++} = \mathbb{R}^n_{++}$ and $\bar{\Delta}^n_+ = \Delta^n_+$.

2.2 Diversity

The notion of diversity entails that no company may ever become too big in terms of relative capitalization. For generalizations to this notion and their implications see [14]. Diversity is a realistic criterion for a market model to satisfy, since it has held empirically in developed equity markets over time and should be expected to continue to hold as long as antitrust regulation prevents capital from concentrating in a single company.

In discussions of diversity, it is useful to adopt the reverse-order-statistics notation. That is, for $x \in \mathbb{R}^n$,

$$x_{(1)} \ge x_{(2)} \ge \ldots \ge x_{(n)}.$$

Definition 2.3. A premodel is *diverse* on $[0,T]$ if there exists $\delta \in (0,1)$ such that a.s.

$$\widetilde{\mu}_{(1),t} < 1 - \delta, \qquad \forall \quad 0 \le t \le T.$$

A premodel is *weakly diverse* on $[0,T]$ if there exists $\delta \in (0,1)$ such that

$$\frac{1}{T}\int_0^T \widetilde{\mu}_{(1),t} dt < 1 - \delta, \quad \text{a.s.}$$

We will not make much use of diversity until later on, but it is good to keep the definition in mind when considering the regulation procedure.

3 Regulation: An Overview

The notion of regulation we introduce consists of confining the market weights (except at exit times) in an open set U^μ by a regulatory procedure that

- conserves the number of companies in the market;
- conserves total market capital;
- conserves portfolio wealth;
- causes a jump in capitalizations.

Upon exit from U^μ, the new relative capitalizations are given by a deterministic mapping $\mathfrak{R}^\mu$ of $\widetilde{\mu}$ at its exit point. Then diffusion occurs according to the SDE (2.1) until $\widetilde{\mu}$ exits from U^μ again. Applying the mapping $\mathfrak{R}^\mu$ every time $\widetilde{\mu}$ exits from U^μ will lead to the regulated capitalization process Y.

Definition 3.1. A *regulation rule* $\mathfrak{R}^\mu$ with respect to the open, nonempty *regulatory set* $U^\mu \subseteq O^\mu \subseteq \Delta^n_+$ is a Borel function

$$\mathfrak{R}^\mu : \partial U^\mu \to U^\mu.$$

The regulation rule $(U^\mu, \mathfrak{R}^\mu)$ uniquely determines the following set and mapping of stock capitalizations, preserving total market capital:

$$\begin{aligned} U^x &:= \mu^{-1}(U^\mu) \subseteq O^x, \\ \mathfrak{R}^x &: \partial U^x \to U^x, \\ \mathfrak{R}^x(x) &:= \left(\sum_{i=1}^n x_i \right) \mathfrak{R}^\mu(\mu(x)). \end{aligned}$$

The inclusion $U^x \subseteq O^x$ follows from our assumption that O^x is conic, which implies $O^x = \mu^{-1}(O^\mu)$. By basic topology, since Δ^n_+ and $\mathbb{R}^n_{++}$ are open sets in $\mathbb{R}^n$, then a subset of either, for example U^μ or U^x, is open in its respective subspace topology if and only if it is open $\mathbb{R}^n$ (see section 16 of [21]). The set U^x is conic, that is $x \in U^x \Rightarrow \lambda x \in U^x$, $\forall \lambda > 0$, allowing any total market value for a given $\mu \in U^\mu$. Therefore the market capitalization M is a degree of freedom for the regulatory mapping in the sense that $\mu(\mathfrak{R}^x(x)) = \mu(\mathfrak{R}^x(\lambda x))$, $\forall \lambda > 0$. Specification of $(U^x, \mathfrak{R}^x)$ or $(U^\mu, \mathfrak{R}^\mu)$ uniquely determines the other, so we refer to either as "regulation rules" and in discussion drop the labels and refer to them as $(U, \mathfrak{R})$.

We first describe heuristically how a single regulatory event is implemented in the premodel. In the next section we use this idea inductively to mathematically construct the regulated capitalization process. The regulation rule is first applied at the exit time

$$\widetilde{\varsigma} := \inf\Big\{ t > 0 \mid \widetilde{\mu}_t \notin U^\mu \Big\} = \inf\Big\{ t > 0 \mid \widetilde{X}_t \notin U^x \Big\},$$

which is the hitting time of a closed ($\mathbb{R}^n$-topology) set by the continuous process $\widetilde{\mu}$. Therefore $\widetilde{\varsigma}$ is an $\mathbb{F}$-stopping time since the augmented Brownian filtration satisfies the usual assumptions of completeness and right-continuity. On $\{\widetilde{\varsigma} = \infty\}$ regulation is never applied since the market weights remain in U^μ for all time. The regulation procedure results in a new capital distribution at $\widetilde{\varsigma}^+$, conserving total market capital while satisfying $\widetilde{\mu}_{\widetilde{\varsigma}+} = \mathfrak{R}^{\mu}(\widetilde{\mu}_{\widetilde{\varsigma}})$ a.s. The values for the capitalizations and market weights at $\widetilde{\varsigma}^{+}$ are given by

$$\widetilde{X}_{\widetilde{\varsigma}+} = \mathfrak{R}^{x}(\widetilde{X}_{\widetilde{\varsigma}}), \qquad \widetilde{M}_{\widetilde{\varsigma}+} = \widetilde{M}_{\widetilde{\varsigma}}, \qquad \widetilde{\mu}_{\widetilde{\varsigma}+} = \mathfrak{R}^{\mu}(\widetilde{\mu}_{\widetilde{\varsigma}}).$$

After $\widetilde{\varsigma}$ the regulated market model "resets" in strongly Markovian fashion as if starting afresh from $\mathfrak{R}^{x}(\widetilde{X}_{\widetilde{\varsigma}})$ until exit from U^x again. In the next section we will apply this procedure inductively in defining the regulated capitalization process.

Remark 3.2. The assumptions made in section 2 imply that the push-forward of P by $\widetilde{X}$ onto canonical space $C([0,\infty), O^x)$, with Borel sigma algebra generated by the coordinate mappings, is the unique solution to the martingale problem on domain O^x corresponding to the SDE (2.1). The canonical process has the strong Markov property (see section 1.12 of [25]), so in this context our modeling assumption that $\widetilde{X}$ "forgets the past" at $\widetilde{\varsigma}^{+}$ is a natural one. The mathematics of the regulation procedure could just as well be applied to continuous semimartingales that are not Markovian. When the premodel is not Markovian, however, there is no compelling reason why mapping $\widetilde{X}$ back into U^x at $\widetilde{\varsigma}^{+}$ should cause $\widetilde{X}$ to "forget the past" and follow dynamics as if it were starting from $\mathfrak{R}^{x}(\widetilde{X}_{\widetilde{\varsigma}})$. An example illustrating this is a stochastic volatility model, where the volatility process $\widetilde{\sigma}$ is not completely determined by $\widetilde{X}$. In such a model, it is not clear what value $\widetilde{\sigma}_{\widetilde{\varsigma}+}$ should reasonably take. While we could specify the dynamics of $\widetilde{X}$ after $\widetilde{\varsigma}$ from a large number of possibilities, this would be an extra modeling assumption, and there is no reason for saying that any one such choice is *the* natural one for a regulated market model.

The economic motivation behind the regulated market models presented in this paper is to study markets with the feature that companies may merge and split, possibly forced to do so by a regulator, with an aim to explore the ramifications for diversity and arbitrage in these markets. In order to avoid what the authors believe to be unnecessary mathematical complications in the study of these notions, we require that splits and merges only occur simultaneously and in pairs, so that the number of companies in the economy remains a constant. For example, the biggest company may split into two, and simultaneously the smallest two merge into one.

The following remark gives an economic argument for why antitrust regulation in the form of company breakup need not create or destroy value. A similar rationale could be given for merges.

Remark 3.3. The existence of antitrust regulation stems from recognition of the power of monopolistic companies to exploit consumers, of their lack of incentive to function efficiently or to innovate, and of their ability to stifle newcomers. Therefore, if a company becomes an abusive monopoly, then a regulator creates net value for the economy by splitting the company into parts. As a result of such a split, the industry is competitive and vibrant again, and innovation and growth may occur as usual. If a regulator splits a company too soon, value might be lost because the costs of reshuffling administration and loss of economy of scale may outweigh the economic value of increased competition. It may be put forth that an ideal regulator would split a company at just the balance point, that is at the point where value is neither created nor destroyed by splitting. Efficiently priced stock in such an economy would then satisfy conservation of total market value at such a regulatory event.

The regulated model includes the rule that portfolio wealth is conserved at a regulatory event. So, even if a portfolio has all of its money invested in a single $\widetilde{X}_i$, whose value is reduced at regulation, then still the wealth of the portfolio is unchanged. The mathematical consequence of wealth conservation is that the portfolio gains process is not the stochastic integral with respect to the regulated capitalization process, but instead with respect to the net capitalization process defined in the next section. The following remark motivates the modeling assumption of wealth-conservation.

Remark 3.4. Consider a company being split into two smaller companies with capitalization fractions ρ and $1-\rho$ relative to the parent company. If an investor's money in the parent company is also broken up so that the investor receives fraction ρ invested in the first offspring and $1-\rho$ invested in the second offspring immediately following the split, then her portfolio wealth is conserved. In general if every investor's equity capital undergoes the mapping $\mathfrak{R}^x$ at $\widetilde{\varsigma}^+$, then the stock capitalizations immediately following a regulatory event equal the sums of investors' capital in each one.

This mapping of portfolio wealth does not impose any constraints on the trading strategies or portfolios available in the regulated market. Since trading occurs in continuous time, any investor may simply rearrange all of her money at $\widetilde{\varsigma}^+$. That our investor may do this without affecting market prices reflects the assumption that stock capitalizations are exogenously determined, that is, our investor is small relative to the market, and her behavior has negligible impact on asset prices.

A consequence of the conservation of portfolio wealth at regulation is that the natural interpretation of a passive (buy-and-hold) trading strategy in the regulated model is one which is piecewise constant, jumping only at regulatory events. Specifically, $\widetilde{H}_{\widetilde{\varsigma}+}$ must solve

$$\widetilde{H}_{\widetilde{\varsigma}+} \boxdot \mathfrak{R}^x(\widetilde{X}_{\widetilde{\varsigma}}) = \mathfrak{R}^x(\widetilde{H}_{\widetilde{\varsigma}} \boxdot \widetilde{X}_{\widetilde{\varsigma}}).$$

This prescription implies that an investor initially holding the market portfolio $\widetilde{\mu}$ and not trading it will still hold the market portfolio at $\widetilde{\varsigma}^+$ after regulation.

4 Regulated Markets

In this section we will rigorously construct the regulated stock process by means of induction via the diffusion-regulation cycle outlined in the previous section. Since the SDE (2.1) for $\widetilde{X}$ satisfies strong existence and pathwise uniqueness, then we need not pass to a new probability space to construct the regulated model. Extensions are possible when (2.1) merely satisfies weak existence and weak uniqueness, but we do not pursue these generalizations here. Define

$$\begin{aligned} W^1 &:= W, \qquad X^1 := \widetilde{X}, \\ \tau_0 &:= 0, \qquad \tau_1 := \varsigma_1 := \inf\left\{t > 0 \mid \mu(X_t^1) \notin U^\mu\right\}. \end{aligned}$$

The process X^1 will serve as the first piece of the regulated capitalization process on the stochastic interval $[0,\tau_1]:= \{(t,\omega) \in [0,\infty) \times \Omega \mid 0 \le t \le \tau_1(\omega)\}$. At τ_1, X^1 has just exited U^x, so the regulation procedure maps the capitalization process to $\mathfrak{R}^x(X^1_{\varsigma_1})$, and the regulated process continues from that point according to the dynamics given by the SDE (2.1). This cycle of diffusion and regulation repeats as many times as necessary. To implement this define the following variables and processes inductively, $\forall k \in \mathbb{N}$ on $\{\tau_{k-1} < \infty\}$, terminating if $P(\tau_{k-1} = \infty) = 1$:

$$\begin{aligned}
W_t^k &:= W_{\tau_{k-1}+t} - W_{\tau_{k-1}}, \quad \forall t \geq 0, \\
dX_t^k &= X_t^k \boxdot \left(b(X_t^k)dt + \sigma(X_t^k)dW_t^k \right), \\
X_0^k &= \begin{cases} y_0 \in U^x, & \text{for } k = 1, \\ \mathfrak{R}^x(X_{\varsigma_{k-1}}^{k-1}), & \text{for } k > 1, \end{cases} \\
\varsigma_k &:= \inf \left\{ t > 0 \mid X_t^k \notin U^x \right\}, \\
\tau_k &:= \sum_{j=1}^{k} \varsigma_j.
\end{aligned} \tag{4.1}$$

If for some $k \in \mathbb{N}$ the induction terminates because $P(\tau_{k-1} = \infty) = 1$, then on $\{\tau_{k-1} < \infty\}$, $\forall m \geq k$ define $X^m \equiv y_0$, $\tau_m = \infty$, $\varsigma_m = 0$. Use these same definitions $\forall m \in \mathbb{N}$ on $\{\tau_{m-1} = \infty\}$. These cases are included for completeness and their specifics are irrelevant for the subsequent development.

By the strong Markov property and stationarity of Brownian increments, if $P(\tau_{k-1} < \infty) > 0$, then for

$$\mathcal{F}_t^k := \mathcal{F}_{\tau_{k-1}+t}, \qquad \mathbb{F}^k := \{\mathcal{F}_t^k\}_{t \geq 0},$$

$(W^k, \mathbb{F}^k)$ is a Brownian motion on $\{\tau_{k-1} < \infty\}$, that is, on $(\Omega \cap \{\tau_{k-1} < \infty\}, \mathcal{F} \cap \{\tau_{k-1} < \infty\})$. The SDE (4.1) for $k \geq 2$ has the same form as the SDE (2.1) for $\widetilde{X}$, but with W^k in place of W, and with initial condition $X_0^k = \mathfrak{R}^x(X_{\varsigma_{k-1}}^k)$ a.s. on $\{\tau_{k-1} < \infty\}$. Therefore on $\{\tau_{k-1} < \infty\}$ by strong existence, there exists X^k adapted to $\mathbb{F}^k$ satisfying (4.1).

Each ς_k is a stopping time with respect to $\mathbb{F}^k$, since it is the hitting time of the closed ($\mathbb{R}^n$-topology) set $\mathbb{R}^n \setminus U^x$ by continuous process X^k. Since $\mathfrak{R}^x(X_{\varsigma_{k-1}}^{k-1}) \in U^x$, then $\varsigma_k > 0$ and $\tau_k > \tau_{k-1}$ both a.s. on $\{\tau_{k-1} < \infty\}$, $\forall k : P(\tau_{k-1} < \infty) > 0$. Each τ_k is an $\mathbb{F}$-stopping time, which is proved in Lemma 4.2.

Under this construction there is the possibility of explosion, that is, of $\lim_{k \to \infty} \tau_k < \infty$. To characterize this possibility we first define the following jump process and variables:

$$N_t := \sum_{k=1}^{\infty} \mathbf{1}_{\{t > \tau_k\}}, \quad N_\infty := \lim_{t \to \infty} N_t, \quad \tau_\infty := \lim_{k \to \infty} \tau_k.$$

The event $\{N_\infty = k\}$ corresponds to exactly k regulations occurring eventually, so $\tau_{k+1} = \infty = \tau_\infty$ on $\{N_\infty = k\}$.

We are now ready to define the regulated capitalization process Y by pasting together and shifting the $\{X^k\}_1^\infty$ at the $\{\tau_k\}_1^\infty$ as follows.

Definition 4.1. With respect to regulation rule $(U, \mathfrak{R})$ and initial point $y_0 \in U^x$, the *regulated capitalization process* is defined as

$$Y_t(\omega) := \begin{cases} X_0^1 \mathbf{1}_{\{0\}}(t) + \sum_{k=1}^{\infty} \mathbf{1}_{(\tau_{k-1}, \tau_k]}(t, \omega) X_{t-\tau_{k-1}}^k(\omega), & \forall (t, \omega) \in [0, \tau_\infty), \\ X_0^1, & \forall (t, \omega) \notin [0, \tau_\infty), \end{cases} \tag{4.2}$$

where $P(X_0^1 = y_0) = 1$. If $P(\tau_\infty = \infty) = 1$, then we call the triple $(y_0, U, \mathfrak{R})$ *viable* for the premodel.

The following lemma makes some necessary technical characterizations of the regulated market.

Lemma 4.2. *The τ_k are $\mathbb{F}$-stopping times, $\forall k \in \mathbb{N}$, and Y and N are $\mathbb{F}$-adapted.*

Proof. For fixed $k \in \mathbb{N}$, consider the logical structure

$$[\tau_{k-1} \text{ is an } \mathbb{F}\text{-stopping time}] \overset{(I)}{\Rightarrow} \Big[0 \vee (t-\tau_{k-1}) \text{ is an } \mathbb{F}^k\text{-stopping time, } \forall t \in \mathbb{R}_+\Big]$$
$$\overset{(II)}{\Rightarrow} \Big[\mathbf{1}_{[\tau_{k-1},\infty)} X^k_{\cdot-\tau_{k-1}} \text{ is } \mathbb{F}\text{-adapted}\Big] \overset{(III)}{\Rightarrow} [\tau_k \text{ is an } \mathbb{F}\text{-stopping time}]\,,$$

where the implications use all of the preceding claims as suppositions. Starting with $\tau_0 = 0$, which is trivially an $\mathbb{F}$-stopping time, we will use induction to show that all the claims are true, $\forall k \in \mathbb{N}$. Implication (I) holds because $\mathcal{F}^k_0 = \mathcal{F}_{\tau_{k-1}}$ and $0 \vee (t-\tau_{k-1}) \in \mathcal{F}_{\tau_{k-1}}$, so is an $\mathbb{F}^k$-stopping time, $\forall t \in \mathbb{R}_+$. To establish implication (II) we will use that $\mathcal{F}^k_{0\vee(t-\tau_{k-1})} = \mathcal{F}_{\tau_{k-1}\vee t}$, $\forall t \in \mathbb{R}_+$, which we prove at the end. Supposing this, then by stopping $X^k_{0\vee(t-\tau_{k-1})} \in \mathcal{F}^k_{0\vee(t-\tau_{k-1})} = \mathcal{F}_{\tau_{k-1}\vee t}$ since X^k is $\mathbb{F}^k$-adapted. This implies that $\mathbf{1}_{[\tau_{k-1},\infty)} X^k_{\cdot-\tau_{k-1}} = \mathbf{1}_{[\tau_{k-1},\infty)} X^k_{0\vee(\cdot-\tau_{k-1})}$ is adapted to $\mathbb{F}$, proving implication (II). For implication (III), by the definition of τ_k and ς_k, using the conventions $\inf\{\emptyset\} = \infty$ and $\infty + r = \infty$, $\forall r \in \mathbb{R}$, the following holds $\forall \omega \in \Omega$:

$$\begin{aligned}\tau_k = \tau_{k-1} + \varsigma_k &= \tau_{k-1} + \inf\{s > 0 \mid X^k_s \notin U^x\},\\ &= \inf\{t > 0 \mid y_0 \mathbf{1}_{[0,\tau_{k-1})}(t) + \mathbf{1}_{[\tau_{k-1},\infty)}(t) X^k_{t-\tau_{k-1}} \notin U^x\}.\end{aligned}$$

This is a hitting time of closed ($\mathbb{R}^n$-topology) set $\mathbb{R}^n \backslash U^x$ by $\mathbb{F}$-adapted, càdlàg process $(y_0 \mathbf{1}_{[0,\tau_{k-1})} + \mathbf{1}_{[\tau_{k-1},\infty)} X^k_{\cdot-\tau_{k-1}})$, and is therefore a stopping time since $\mathbb{F}$ satisfies the usual assumptions. This completes the induction.

The counting process N is $\mathbb{F}$-adapted since the τ_k are stopping times. For any $k \in \mathbb{N}$, $t \geq 0$, on the event $\{N_t = k\}$ we have

$$Y_t = X^{k+1}_{t-\tau_k} = \mathbf{1}_{[\tau_k,\infty)} X^{k+1}_{t-\tau_k} \in \mathcal{F}_t,$$

whereas on the event $\{N_t = \infty\}$ we have $Y_t = X^1_0 \in \mathcal{F}_0$. Therefore by partitioning with $\{N_t = k\}$ for $k \in \mathbb{N} \cup \{+\infty\}$, this implies that Y is adapted to $\mathbb{F}$.

Proof of $\mathcal{F}^k_{0\vee(t-\tau_{k-1})} = \mathcal{F}_{\tau_{k-1}\vee t}$. By the definition of a stopped filtration,

$$\begin{aligned} A \in \mathcal{F}^k_{0\vee(t-\tau_{k-1})} &\iff A \cap \{0 \vee (t-\tau_{k-1}) \leq s\} \in \mathcal{F}^k_s = \mathcal{F}_{\tau_{k-1}+s}, \quad \forall s \geq 0,\\ &\iff A \cap \{(t-\tau_{k-1}) \leq s\} \cap \{(\tau_{k-1}+s) \leq r\} \in \mathcal{F}_r, \quad \forall r \geq 0,\ \forall s \geq 0,\\ &\iff A \cap \{(t-s) \leq \tau_{k-1} \leq (r-s)\} \in \mathcal{F}_r, \quad \forall r \geq t,\ 0 \leq s \leq r. \end{aligned} \tag{4.3}$$

For $\mathcal{F}_{\tau_{k-1}\vee t}$ we have that

$$\begin{aligned} B \in \mathcal{F}_{\tau_{k-1}\vee t} &\iff B \cap \{\tau_{k-1} \vee t \leq r\} \in \mathcal{F}_r, \quad \forall r \geq 0,\\ &\iff B \cap \{\tau_{k-1} \leq r\} \in \mathcal{F}_r, \quad \forall r \geq t. \end{aligned} \tag{4.4}$$

To prove $\mathcal{F}_{\tau_{k-1}\vee t} \subseteq \mathcal{F}^k_{0\vee(t-\tau_{k-1})}$ we use that since τ_{k-1} is an $\mathbb{F}$-stopping time, then (4.4) implies

$$B \in \mathcal{F}_{\tau_{k-1}\vee t} \Rightarrow B \cap \{t-s \leq \tau_{k-1} \leq r-s\} \in \mathcal{F}_r, \quad \forall r \geq t,\ 0 \leq s \leq r.$$

To prove $\mathcal{F}_{\tau_{k-1}\vee t} \supseteq \mathcal{F}^k_{0\vee(t-\tau_{k-1})}$ we use (4.3) and that

$$\bigcup_{\substack{s\in[0,r]\cap\mathbb{Q},\\ s=r}} A \cap \{t-s \leq \tau_{k-1} \leq r-s\} = A \cap \{\tau_{k-1} \leq r\} \in \mathcal{F}_r, \quad \forall r \geq t.$$

$\square$

4.1 Investment in the Regulated Market

As remarked earlier, in the regulated model wealth is unaltered by a regulatory event. Specifically, the wealth process $V^{w,H}$ of admissible trading strategy H does not jump upon redistribution of market capital at τ_k^+. This implies that the capital gains of a trading strategy can't be the stochastic integral of a trading strategy with respect to the regulated capitalization process. In order to recover the useful tool of representing the capital gains process as a stochastic integral, it is helpful to define a net capitalization process $\widehat{Y}$, which only accounts for the non-regulatory movements of Y.

Definition 4.3. The *net capitalization process* $\widehat{Y}$ is defined as

$$\widehat{Y}_t := \begin{cases} Y_t - \sum_{k=1}^{N_t}(\Re^x(Y^k_{\tau_k}) - Y_{\tau_k}), & \forall (t,\omega) \in [0,\tau_\infty), \\ X^1_0, & \forall (t,\omega) \notin [0,\tau_\infty). \end{cases} \tag{4.5}$$

The process $\widehat{Y}$ is $\mathbb{F}$-adapted since Y and N are adapted by Lemma 4.2. If the regulated market is viable, then a.s. $\widehat{Y}$ has continuous paths since then a.s. Y has piecewise continuous paths, jumping only at the τ_k. The following representation of $\widehat{Y}$ will also be useful and is easily obtainable from the definitions of Y and $\widehat{Y}$.

$$\widehat{Y}_t := X^1_0 + \sum_{k=1}^{N_t+1} (X^k_{(t-\tau_{k-1})\wedge\varsigma_k} - X^k_0), \quad \forall (\omega, t) \in [0,\tau_\infty). \tag{4.6}$$

The net capitalization process can further be characterized by the following proposition.

Proposition 4.4. *The net capitalization process satisfies*

$$d\widehat{Y}_t = Y_t \boxdot \left[b(Y_t)dt + \sigma(Y_t)dW_t\right], \quad \text{on } [0,\tau_\infty).$$

That is, for any stopping time $\alpha \in [0,\tau_\infty)$, $\widehat{Y}_{\cdot\wedge\alpha}$ *satisfies*

$$\widehat{Y}_{t\wedge\alpha} - \widehat{Y}_0 = \int_0^{t\wedge\alpha} Y_u \boxdot b(Y_u)du + \int_0^{t\wedge\alpha} Y_u \boxdot \sigma(Y_u)dW_u, \quad \forall t \geq 0.$$

Proof. For any stopping time $\alpha \in [0,\tau_\infty)$, the variable $0 \vee (\alpha - \tau_{k-1})$ satisfies $\{0 \vee (\alpha - \tau_{k-1}) \leq t\} = \{\alpha \leq t + \tau_{k-1}\} \in \mathcal{F}_{t+\tau_{k-1}} = \mathcal{F}^k_t$ since α and τ_{k-1} are $\mathbb{F}$-stopping times. Therefore $0 \vee (\alpha - \tau_{k-1})$ is an $\mathbb{F}^k$-stopping time and so $0 \vee (t \wedge \alpha - \tau_{k-1}) \wedge \varsigma_k = 0 \vee [(t - \tau_{k-1}) \wedge (\alpha - \tau_{k-1})] \wedge \varsigma_k$ is also an $\mathbb{F}^k$-stopping time. This along with (4.6) and (4.1) implies that

$$\begin{aligned} \widehat{Y}_{t\wedge\alpha} &= X^1_0 + \sum_{k=1}^{N_{t\wedge\alpha}+1} \int_0^{0\vee(t\wedge\alpha-\tau_{k-1})\wedge\varsigma_k} dX^k_s, \\ &= \widehat{Y}_0 + \sum_{k=1}^{N_{t\wedge\alpha}+1} \int_0^{0\vee(t\wedge\alpha-\tau_{k-1})\wedge\varsigma_k} X^k_s \boxdot b(X^k_s)ds + \int_0^{0\vee(t\wedge\alpha-\tau_{k-1})\wedge\varsigma_k} X^k_s \boxdot \sigma(X^k_s)dW^k_s. \end{aligned}$$

Changing variables to $u^k := s + \tau_{k-1}$, which is an $\mathbb{F}$-stopping time, and noting that on the stochastic interval $(0, [0 \vee (t \wedge \alpha - \tau_{k-1})] \wedge \varsigma_k)$ we have $W^k_s = W_{\tau_{k-1}+s} - W_{\tau_{k-1}} = W_{u^k} - W_{\tau_{k-1}}$ and $X^k_s =$

$X^k_{u^k-\tau_{k-1}} = Y_{u^k}$, it follows that

$$\begin{aligned}
\widehat{Y}_{t\wedge\alpha} &= \widehat{Y}_0 + \sum_{k=1}^{N_{t\wedge\alpha}} \left[\int_{\tau_{k-1}}^{\tau_k} Y_{u^k} \boxdot b(Y_{u^k}) du^k + \int_{\tau_{k-1}}^{\tau_k} Y_{u^k} \boxdot \sigma(Y_{u^k}) dW_{u^k} \right] \\
&\quad + \int_{\tau_{N_{t\wedge\alpha}}}^{t\wedge\alpha} Y_v \boxdot b(Y_v) dv + \int_{\tau_{N_{t\wedge\alpha}}}^{t\wedge\alpha} Y_v \boxdot \sigma(Y_v) dW_v, \\
&= \widehat{Y}_0 + \int_0^{t\wedge\alpha} Y_u \boxdot b(Y_u) du + \int_0^{t\wedge\alpha} Y_u \boxdot \sigma(Y_u) dW_u,
\end{aligned}$$

where $v := s + \tau_{N_{t\wedge\alpha}}$. □

The net capitalization process is the right choice to fulfill the role of integrator for a trading strategy in the regulated market. The self-financing condition in the regulated model is taken to be the natural analog of the usual self-financing condition.

Definition 4.5. In a viable regulated market, a wealth process $V^{w,H}$ corresponding to $\widehat{Y}$-integrable trading strategy H is called *self-financing* in the regulated market if

$$V_t^{w,H} = w + (H \cdot \widehat{Y})_t, \quad \forall t \geq 0.$$

This condition is simply the usual self-financing condition, incorporating the rule that portfolio wealth is conserved at regulatory events (see remark 3.4 for a discussion). This is supported by

$$V_t^{w,H} = w + \sum_{k=1}^{N_t} \int_{\tau_{k-1}^+}^{\tau_k} H_s dY_s + \int_{\tau_{N_t}^+}^{t} H_s dY_s,$$

holding in a viable regulated market. As in the premodel, in a viable regulated model we will henceforth assume that all wealth processes are self-financing, and that all trading strategies are $\widehat{Y}$-admissible, which means that H is $\widehat{Y}$-integrable, and $H \cdot \widehat{Y}$ is a.s. bounded from below uniformly in time, paralleling Definition 2.1.

A portfolio in the regulated model will be denoted by π, and is a process meeting the requirements of Definition 2.2. A portfolio π represents the fractional amount of total wealth invested in the regulated stocks Y. For initial wealth $w \in \mathbb{R}_{++}$, the wealth process $V^{w,\pi}$ corresponding to π is defined to be the solution to

$$\begin{aligned}
dV_t^{w,\pi} &= V_t^{w,\pi} \sum_{i=1}^n \pi_{i,t} \frac{d\widehat{Y}_{i,t}}{Y_{i,t}}, \\
&= (V^{w,\pi}) \pi_t' \left[b_t dt + \sigma_t dW_t \right],
\end{aligned} \tag{4.7}$$

where b and σ denote the processes $b(Y)$ and $\sigma(Y)$, respectively. By use of Itô's lemma, $V^{w,\pi}$ can be verified to be

$$V_t^{w,\pi} = w \exp\left\{ \int_0^t \gamma_{\pi,s} ds + \int_0^t \pi_s' \sigma_s dW_s \right\}, \quad \forall t \geq 0, \tag{4.8}$$

where

$$\gamma_\pi := \pi' b - \frac{1}{2} \pi' a \pi, \qquad a := \sigma\sigma'.$$

Paralleling the premodel, the definitions of the wealth process corresponding to a portfolio $V^{w,\pi}$ and that corresponding to a trading strategy $V^{w,H}$ are consistent in the sense that any portfolio has an a.s. unique corresponding admissible trading strategy yielding the same wealth process from the same initial wealth. The corresponding trading strategy $H^{w,\pi}$ can be obtained from

$$(V^{w,\pi})\pi = H^{w,\pi} \boxdot Y, \tag{4.9}$$

from which it follows that

$$\begin{aligned} dV_t^{w,H^{w,\pi}} &= (H_t^{w,\pi})' d\hat{Y}_t, \\ &= (V^{w,\pi})\pi_t' \left[b_t dt + \sigma_t dW_t\right], \end{aligned}$$

in agreement with (4.7). As in the premodel, the form of (4.8) guarantees that $H^{w,\pi}$ is admissible since $H^{w,\pi} \cdot \widehat{Y}$ is uniformly bounded from below by $-w$.

The market portfolio is the portfolio with the same weights μ as the market. Note that unlike $\widetilde{H}^{w,\widetilde{\mu}}$, which is constant, $H^{w,\mu}$ is piecewise constant, jumping at the τ_k. All portfolios, including the market portfolio, have wealth processes of identical functional form (compare (4.8) and (2.6)) in the regulated model and in the premodel. Therefore, from a mathematical viewpoint the differences in investment opportunities in these markets are completely due to the differences in dynamics of $b(Y)$ and $\sigma(Y)$ compared to $b(\widetilde{X})$ and $\sigma(\widetilde{X})$, which are due in turn to confining $\widetilde{X}$ to U^x to obtain Y.

4.2 Split-Merge Regulation

The exemplar for regulation used in this paper is the split-merge regulation rule. The basic economic motivation behind split-merge regulation is that it provides a means for regulators to control the size of the largest company in the economy. At τ_k, the largest company is split into two new companies of equal capitalization. In order to avoid the mathematical complications of a market model with a variable number of companies, we also impose that at τ_k the smallest two companies merge, so that the total number of companies is a constant, n.

A natural trigger for when regulators might force a large company to split is company size. For example, regulation may be triggered when the biggest company reaches $1-\delta$ in relative capitalization for some $\delta \in (0, \frac{n-1}{n})$, where this range is chosen because $[\delta \geq \frac{n-1}{n}] \Leftrightarrow 1-\delta \leq \frac{1}{n}$, and each of the n companies must be allowed to have relative capitalization at least $\frac{1}{n}$.

The purpose of this subsection is to define the class of split-merge regulation rules and to find sufficient conditions for the viability of this class. These results are summarized in Lemma 4.10.

To identify which company by index occupies the kth rank at time t, we use the random function $p_t(\cdot)$ so that $\mu_{p_t(k),t} = \mu_{(k),t}$, for $1 \leq k \leq n$. Similarly, for the vector $x := (x_1, \ldots, x_n)$ we use $p(\cdot)$ satisfying $x_{p(k)} = x_{(k)}$, for $1 \leq k \leq n$. In the event that several components are tied, for example $x_{(k)} = \ldots x_{(k+j)}$, then ties are settled as $p(k) < \ldots < p(k+j)$.

To define the notion of split-merge regulation, we first define a regulation prerule, which captures the essential idea but still requires some technical refinement.

Definition 4.6. In a market where $n \geq 3$, a *split-merge regulation prerule* $(U^\mu, \check{\mathfrak{R}}^\mu)$ with respect to open, nonempty regulatory set $U^\mu \subseteq O^\mu$ is a mapping

$$\check{\mathfrak{R}}^\mu : \bar{U}^\mu \to \Delta_+^n$$

such that

$$\check{\mathfrak{R}}^\mu(\mu) = \mu, \quad \forall \mu \in U^\mu,$$

and $\check{\mathfrak{R}}^\mu \restriction_{\partial U^\mu}$ is specified by the map:

$$\begin{aligned}
\mu_{p(1)} &\mapsto \frac{\mu_{p(1)}}{2}, \\
\mu_{p(n-1)} &\mapsto \frac{\mu_{p(1)}}{2}, \\
\mu_{p(n)} &\mapsto \mu_{p(n)} + \mu_{p(n-1)}, \\
\mu_{p(k)} &\mapsto \mu_{p(k)}, \quad \forall k : 2 \leq k < n-1.
\end{aligned}$$

The split-merge regulation prerule can be interpreted as splitting the largest company in half into two new companies and forcing the smallest two companies to merge into a new company. The condition $n \geq 3$ insures that these companies are distinct. The new companies from the split are assigned the indices of the previous largest and the previous second smallest companies. The new company from the merge is assigned the index of the previous smallest company.

Remark 4.7. Due to the interchange of indices, this interpretation makes economic sense only in a market model where the companies are taken to be generic, that is, they have no firm-specific (index-specific) properties. For example, in a market model where sector-specific correlations are being modeled, it would not make sense for an oil company resulting from a split to take over the index of a technology company freed up from a merge, since the subsequent correlations would not be realistic. The examples in this paper focus on generic market models, so this interpretation is sensible for them.

A split-merge regulation prerule $(U^\mu, \check{\mathfrak{R}}^\mu)$ is not quite suitable for our notion of split-merge regulation, because in the event that $\mu_{(1),\tau_k} = \mu_{(2),\tau_k} = \ldots = \mu_{(j),\tau_k}$, we desire that all of these largest companies be broken up, not just one of them. This can be easily accomplished, however, by repeating the procedure n times.

Definition 4.8. If $n \geq 3$ and split-merge regulation prerule $(U^\mu, \check{\mathfrak{R}}^\mu)$ is into $\bar{U}^\mu$, then we may define

$$\mathfrak{R}^\mu := (\underbrace{\check{\mathfrak{R}}^\mu \circ \ldots \circ \check{\mathfrak{R}}^\mu}_{n \text{ compositions}}) \restriction_{\partial U^\mu} .$$

If $\mathfrak{R}^\mu$ is into U^μ, then we may restrict the codomain to U^μ, and we call the resulting function $(U^\mu, \mathfrak{R}^\mu)$ the *split-merge regulation rule* associated with $(U^\mu, \check{\mathfrak{R}}^\mu)$.

Note that the above definition implies that when a split-merge regulation rule exists, it is a regulation rule. The following technical lemma will be handy for verifying the viability of split-merge rules. We use the notation $C_b^2(\Delta_+^n, \mathbb{R})$ to denote the continuous bounded functions from Δ_+^n to $\mathbb{R}$ with partial derivatives continuous and bounded through 2nd order.

Lemma 4.9. *If the SDE (2.1) has drift $b(\cdot)$ and volatility $\sigma(\cdot)$ functions which are bounded on U^x, and there exists a function $G \in C_b^2(\Delta_+^n, \mathbb{R})$ such that the regulation rule $(U, \mathfrak{R})$ satisfies either*

$$\inf \{G(\mathfrak{R}^\mu(\mu)) - G(\mu) \mid \mu \in \partial U^\mu\} > 0 \quad \text{or} \quad \sup \{G(\mathfrak{R}^\mu(\mu)) - G(\mu) \mid \mu \in \partial U^\mu\} < 0,$$

where ∂U^μ is the boundary of the set U^μ taken as a subset of the space Δ_+^n, then the regulated market is viable.

Proof. For $\mu_t := \mu(Y_t)$, let $G_t := G(\mu_t)$, $\forall(\omega, t) \in [0, \tau_\infty)$. By Definition 4.3 of $\widehat{Y}$ and Proposition 4.4, we can decompose $G_{t\wedge\tau_k}$ as

$$G_{t\wedge\tau_k} = G_0 + \sum_{m=1}^{k} \int_{t\wedge\tau_{m-1}^+}^{t\wedge\tau_m} dG_t + \sum_{m=1}^{N_t\wedge(k-1)} \left[G(\mathfrak{R}^\mu(\mu_{\tau_m})) - G_{\tau_m}\right]. \tag{4.10}$$

On (τ_{k-1}, τ_k) by Itô's lemma, the process μ obeys

$$\begin{aligned} d\mu_t &= \mu_t \boxdot \left[\left(b_t - a_t\mu_t - \mathbf{1}_n[\mu_t' b_t - \mu_t' a_t \mu_t]\right) dt + \left(\sigma_t - \mathbf{1}_n \mu_t' \sigma_t\right) dW_t\right], \\ &= B_t dt + R_t dW_t, \end{aligned}$$

where $\mathbf{1}_n$ is the column vector of n ones. The processes B and R are bounded on $(0, \tau_\infty)$, since $b(\cdot)$ and $\sigma(\cdot)$ are uniformly bounded on U^x. Defining $\widehat{G}_t := G_t - \sum_{m=1}^{N_t} [G(\mathfrak{R}^\mu(\mu_{\tau_m})) - G_{\tau_m}]$, $\forall(t, \omega) \in [0, \tau_\infty)$, then by Itô's lemma $\widehat{G}$ is an Itô process on $[0, \tau_\infty)$, and so there exist processes C and S taking values in $\mathbb{R}^n$ and $\mathbb{R}^{n\times n}$, respectively, such that

$$d\widehat{G}_t = C_t dt + S_t dW_t, \quad \text{on } (0, \tau_\infty).$$

The integrands C and S are uniformly bounded on $(0, \tau_\infty)$ since the first and second derivatives of $G(\cdot)$ are by assumption bounded on Δ^n_+, and B, R above are uniformly bounded on $(0, \tau_\infty)$. This implies that $\int_0^{t\wedge\tau_\infty} C_s ds$ and $\int_0^{t\wedge\tau_\infty} S_s dW_s$ are well-defined for all $t > 0$ by the theories of Lebesgue and stochastic integration. Therefore $\lim_{k\to\infty}(\mathbf{1}_{\{\tau_\infty<\infty\}}\widehat{G}_{\tau_k}) \in \mathbb{R}$ a.s.

By (4.10) and the definition of $\widehat{G}$, we have:

$$G_{\tau_k} = \widehat{G}_{\tau_k} + \sum_{m=1}^{k-1} \left[G(\mathfrak{R}^\mu(\mu_{\tau_m})) - G_{\tau_m}\right]. \tag{4.11}$$

On $\{\tau_\infty < \infty\}$ by assumption either

$$\sum_{m=1}^{k-1} \left[G(\mathfrak{R}^\mu(\mu_{\tau_m})) - G_{\tau_m}\right] \underset{k\to\infty}{\longrightarrow} \infty, \quad \text{a.s.},$$

or

$$\sum_{m=1}^{k-1} \left[G(\mathfrak{R}^\mu(\mu_{\tau_m})) - G_{\tau_m}\right] \underset{k\to\infty}{\longrightarrow} -\infty, \quad \text{a.s.}$$

But in (4.11) $\{\widehat{G}_{\tau_k}\}_1^\infty$ converges in $\mathbb{R}$ a.s. on $\{\tau_\infty < \infty\}$, and $G(\cdot)$ is a bounded function by assumption, so (4.11) implies that $P(\tau_\infty < \infty) = 0$. $\square$

We turn now to the question of identifying suitable regulatory sets U for split-merge regulation that are both economically compelling and generate viable split-merge rules.

Lemma 4.10. *Suppose the following hold:*

(i) $n \geq 3$.

(ii) $\delta \in (0, \frac{n-1}{n+1})$.

(iii) The regulatory set,

$$U^\mu := \{\mu \in \Delta^n_+ \mid \mu_{(1)} < 1 - \delta\},$$

satisfies $U^\mu \subseteq O^\mu$.

(iv) $(U^\mu, \check{\mathfrak{R}}^\mu)$ is a split-merge regulation prerule.

(v) The functions $b(\cdot)$ and $\sigma(\cdot)$ are bounded on U^x.

Then the split-merge rule $(U^\mu, \mathfrak{R}^\mu)$ associated with $(U^\mu, \check{\mathfrak{R}}^\mu)$ exists and is viable.

Proof. Fix $n \geq 3$, and $\delta \in (0, \frac{n-1}{n+1})$. Recalling that U^μ is taken as a subset of the space Δ^n_+, we have

$$\partial U^\mu = \{\mu \in \Delta^n_+ \mid \mu_{(1)} = 1 - \delta\}.$$

The set U^μ is non-empty and open, and by assumption satisfies $U^\mu \subseteq O^\mu$. To check that $\check{\mathfrak{R}}^\mu$ is into $\bar{U}^\mu$, note that $\mu \in \partial U^\mu \Rightarrow \mu_{(1)} = 1 - \delta \Rightarrow \mu_{(n)} + \mu_{(n-1)} \leq \frac{2\delta}{n-1} < \frac{2}{n+1} < 1 - \delta$, where the first inequality follows from $\sum_{j=1}^n \mu_j - \mu_{(1)} = \delta$, implying that the smallest two weights can sum to at most $\frac{2\delta}{n-1}$, and the second inequality follows from $\delta \in (0, \frac{n-1}{n+1})$. This implies that all of the "new companies" created by $\check{\mathfrak{R}}$ are of relative size strictly smaller than $1 - \delta$. So $[\check{\mathfrak{R}}(\mu)]_{(1)} \leq 1 - \delta$ which implies that $\check{\mathfrak{R}}$ is into $\bar{U}^\mu$. If there were k companies of relative size $1 - \delta$ for $\mu \in \partial U^\mu$, then $\check{\mathfrak{R}}^\mu(\mu)$ has $k - 1$ companies of relative size $1 - \delta$. Therefore, applying the n-fold composition $(\check{\mathfrak{R}}^\mu \circ \ldots \circ \check{\mathfrak{R}}^\mu)$ to $\mu \in \bar{U}^\mu$ results in no companies of relative size $1 - \delta$. This implies that $\mathfrak{R}^\mu$ of Definition 4.8 is into U^μ, making $(U^\mu, \mathfrak{R}^\mu)$ a regulation rule and therefore a split-merge rule.

Consider the entropy function

$$S : \mathbb{R}^n_{++} \to \mathbb{R},$$
$$S(x) = -\sum_{i=1}^n x_i \log x_i.$$

We examine the change in entropy resulting from $\check{\mathfrak{R}}$. For $\mu \in \partial U^\mu$ we have $\mu_{(1)} = 1 - \delta$, and so

$$\begin{aligned} S(\check{\mathfrak{R}}^\mu(\mu)) - S(\mu) &= -\left[2\frac{\mu_{(1)}}{2}\log(\frac{\mu_{(1)}}{2}) + (\mu_{(n)} + \mu_{(n-1)})\log(\mu_{(n)} + \mu_{(n-1)})\right] \\ &\quad + \left[\mu_{(1)}\log\mu_{(1)} + \mu_{(n)}\log\mu_{(n)} + \mu_{(n-1)}\log\mu_{(n-1)}\right], \\ &= (1-\delta)\log 2 - (\mu_{(n)} + \mu_{(n-1)})\log(\mu_{(n)} + \mu_{(n-1)}) \\ &\quad + 2\left(\frac{\mu_{(n)}\log\mu_{(n)} + \mu_{(n-1)}\log\mu_{(n-1)}}{2}\right). \end{aligned}$$

Applying Jensen's inequality to the convex function $x \mapsto x\log x$, we get

$$\begin{aligned} S(\check{\mathfrak{R}}^\mu(\mu)) - S(\mu) &\geq (1-\delta)\log 2 + (\mu_{(n)} + \mu_{(n-1)})\left[-\log(\mu_{(n)} + \mu_{(n-1)}) + \log(\frac{\mu_{(n)} + \mu_{(n-1)}}{2})\right], \\ &= (1-\delta)\log 2 - (\mu_{(n)} + \mu_{(n-1)})\log 2, \\ &\geq \log 2\left[1 - \delta - \frac{2\delta}{n-1}\right] > 0, \end{aligned}$$

where the second to last inequality follows from the fact that $\sum_{j=1}^n \mu_j - \mu_{(1)} = \delta$, so the smallest two weights can sum to at most $\frac{2\delta}{n-1}$. The last inequality follows from the supposition that $\delta \in (0, \frac{n-1}{n+1})$. From this, the change in entropy of $\mathfrak{R}$ can be seen to satisfy

$$S(\mathfrak{R}^\mu(\mu)) - S(\mu) \geq \left[1 - \delta\left(\frac{n+1}{n-1}\right)\right] \log 2 > 0, \quad \forall \mu \in \partial U^\mu.$$

For $\varepsilon \in \mathbb{R}_{++}$, we may define the shifted entropy function

$$S^{(\varepsilon)} : \Delta_+^n \to \mathbb{R}$$

$$S^{(\varepsilon)}(\mu) := S(\varepsilon \mathbf{1}_n + \mu) = -\sum_{i=1}^n (\mu_i + \varepsilon)\log(\mu_i + \varepsilon),$$

where $\mathbf{1}_n$ is the column vector of n ones. For any $\kappa \in (0, \infty)$, the entropy function S restricted to domain $\{\mu : 0 < \mu_i \leq \kappa, \text{ for } 1 \leq i \leq n\}$ is uniformly continuous, so therefore $\varepsilon \in (0,1)$ can be chosen such that

$$\inf\left\{S^{(\varepsilon)}(\mathfrak{R}^\mu(\mu)) - S^{(\varepsilon)}(\mu) \mid \mu \in \partial U^\mu\right\} > 0. \tag{4.12}$$

The shifted entropy function satisfies $S^{(\varepsilon)} \in C_b^2(\Delta_+^n, \mathbb{R})$, so for an SDE (2.1) with $b(\cdot)$ and $\sigma(\cdot)$ bounded on U^x, an application of Lemma 4.9 with $G = S^{(\varepsilon)}$ proves the viability of $(U, \mathfrak{R})$. □

5 Arbitrage

5.1 FTAP in the Premodel and Regulated Model

We begin with the notions of arbitrage, relative arbitrage, and no free lunch with vanishing risk (NFLVR). Then we recall the fundamental theorem of asset pricing (FTAP) for the premodel and derive a corresponding FTAP for the regulated model.

Definition 5.1. An *arbitrage* over $[0,T]$ is an admissible trading strategy $\widetilde{H}$ such that

$$P[(\widetilde{H} \cdot \widetilde{X})_T \geq 0] = 1 \quad \text{and} \quad P[(\widetilde{H} \cdot \widetilde{X})_T > 0] > 0. \tag{5.1}$$

A *relative arbitrage* over $[0,T]$ with respect to portfolio $\widetilde{\eta}$ is a portfolio $\widetilde{\pi}$ such that

$$P(\widetilde{V}_T^{1,\widetilde{\pi}} \geq \widetilde{V}_T^{1,\widetilde{\eta}}) = 1 \quad \text{and} \quad P(\widetilde{V}_T^{1,\widetilde{\pi}} > \widetilde{V}_T^{1,\widetilde{\eta}}) > 0. \tag{5.2}$$

The corresponding notions of *strong arbitrage* and *strong relative arbitrage* are defined by making the first inequalities of (5.1) and (5.2) strict, respectively.

The condition NFLVR is a strengthening of the no arbitrage condition, roughly implying that not only are there no arbitrages, but no "approximate arbitrages." More specifically, a free lunch with vanishing risk (FLVR) is a sequence of trading strategies with uniformly bounded loss approximating an arbitrage. To fully motivate the NFLVR condition would take us too far from our primary purpose here, so we refer the interested reader to [3–5] for a complete exposition. Our definition is adapted from the monograph by Delbaen and Schachermayer [5].

Definition 5.2. For $T \in \mathbb{R}_{++}$ define

$$\widetilde{K} := \left\{ (\widetilde{H} \cdot \widetilde{X})_T \mid \widetilde{H} \text{ admissible} \right\},$$

which is a convex cone of random variables in $L^0(\Omega, \mathcal{F}_T, P)$, and

$$\widetilde{C} := \left\{ \widetilde{g} \in L^\infty(\mathcal{F}_T, P) \mid \widetilde{g} \leq \widetilde{f} \text{ for some } \widetilde{f} \in \widetilde{K} \right\}.$$

The condition *no free lunch with vanishing risk (NFLVR)* over $[0,T]$ with respect to $\widetilde{X}$ is

$$\bar{\widetilde{C}} \cap L^\infty_+(\mathcal{F}_T, P) = \{0\},$$

where $\bar{\widetilde{C}}$ denotes the closure of $\widetilde{C}$ with respect to the norm topology of $L^\infty(\mathcal{F}_T, P)$.

The Definitions 5.1 and 5.2 above are cast for the premodel. For the definitions of these notions in a viable regulated model, simply replace $\widetilde{X}$ with $\widehat{Y}$ and remove all other "$\widetilde{\ }$".

Investigations pertaining to relative arbitrage include E. R. Fernholz, Karatzas, and Kardaras [14], E. R. Fernholz and Karatzas [12], Banner and D. Fernholz [6], Ruf [29], and Mijatović and Urusov [20], to name a few. An arbitrage is essentially a relative arbitrage with respect to the money market account, modulo the uniform boundedness requirement of portfolios and their prohibition from investing in the money market, both of which can be relaxed as in D. Fernholz and Karatzas [7]. The existence of a relative arbitrage does not imply the existence of an arbitrage as illustrated by examples, often termed "bubble markets" (see [2,24,26,27]) where there exists an equivalent measure under which the stock process is a strict local martingale. In particular, if π is a relative arbitrage with respect to η, then the trading strategy $H := H^{1,\pi} - H^{1,\eta}$ need not satisfy the requirement that $H \cdot \widetilde{X}$ be uniformly bounded from below.

We assumed earlier that $\sigma(x)$ has rank n, $\forall x \in O^x$. This allows us to define

$$\begin{aligned} \theta :& O^x \to \mathbb{R}^n, \\ \theta(x) :=& \sigma^{-1}(x) b(x). \end{aligned} \tag{5.3}$$

In this setting the premodel is a complete market, and the market price of risk $\widetilde{\theta} := \theta(\widetilde{X})$ in the premodel and $\theta := \theta(Y)$ in the regulated model are the unique solutions, up to indistinguishability, of the market price of risk equations

$$\widetilde{\sigma}_t \widetilde{\theta}_t = \widetilde{b}_t, \qquad \text{and} \qquad \sigma_t \theta_t = b_t, \qquad \forall t \geq 0, \tag{5.4}$$

respectively. Fixing $T \in \mathbb{R}_{++}$, if basic square integrability holds for $\widetilde{\theta}$, namely

$$\int_0^T |\widetilde{\theta}_t|^2 dt < \infty, \quad \text{a.s.}, \tag{5.5}$$

where $|x|^2 = x'x$, then we may define the local martingale and supermartingale $\widetilde{Z}$ by

$$\widetilde{Z}_t := \mathcal{E}(-\widetilde{\theta} \cdot W)_t = \exp\left\{ -\int_0^t \widetilde{\theta}'_s dW_s - \frac{1}{2} \int_0^t |\widetilde{\theta}_s|^2 ds \right\}, \quad 0 \leq t \leq T. \tag{5.6}$$

The stochastic exponential $\mathcal{E}(S)$ of a continuous semimartingale S is given by

$$\mathcal{E}(S) = \exp\left\{ S - \frac{1}{2} \langle S, S \rangle \right\},$$

where $\langle S, S\rangle$ is the predictable quadratic variation of S (see [28] for more details).

If (5.5) fails, then no equivalent local martingale measure (ELMM) is possible for $\widetilde{X}$. For further details on the relationships amongst NFLVR, no arbitrage, and the integrability of $\left|\widetilde{\theta}\right|^2$, see Levental and Skorohod [19] which contains some constructive examples. When (5.5) holds, then the uniqueness of $\widetilde{\theta}$ as a solution to (5.4) implies that $\widetilde{Z}$ is the only candidate for an ELMM for the premodel. The fundamental theorem of asset pricing then hinges on whether or not $\widetilde{Z}$ is a martingale.

Theorem 5.3 (FTAP premodel)**.** *The following equivalence holds for any $T \in \mathbb{R}_{++}$:*

$$\widetilde{X} \text{ satisfies NFLVR over } [0,T] \iff \begin{cases} 1. \ \int_0^T |\widetilde{\theta}_t|^2 dt < \infty, \quad a.s., \\ 2. \ \{\widetilde{Z}_t\}_{0\le t\le T} \text{ is a martingale.} \end{cases}$$

When these conditions are satisfied, then $\widetilde{Q}_T$ given by $\frac{d\widetilde{Q}_T}{dP} := \widetilde{Z}_T$ is the unique ELMM for $\{\widetilde{X}_t\}_{0\le t\le T}$.

Proof. The proof is similar to that of Theorem 5.4. ☐

The FTAP in the regulated model closely parallels the unregulated case. If the regulated model is viable, and

$$\int_0^T |\theta_t|^2 dt < \infty, \quad \text{a.s.}, \tag{5.7}$$

then define $Z_t := \mathcal{E}(-\theta \cdot W)_t$, for $0 \le t \le T$. This familiar exponential martingale form of Z allows usage of the usual tools, such the Novikov criterion,

$$E\left[\exp\left\{\frac{1}{2}\int_0^T |\theta_s|^2 \, ds\right\}\right] < \infty, \tag{5.8}$$

to provide sufficient (although not necessary) conditions for $\{Z_t\}_{0\le t\le T}$ to be a martingale (see page 141 of [28]). As is the case for unregulated models, this martingality is an equivalent condition for NFLVR, providing an FTAP for the regulated model.

Theorem 5.4 (FTAP regulated model)**.** *For a viable regulated model, the following equivalence holds for any $T \in \mathbb{R}_{++}$:*

$$\widehat{Y} \text{ satisfies NFLVR over } [0,T] \iff \begin{cases} 1. \ \int_0^T |\theta_t|^2 dt < \infty, \quad a.s., \\ 2. \ \{Z_t\}_{0\le t\le T} \text{ is a martingale.} \end{cases}$$

When these conditions are satisfied, then Q_T given by $\frac{dQ_T}{dP} := Z_T$ is the unique ELMM for $\{\widehat{Y}_t\}_{0\le t\le T}$.

Proof. By the FTAP for locally bounded processes [3], NFLVR for $\widehat{Y}$ is equivalent to the existence of a local martingale measure for $\widehat{Y}$ (which is continuous). It remains to prove that the existence of a local martingale measure for $\widehat{Y}$ is equivalent to $\int_0^T |\theta_t|^2 dt < \infty$, a.s., and $\mathcal{E}(-\theta \cdot Y)_{0\le t\le T}$ being a martingale. The final claim will follow from the proof of the equivalence. Assuming the right hand side of the equivalence, fix $T \in \mathbb{R}_{++}$ and define measure Q_T by $\frac{dQ_T}{dP} := Z_T$. By supposition $(-\theta \cdot W)_T \in \mathbb{R}$, so $Z_T > 0$ a.s., implying that $Q_T \sim P$. Furthermore,

$$\begin{aligned} d[\widehat{Y}_t Z_t] &= Z_t d\widehat{Y}_t + \widehat{Y}_t dZ_t + (dZ_t)(d\widehat{Y}_t), \\ &= Z_t \left[Y_t \boxdot (b_t dt + \sigma_t dW_t) - Y_t \theta_t' dW_t - Y_t \boxdot \sigma_t \theta_t dt \right], \\ &= Z_t \left[Y_t \boxdot (\sigma_t dW_t) - Y_t \theta_t' dW_t \right], \end{aligned} \tag{5.9}$$

where we used Itô's lemma, Proposition 4.4, and the fact that θ solves the market price of risk equation (5.4). The above shows that $\widehat{Y}Z$ is a stochastic integral with respect to W and is therefore a P-local martingale. Therefore $\widehat{Y}$ is a Q_T-local martingale since any process $\{L_t\}_{0\le t\le T}$ is a Q_T martingale if and only if $\{L_t Z_t\}_{0\le t\le T}$ is a P-martingale.

For the converse, assume that there exists an ELMM $\check{Q}_T$ on $[0,T]$ with Radon-Nikodym derivative $\frac{d\check{Q}_T}{dP} = \check{Z}_T > 0$, a.s. Then there exists a continuous martingale $\{\check{Z}_t\}_{0\le t\le T}$ satisfying $\check{Z}_t := E[\check{Z}_T \mid \mathcal{F}_t]$. $\{\check{Z}_t\}_{0\le t\le T}$ has a.s. strictly positive paths since it is continuous and $\check{Z}_T > 0$ a.s. By the martingale representation theorem on the augmented Brownian filtration (Theorem 5.49 of [15]), $d\check{Z}_t = \phi_t' dW_t$ for some W-integrable process ϕ. Defining $\check{\theta} := \frac{\phi}{\check{Z}}$, then by Itô's lemma we have

$$d\log \check{Z}_t = \check{\theta}_t' dW_t - \frac{1}{2}\left|\check{\theta}_t\right|^2 dt, \quad 0 \le t \le T.$$

Therefore $\int_0^T \left|\check{\theta}_t\right|^2 dt < \infty$, a.s., and $\check{Z} = \mathcal{E}(-\check{\theta}\cdot W)$. By supposition, $\{\widehat{Y}_t\}_{0\le t\le T}$ is a $\check{Q}_T$-local martingale, so $\{\widehat{Y}_t\check{Z}_t\}_{0\le t\le T}$ is a P-local martingale. Therefore, by (local) martingale representation, $d[\widehat{Y}_t\check{Z}_t] = C_t dW_t$ for some W-integrable process C. Comparison with (5.9) implies that $\check{\theta}$ solves the market price of risk equation, that is $\sigma_t\check{\theta}_t = b_t$, $dt\times dP$-a.e. Since θ is the unique solution to the market price of risk equation, we have $\theta = \check{\theta}$, $dt\times dP$-a.e., implying that $Z_T = \check{Z}_T$ a.s. Therefore, $\check{Q}_T = Q_T$, so we have uniqueness. ∎

While the NFLVR condition is of theoretical interest, it is not necessarily of practical relevance. If we put ourselves in the situation of having to select from some set of candidate market models, some of which satisfy NFLVR and others of which do not, it may be a hopeless task to figure out whether financial data support or refute NFLVR. In fact, example 4.7 of Karatzas and Kardaras [17] shows that two general semimartingale models on the same stochastic basis may possess the same triple of predictable characteristics, with one admitting an arbitrage while the other does not. Even if we have reason to believe that a model admitting arbitrage or relative arbitrage is an accurate one, it may be the case that the arbitrage portfolios depend in a delicate way on the parameters of the model, b and σ here. In such a case any attempts to estimate these parameters from observed data would likely be too imprecise to lead to an investment strategy that could convincingly be called an approximation to an arbitrage.

In contrast to this, the condition of diversity is supported by world market data and the existence of antitrust laws in developed markets. The condition of uniform ellipticity of the covariance is not as readily apparent, but seems to be a reasonable manifestation of the idea that there is always at least some baseline level of volatility in markets. The significance of these two conditions is that in unregulated market models together they imply the existence of a long-only relative arbitrage portfolio that is functionally generated from the market weights [10,12,13], not requiring estimation of b or σ. It is therefore of great interest whether or not this implication carries over to regulated markets. The following parallel results provide comparable sufficient conditions for no relative arbitrage of any type in the premodel and the regulated model.

Corollary 5.5. *If $\widetilde{X}$ satisfies NFLVR over $[0,T]$, and $\sigma(\cdot)$ is bounded on O^x, then the unique ELMM of Theorem 5.3 is an EMM, and no portfolio is a relative arbitrage with respect to any other portfolio over $[0,T]$ in the premodel.*

Proof. The proof is similar to that of Corollary 5.6. ∎

Corollary 5.6. *If the regulated model is viable, $\widehat{Y}$ satisfies NFLVR over $[0,T]$, and $\sigma(\cdot)$ is bounded on U^x, then the unique ELMM of Theorem 5.4 is an EMM, and no portfolio is a relative arbitrage with respect to any other portfolio over $[0,T]$ in the regulated model.*

Proof. To prove the martingality, let Q_T be given by $\frac{dQ_T}{dP} = Z_T = \mathcal{E}(-\theta \cdot W)_T$. Proposition 4.4 and the market price of risk equation (5.4) imply that

$$d\widehat{Y}_t = Y_t \boxdot (\sigma_t dW_t^{(Q_T)}), \quad 0 \le t \le T,$$

where

$$W_t^{(Q_T)} := W_t + \int_0^t \theta_s ds, \quad 0 \le t \le T$$

is a Q_T-Brownian motion by Girsanov's theorem. This implies that

$$dV_t^{w,\pi} = V_t^{w,\pi} \pi_t' \sigma_t dW^{(Q_T)}, \quad 0 \le t \le T.$$

Therefore $V^{w,\pi}$ is an exponential Q_T-local martingale. Since $Y \in U^x$, $dt \times dP$-a.e., this implies that $\sigma = \sigma(Y)$ is bounded, $dt \times dP$-a.e. The portfolio π is uniformly bounded by definition, so $\{V_t^{w,\pi}\}_{0 \le t \le T}$ is a Q_T-martingale by the Novikov criterion (5.8).

Now suppose that π is a relative arbitrage with respect to η. Then by $Q_T \sim P$ it follows that

$$Q_T(V_T^{w,\pi} \ge V_T^{w,\eta}) = 1 \quad \text{and} \quad Q_T(V_T^{w,\pi} > V_T^{w,\eta}) > 0.$$

However $\{V_t^{w,\pi}\}_{0 \le t \le T}$ and $\{V_t^{w,\eta}\}_{0 \le t \le T}$ are both Q_T-martingales, so their difference is also a Q_T-martingale, with $E^{Q_T}[V_T^{w,\pi} - V_T^{w,\eta}] = w - w = 0$. This contradicts the relative arbitrage property above, so this market admits no pair of relative arbitrage portfolios. □

5.2 Diversity, Intrinsic Volatility, and Relative Arbitrage

The works by Robert Fernholz et al. [9,11,12] on diversity and arbitrage prove that for unregulated markets, over an arbitrary time horizon, there exist strong relative arbitrage portfolios with respect to the market portfolio in any weakly diverse market satisfying certain assumptions and regularity conditions. Furthermore, they show how such relative arbitrages can be constructed as long-only portfolios which are functionally generated from $\widetilde{\mu}$, not requiring knowledge of $\widetilde{b}$ or $\widetilde{\sigma}$. A sufficient set of assumptions and regularity are given by the following.

Assumption 5.7. *(i) The capitalizations are modeled by an Itô process*

$$d\widetilde{X}_t = \widetilde{X}_t \boxdot \left(\widetilde{b}_t dt + \widetilde{\sigma}_t dW_t\right),$$
$$\widetilde{X}_0 = x_0 \in \mathbb{R}^n_{++},$$

where $\widetilde{b}$ and $\widetilde{\sigma}$ are progressively measurable processes satisfying $\forall T \in \mathbb{R}_{++}$,

$$\sum_{i=1}^n \left(\int_0^T |\widetilde{b}_{i,t}| dt + \sum_{\nu=1}^d \int_0^T \left|\widetilde{\sigma}_{i\nu,t}\right|^2 dt \right) < \infty, \quad a.s.$$

(ii) The capitalizations' covariance process is uniformly elliptic:

$$\exists \varepsilon > 0: \ a.s. \quad \varepsilon |\xi|^2 \le \xi \widetilde{\sigma}_t \widetilde{\sigma}_t' \xi, \quad \forall t \ge 0, \ \forall \xi \in \mathbb{R}^n.$$

(iii) Companies pay no dividends and therefore can't regulate their size by this means.

(iv) The number of companies is a constant.

(v) The market is weakly diverse.

(vi) Trading may occur in continuous time, in arbitrary quantities, is frictionless, and does not impact prices.

These conditions have been generalized by Fernholz and Karatzas [12]. They have shown that the uniform ellipticity assumption may be dropped, and the market need not be weakly diverse if it satisfies one of several notions of "sufficient intrinsic volatility." One measure of the intrinsic volatility in the market is the excess growth rate of the market portfolio,

$$\gamma^*_{\widetilde{\mu},t} = \frac{1}{2}\left(\sum_{i=1}^{n} \widetilde{\mu}_{i,t}\widetilde{a}_{ii,t} - \widetilde{\mu}_t'\widetilde{a}_t\widetilde{\mu}_t\right).$$

The following proposition provides an example of a "sufficient intrinsic volatility" type condition.

Proposition 5.8 (adapted from Proposition 3.1 of [12])**.** *Assume an unregulated market model satisfies items (i), (iii), (iv), and (vi) of Assumption 5.7. Additionally suppose there exists a continuous, strictly increasing function* $\widetilde{\Gamma} : [0,\infty) \to [0,\infty)$ *with* $\widetilde{\Gamma}(0) = 0$, $\widetilde{\Gamma}(\infty) = \infty$, *and satisfying a.s.*

$$\widetilde{\Gamma}(t) \le \int_0^t \widetilde{\gamma}^*_{\widetilde{\mu},s} ds < \infty, \qquad \text{for all} \quad 0 \le t < \infty. \tag{5.10}$$

Then there exists a functionally generated, long-only portfolio that is a strong relative arbitrage with respect to the market portfolio over sufficiently long horizon.

Proof. See Proposition 3.1 of [12]. ☐

A diverse regulated market is simply a regulated market in which μ in place of $\widetilde{\mu}$ satisfies Definition 2.3. By Lemma 3.4 of [13] (the proof of which is merely algebraic and has nothing to do with whether the market model is regulated or not) in a uniformly elliptic, diverse market (regulated market), $\widetilde{\gamma}^*_{\widetilde{\mu}}$ (γ^*_μ) satisfies

$$\frac{\varepsilon\delta}{2} \le \widetilde{\gamma}^*_{\widetilde{\mu},t}, \quad \forall t \ge 0 \qquad \left(\frac{\varepsilon\delta}{2} \le \gamma^*_{\mu,t}, \quad \forall t \ge 0\right). \tag{5.11}$$

In this equation ε satisfies $\varepsilon\,|\xi|^2 \le \xi\widetilde{\sigma}_t\widetilde{\sigma}_t'\xi$ ($\varepsilon\,|\xi|^2 \le \xi\sigma_t\sigma_t'\xi$), $\forall t \ge 0$, $\forall \xi \in \mathbb{R}^n$, and δ satisfies $\mu_{(1)} \le 1-\delta$ ($\widetilde{\mu}_{(1)} \le 1-\delta$). This implies that in any uniformly elliptic, diverse market (regulated market), that (5.10) (its regulated market counterpart) is satisfied by $\widetilde{\Gamma}(t) = \frac{\varepsilon\delta}{2}t$ ($\Gamma(t) = \frac{\varepsilon\delta}{2}t$). In the examples of section 6, NFLVR and no relative arbitrage hold for the regulated markets, while diversity and uniform ellipticity also hold, implying that (5.10) is satisfied in these cases. Therefore, in contrast to the premodel, the conditions of weak diversity and uniform ellipticity or, *a fortiori,* sufficient intrinsic volatility, are not sufficient for the existence of relative arbitrage in the regulated model.

6 Examples of Regulated Markets

In this section we apply split-merge regulation to geometric Brownian motion (GBM) and a log-pole market as premodels. In both cases the regulated market is diverse, uniformly elliptic, and therefore satisfies the regulated market analog of the sufficient intrinsic volatility condition (5.10). In both cases the regulated markets satisfy NFLVR and admit no pair of relative arbitrage portfolios.

6.1 Geometric Brownian Motion

Consider the case where the unregulated capitalization process is a GBM,

$$d\widetilde{X}_t = \widetilde{X}_t \boxdot [bdt + \sigma dW_t],$$
$$\widetilde{X}_0 = x_0 \in O^x = \mathbb{R}^n_{++},$$

for some $n \geq 3$, $b \in \mathbb{R}^n$, and $\sigma \in \mathbb{R}^{n\times n}$ of rank n. GBM satisfies NFLVR on all $[0,T]$, $T \in \mathbb{R}_{++}$ and has constant volatility, so it is not weakly diverse on any $[0,T]$ and admits no pair of relative arbitrage portfolios (see section 6 of [13]). Select $\delta \in (0, \frac{n-1}{n+1})$ and define the regulatory set

$$U^\mu := \{\mu \in \Delta^n_+ \mid \mu_{(1)} < 1-\delta\}.$$

By Lemma 4.10 the associated split-merge rule exists and is viable. Since $\theta := \sigma^{-1}b$ is a constant, the Novikov criterion (5.8) for $Z := \mathcal{E}(-\theta \cdot W)$ is satisfied, and Z is therefore a martingale. This implies that for any $T \in \mathbb{R}_{++}$, Q_T specified by $\frac{dQ_T}{dP} := Z_T$ is an ELMM for $\{\widehat{Y}_t\}_{0\leq t\leq T}$ by Theorem 5.4. Furthermore, $\{\widehat{Y}_t\}_{0\leq t\leq T}$ is a Q_T-martingale, and the regulated market is free of relative arbitrage by Corollary 5.6. The regulated market is diverse since $P(\mu_t \in \bar{U}^\mu,\ \forall t \geq 0) = P(\mu_{(1),t} \leq 1-\delta,\ \forall t \geq 0) = 1$, which implies that (5.11) and thus (5.10) are satisfied. Therefore, in this regulated market, the notions of sufficient intrinsic volatility and diversity coexist with NFLVR and no relative arbitrage.

6.2 Log-Pole Market

So-called "log-pole" market models provide examples of diverse, unregulated markets. Diversity is maintained in these markets by means of a log-pole-type singularity in the drift of the largest capitalization, diverging to $-\infty$ as the largest weight $\mu_{(1)}$ approaches the diversity cap $1-\delta$. This suggests that it becomes unfavorable at some point to hold significant quantities of the largest company in a portfolio. Explicit portfolios which are relative arbitrages with respect to the market portfolio over any prespecified time horizon may be formed by down-weighting the largest company in a controlled manner [13, 14]. But when regulation is applied, keeping the largest weight $\mu_{(1)}$ away from $1-\delta$, then these arbitrage opportunities vanish.

Following section 9 of [13] (see [14] for more details and generality) fix $n \geq 3$, $\delta \in (0, \frac{1}{2})$ and consider the unregulated capitalization process $\widetilde{X}$, the pathwise unique strong solution to

$$d\widetilde{X}_t = \widetilde{X}_t \boxdot \left(b(\widetilde{X}_t)dt + \sigma W_t\right),$$
$$\widetilde{X}_0 = x_0 \in O^x := \{x_0 \in \mathbb{R}^n_{++} \mid \mu_{(1)}(x_0) < 1-\delta\},$$

where $\sigma \in \mathbb{R}^{n\times n}$ is rank n. The function $b(\cdot)$ is given by

$$b_i(x) := \frac{1}{2}a_{ii} + g_i 1_{\mathcal{Q}_i^c}(x) - \frac{c}{\delta}\frac{1_{\mathcal{Q}_i}(x)}{\log\left((1-\delta)/\mu_i(x)\right)}, \qquad 1 \leq i \leq n,$$

where $\{g_i\}_1^n$ are non-negative numbers, c is a positive number, and

$$\mathcal{Q}_1 := \left\{x \in \mathbb{R}^n_{++} \mid x_1 \geq \max_{2\leq j\leq n} x_j\right\}, \qquad \mathcal{Q}_n := \left\{x \in \mathbb{R}^n_{++} \mid x_n > \max_{1\leq j\leq m-1} x_j\right\},$$
$$\mathcal{Q}_i := \left\{x \in \mathbb{R}^n_{++} \mid x_i > \max_{1\leq j\leq i-1} x_j, \quad x_i \geq \max_{i+1,\leq j\leq n} x_j\right\}, \quad \text{for } i = 2, \ldots, n-1.$$

When $x \in \mathcal{Q}_i$, then x_i is the largest of the $\{x_j\}_1^n$ with ties going to the smaller index. In this model each company behaves like a geometric Brownian motion when it is not the largest. The largest company is repulsed away from the log-pole-type singularity in its drift at $1-\delta$. Strong existence and pathwise uniqueness for this SDE are guaranteed for any x_0 in O^x by [32] (see also [14]). The capitalizations satisfy $P(\widetilde{X}_t \in O^x, \quad \forall t \geq 0) = 1$, so this premodel is diverse. The function $b(\cdot)$ is locally bounded since the coefficients of $1_{\mathcal{Q}_i^c}(x)$ and $1_{\mathcal{Q}_i}(x)$ are continuous on O^x, and the singularity at $\mu_{(1)}(x) = 1-\delta$ is away from the boundary of each $\mathcal{Q}_i$ for $\delta \in (0, \frac{1}{2})$. Since the market is diverse and has constant volatility, then by the results of Fernholz [11,13] over arbitrary horizon the market admits long-only relative arbitrage portfolios which are functionally generated from the market portfolio. Furthermore since σ is a constant, Corollary 5.5 implies that $\widetilde{X}$ has no ELMM, so admits a FLVR.

This model may be regulated in such a way to remove these relative arbitrage opportunities and satisfy NFLVR. Picking $\delta' \in (\delta, \frac{n-1}{n+1})$ and $x_0 \in U^x$, define the regulatory set to be

$$U^\mu := \{\mu \in \Delta_+^n \mid \mu_{(1)} < 1-\delta'\} \subseteq O^\mu.$$

The associated split-merge regulation rule exists and is viable by Lemma 4.10. From the form of $b(\cdot)$, there exists $\kappa \in \mathbb{R}_{++}$ such that

$$|b(x)| \leq \kappa, \qquad \forall x \in \bar{U}^x.$$

Therefore with $\theta := \sigma^{-1} b$, there exists $\Theta \in \mathbb{R}_{++}$ such that a.s.

$$|\ \theta(Y_t)\ | \leq \Theta < \infty, \qquad \forall t \geq 0.$$

This implies that the Novikov criterion (5.8) is satisfied for $Z := \mathcal{E}(-\theta \cdot X)$, and so Z is a martingale. By Theorem 5.4, for any $T \in \mathbb{R}_{++}$, Q_T specified by $\frac{dQ_T}{dP} := Z_T$ is an ELMM for $\{\widehat{Y}_t\}_{0 \leq t \leq T}$. Furthermore, $\{\widehat{Y}_t\}_{0 \leq t \leq T}$ is a Q_T-martingale, and the regulated market is free of relative arbitrage by Corollary 5.6. The diversity of the regulated market implies that (5.11) and thus (5.10) are satisfied. Therefore, in this regulated market, the notions of sufficient intrinsic volatility and diversity coexist with NFLVR and no relative arbitrage.

The pathology of this premodel is that the largest company's drift approaches $-\infty$ as $\mu_{(1)}$ approaches $1-\delta$. The cure is to prevent the largest company from approaching $1-\delta$ by regulation and thus bound the worst expected rate of return. The pathological region of Δ_+^n is excised from μ's state space by the regulation procedure, and the result is an arbitrage-free market.

7 Conclusions

Models in which diversity is maintained by a drift-type condition, whereby the rate of expected return of the largest company must become unboundedly negative compared to the rate of expected return of some other company in the economy, cover only one particular mechanism by which diversity may be achieved. These are reasonable models for markets in which diversity is maintained by some combination of fines on big companies imposed by antitrust regulators, and/or the biggest company consistently delivering less return than the other companies for other reasons. In such markets there is an intuitive disadvantage to holding the stock of the largest company, since its upside potential is limited relative to that of the other companies. Fernholz showed that this is not merely a perceived disadvantage, but that any passive portfolio holding shares of the biggest company can be strictly outperformed by functionally generated portfolios which are relative arbitrages with respect to the former.

If regulators maintain diversity within an equity market by utilizing regulatory breakup, then the situation is quite different. This mechanism need not open the door to arbitrageurs. It entails no systematic debasement of the total capital in the economy and for many models can be shown to be arbitrage-free, admitting an equivalent martingale measure.

The current situation in U.S. markets is that regulatory breakups are very rarely used, except in cases reversing a recent merge or acquisition. This suggests that the previous conclusion of Fernholz, Karatzas et al., that in the past conditions in U.S. markets have likely been compatible with functionally generated relative arbitrage with respect to the market portfolio [12], is not threatened by this result. If, however, regulatory breakup were to become a primary tool of antitrust regulators, then, modulo our assumption of capital conservation, the argument for existence of functionally generated relative arbitrage in diverse markets would be substantially weakened.

A natural future development of regulatory breakup models would be to construct models in which the number of companies is a stochastic process. This would allow splits and merges to occur at different times and permit investigation into market diversity and stability over time.

Another possible extension is to premodels in which the capitalization process is a strong Markov semimartingale with jumps, or even to general semimartingales, paying heed to remark 3.2 on the necessity of specifying extra modeling information in this case.

The notions of diversity combined with uniform ellipticity and to a lesser extent the more general "sufficient intrinsic volatility of the market" are useful conditions in that empirical observations can either support or refute them. This is in contrast to the rather abstract and normative condition of existence of an equivalent martingale measure, for which it may be hopeless to make a case for or against via observed data alone. That these conditions do not imply relative arbitrage in regulated market models prompts the question of whether a general, empirically verifiable condition can be found that is consistent across regulated and unregulated market models.

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